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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 33 (1992), No. 2, 91--97

Persistent URL: `http://dml.cz/dmlcz/701980`

1992 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 33. NO. 2

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Received 10 May 1992

1. Introduction

Investigation of stochastic process is closely related to probability measures defined on their sample paths. Therefore, Borel probability measures considered on a special topological space of functions are of great interest for such purpose. Especially, topological spaces of discontinuous functions are needful.

Original idea of a topological set of discontinuous functions on the interval $\langle 0, 1\rangle$ was introduced by Skorohod (1956). He studied five different distances in his paper. One of them has an equivalent one giving a Polish space; see Billingsley (1968). The notion of Skorohod space was generalized by Straf (1969) and Neuhaus (1971) for a set of discontinuous functions on a rectangle $\langle 0, 1\rangle^k$. Let us denote these spaces by $D_k(0, 1)$. The space $D_1(0, 1)$ coincides with the original definition on the interval $\langle 0, 1\rangle$. A larger space equipped with the same distance as $D_k(0, 1)$ was introduced by Straf (1970).

Skorohod spaces are very useful namely for a study of weak convergence of stochastic processes. There are convergence criteria for $D_1(0, 1)$, survey of which is in Billingsley (1968). Moreover, there is a criterion derived by Bickel and Wichura (1971) for $D_k(0, 1)$. An improvement of that one is given by Lachout (1988).

This paper aims at showing a possibility how to define a metric space of discontinuous function on a general subset of R^k. Unfortunately, completeness need not take place necessarily.

2. Skorohod spaces

This chapter gives a general view of a Skorohod space. A basic space is defined and the others are developed from it by aid of an embedding. Let us denote the set of all real numbers by R and its two-point compactification by R^*. For the sake of a special type of continuity, the following quadrants are important.

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If $t \in (R^*)^k$ and if, for $i = 1, \dots, k$, S_i is one of the relations $<$ and $\geqq$, let

$$Q_{S_1 \dots S_k}(t) = \{s \in R^k | \, s_i S_i t_i , \quad i = 1, \dots k\} .$$

A Polish space of discontinuous functions on R^k is a basic space in our considerations.

Definition 1. *Let us denote by $D(R^k)$ a set of all functions $f : R^k \to R$ keeping the following properties:*

(1) *For every $t \in (R^*)^k$ and an arbitrarily chosen quadrant,*

$$\lim_{\substack{s \to t \\ s \in Q_{S_1 \dots S_k}(t)}} f(s) \text{ exists} .$$

(2) *For every $t \in R^k$,*

$$\lim_{\substack{s \to t \\ s \in Q_{\geqq, \geqq, \dots, \geqq}(t)}} f(s) = f(t) \quad \textit{takes place} .$$

Define a distance d_k of two functions $f, g \in D(R^k)$ as follows

(3)
$$d_k(f, g) = \min \left\{ \varepsilon > 0 , \left| \begin{array}{l} |f(t) - g \circ \lambda(t)| \leqq \varepsilon , \quad \left| \ln \left(\dfrac{\arctan \lambda_i(t_i) - \text{acrtan } \lambda_i(s_i)}{\arctan t_i - \arctan s_i} \right) \right| \leqq \varepsilon \\ \text{for every} \quad t, s \in R^k \quad \text{and some} \quad \lambda \in \Lambda^k \end{array} \right. \right\}$$

where Λ is a set of all injective increasing maps from R to R, i.e. $\lim_{x \to -\infty} \varphi(x) = -\infty$, $\lim_{x \to +\infty} \varphi(x) = +\infty$ *whenever $\varphi \in \Lambda$.*

Theorem 1. *The space $D(R^k)$ equipped with the distance d_k is a Polish space.*

Proof: Every $f \in D(R^k)$ can be extended to $(R^*)^k$ by the following procedure. For $t \in (R^*)^k$

$$\tilde{f}(t) = \lim_{\substack{s \to t \\ s \in Q_{S_1 \dots S_k}(t)}} f(s) , \quad \text{where either} \quad S_i \text{ is } \geqq \text{ if } t_i < +\infty \quad \text{or} \quad S_i \text{ is } < \text{ if } t_i = +\infty .$$

Define $\Phi : D(R^k) \to D_k(0, 1)$ such that

$$\Phi f(t) = \tilde{f}\left(\tan\left(\pi t_i - \frac{\pi}{2}\right), ,,,, \tan\left(\pi t_k - \frac{\pi}{2}\right)\right)$$

for arbitrarily chosen $t \in \langle 0, 1 \rangle^k$, $f \in D(R^k)$ where the convention $\tan(-\pi/2) = -\infty$, $\tan(\pi/2) = +\infty$ is used. Φ is a homeomorphism between $D(R^k)$ and $D_k(0, 1)$. Therefore, $D(R^k)$ is a Polish space since $D_k(0, 1)$ is a Polish space; see Straf (1969), Neuhaus (1971). Q.E.D

Remark that the function arctan used in the definition of the distance d_k may be replaced by an arbitrary increasing bounded continuous map from R to R without any loss of topological structure.

Definition 2. *For a nonempty subset V of R^k, let us denote by $D(V)$ the set of all functions $f : V \to R$ which are restrictions of functions $g \in D(R^k)$ to the set V.*

The set $D(V)$ is a natural candidate for a generalized Skorohod space. But to carry over the distance meets some difficulties.

Definition 3. *Let us call $D(V)$ a Skorohod space with an embedding ψ if*

(4) $\quad D(V) \overset{\psi}{\hookrightarrow} D(R^k)$ *is a* $1-1$ *map such that* $\psi f / V = f$ *for every* $f \in D(V)$.

Definition 3 gives a natural generalization of the notion of the Skorohod space.

Theorem 2. *If $D(V)$ is a Skorohod space with an embedding ψ then the space $D(V)$ equipped with a distance d,*

$$d(f, g) = d_k(\psi f, \psi g),$$

is a separable metric space.

Proof: The space $\psi(D(V))$ equipped with the distance d_k must be a separable metric space. The topological spaces $\psi(D(V))$ equipped with the distance d_k and $D(V)$ equipped with the distance d are homeomorphic. Therefore $D(V)$ equipped with d is a separable metric space. Q.E.D.

The space can be incomplete as the following example shows.

Example: Consider $D(\langle 0, 1\rangle)$ with embedding

$$\psi f(t) = \begin{cases} f(0) & t < 0 \\ f(t) & 0 \leqq t \leqq 1 \\ f(1) & t > 1 \end{cases}$$

and the sequence of functions

$$f_n(t) = \begin{cases} 1 & 0 \leqq t < 1/n \\ 0 & 1/n \leqq t \leqq 1 \end{cases}.$$

It is a Cauchy sequence since $\psi f_n \to h$ where

$$h(t) = \begin{cases} 1 & t < 0 \\ 0 & t \geqq 0 . \end{cases}$$

But $\psi\ (h/\langle 0, 1\rangle) \equiv 0 \not\equiv h$ and thus the sequence of f_n cannot converge.

In the sequel finite unions of open rectangles are considered. The special kind of sets gives a possibility to obtain Polish spaces. That is because a very natural mapping can be employed.

Definition 4. *For a subset T of R^k we denote by $\hat{\psi}$ the mapping $\hat{\psi} : D(T) \to R^{(R^k)}$,*

$$\hat{\psi} f(t) = \begin{cases} \lim\limits_{\substack{s \to t \\ s \in Q_{\geq, \ldots, \geq}(t) \cap T}} f(s) & \text{if } t \in \operatorname{clo}(Q_{\geq, \ldots, \geq}(t) \cap T - \{t\}) \\ 0 & \text{otherwise,} \end{cases}$$

where clo (A) *denotes a closure of the set A.*

Lemma 1. *$\hat{\psi}$ fulfils* (4) *for every finite union of open rectangles* $T = \bigcup_{j=1}^{J} \times_{i=1}^{k} (a_{ij}, b_{ij})$.

Proof:

i) We show $\hat{\psi} f/T = f$.

If $t \in T$ then there exists an open set G such that $t \in G \subset T$. Hence

$$\hat{\psi} f(t) = \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t) \cap T}} = f(t) \quad \text{for each} \quad f \in D(T)\,.$$

ii) Let us show $\hat{\psi} : D(T) \to D(R^k)$.

a) If $t \in R^k - \operatorname{clo}(T)$ then there exists an open set G, $t \in G \subset R^k - \operatorname{clo}(T)$. Hence $\hat{\psi} f(s) = 0$ for each $s \in G$ and $\hat{\psi} f$ is continuous at the point t.

b) If $t \in T$ then there exists an open set G, $t \in G \subset T$. Hence the point t fulfils (1) and (2) since $f \in D(T)$.

c) Let $t \in \partial T$ and $Q_{\geqq, \dots, \geqq}(t) \cap T = \emptyset$. Then

$$\hat{\psi} f(t) = \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t)}} \hat{\psi} f(s) = 0\,.$$

d) Let $t \in \partial T$ and $Q_{\geqq, \dots, \geqq}(t) \cap T \neq \emptyset$. Since T is a finite union of open rectangles there exists an open set G, $t \in G$ such that

$$Q_{\geqq, \dots, \geqq}(t) \cap \operatorname{clo}(T) \cap G = Q_{\geqq, \dots, \geqq}(t) \cap G\,.$$

Hence

$$\hat{\psi} f(t) = \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t) \cap T}} f(s) = \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t) \cap G \cap T}} f(s) =$$

$$= \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t) \cap G}} \hat{\psi} f(s) = \lim_{\substack{s \to t \\ s \in Q_{\geqq, \dots, \geqq}(t)}} \hat{\psi} f(s)\,.$$

e) Let $t \in \partial T$ and $Q_{S_1, \dots, S_k}(t) \cap T = \emptyset$, at least one of S_i is equal to $<$. Hence

$$\lim_{\substack{s \to t \\ s \in Q_{S_1, \dots, S_k}(t)}} \hat{\psi} f(s) = 0 \quad \text{since} \quad Q_{S_1, \dots, S_k}(t) \subset R^k - \operatorname{clo} T\,.$$

f) Let $t \in \partial T$ and $Q_{S_1, \dots, S_k}(t) \cap T \neq \emptyset$, at least one S_i is $<$. Hence there exists an open set G, $t \in G$ such that

$$Q_{S_1, \dots, S_k}(t) \cap G = Q_{S_1, \dots, S_k}(t) \cap G \cap \operatorname{clo} T$$

since T is a finite union of open rectangles.

Therefore

$$\lim_{\substack{s \to t \\ s \in Q_{S_1, \dots, S_k}(t)}} \hat{\psi} f(s) = \lim_{\substack{s \to t \\ s \in Q_{S_1, \dots, S_k}(t) \cap G}} \hat{\psi} f(s) = \lim_{\substack{s \to t \\ s \in Q_{S_1, \dots, S_k}(t) \cap T}} f(s)$$

exists since $f \in D(T)$. Q.E.D.

Theorem 3. *Skorohod space $D(T)$ with embedding $\hat{\psi}$ is a Polish space for each finite union of open rectangles $T = \bigcup_{j=1}^{J} \times_{i=1}^{k} (a_{ij}, b_{ij})$.*

Proof: It is enough to prove a completeness because Lemma 1 and Theorem 2 guarantee that $D(T)$ is a separable metric space.

Let $f_n \in D(T)$ be a Cauchy sequence. Hence $\hat{\psi} f_n$ is a Cauchy sequence and $\hat{\psi} f_n \to h$ since $D(R^k)$ is a Polish space.

Completeness of $D(T)$ will be proved if $\hat{\psi}(h/T) = h$. For that goal it is enough to show that $h(t) = 0$ for each $t \in R^k - \text{clo } T$. Let $t \in R^k - \text{clo } T$. Then there exists an open ball G, $t \in G \subset R^k - \text{clo } T$. Hence $\hat{\psi} f_n(s) = 0$ for every $s \in G$ and $\hat{\psi} f_n \to h$. Therefore $h(t) = 0$ as well. Q.E.D.

The introduced spaces give a generalization of $D_k(0, 1)$ in the following sense.

Theorem 4. *The bijection* $\varrho : D_k(0, 1) \to D((0, 1)^k) : f \mapsto f/(0, 1)^k$ *is continuous. But* ϱ^{-1} *is not continuous.*

Proof: Evidently, ϱ is bijection since f belonging to $D_k(0, 1)$ is determined by its values on $(0, 1)^k$.

a) Consider a sequence of the functions

$$g_n(t) = \begin{cases} 0 & t \in \langle 0, 1/n)^k \\ 1 & \text{otherwise} \end{cases}.$$

These functions belong to $D_k(0, 1)$ and have not limit there. But $\varrho g_n \to h \equiv 1$ in $D((0, 1)^k)$ equipped by $\hat{\psi}$. Thus ϱ^{-1} is not continuous.

b) Let us prove the continuity of ϱ.

Let $f_n \in D_k(0, 1)$ and $f_n \to f$ in $D_k(0, 1)$. Let $\varepsilon > 0$ be arbitrarily chosen but fixed in the sequel.

Then there exists $n_0 \in N$ such that for every $n \geqq n_0$ we have $\lambda^n \in \Lambda_{0,1}^k$, where $\Lambda_{0,1}$ is the set of all continuous increasing maps of $\langle 0, 1\rangle$ into itself i.e. $\lambda(0) = 0$ and $\lambda(1) = 1$ for each $\lambda \in \Lambda_{0,1}$, fulfiling

$$|f_n \circ \lambda^n(t) - f(t)| \leqq \varepsilon \quad \text{for each} \quad t \in \langle 0, 1\rangle^k$$

and
$$\left|\ln\left(\frac{\lambda_i^n(t) - \lambda_i^n(s)}{t - s}\right)\right| \leqq \varepsilon \quad \text{for each} \quad 0 \leqq s < t \leqq 1\,, \quad i = 1, \ldots, k\,.$$

Define maps φ^n of R^k into itself by the following prescription

$$\varphi_i^n(t) = \begin{cases} \lambda_i^n(t) & \text{if} \quad 0 \leqq t \leqq 1 \\ t & \text{otherwise} \end{cases} \quad \text{for each} \quad i = 1, \ldots, k\,.$$

Certainly, φ^n belongs to Δ^k and $|\hat{\psi}\varrho f_n \circ \lambda^n(t) - \hat{\psi}\varrho f(t)| \leqq \varepsilon$ for each $t \in R^k$. It remains to prove that the other part of (3) is small as well.

Evidently we have for every $i = 1, \ldots, k$

$$\sup_{s<t} \left|\ln\left(\frac{\arctan \varphi_i^n(t) - \arctan \varphi_i^n(s)}{\arctan t - \arctan s}\right)\right| =$$

$$\sup_{0 \leqq s < t \leqq 1} \left|\ln\left(\frac{\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s)}{\arctan t - \arctan s}\right)\right|.$$

Fix $0 \leqq s < t \leqq 1$ and $i = 1, \ldots, k$. Hence we obtain

$$\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s) = \int_{\lambda_i^n(s)}^{\lambda_i^n(t)} \frac{1}{1 + x^2}\, dx \leqq \int_{e^{-\varepsilon}s}^{e^{-\varepsilon}s + e^{\varepsilon}(t-s)} \frac{1}{1 + x^2}\, dx$$

because of $\lambda_i^n(t) - \lambda_i^n(s) \leqq e^{\varepsilon}(t - s)$ and $0 < e^{-\varepsilon}s \leqq \lambda_i^n(s)$. Thus

$$\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s) \leqq e^{-\varepsilon} \int_s^{s + e^{2\varepsilon}(t-s)} \frac{1}{1 + e^{-2\varepsilon}x^2}\, dx \leqq$$

$$\leqq \int_s^t \frac{1}{1 + x^2}\, dx + \int_s^t \frac{(1 - e^{-2\varepsilon})\, x^2}{(1 + x^2)(1 + e^{-2\varepsilon}x^2)}\, dx + \int_t^{s + e^{2\varepsilon}(t-s)} \frac{1}{1 + e^{-2\varepsilon}x^2}\, dx \leqq$$

$$\leqq \arctan t - \arctan s + (1 - e^{-2\varepsilon})(t - s) + (e^{2\varepsilon} - 1)(t - s) =$$

$$= \arctan t - \arctan s + (e^{2\varepsilon} - e^{-2\varepsilon})(t - s)\,.$$

Similarly we obtain a lower limit

$$\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s) =$$

$$= \int_{\lambda_i^n(s)}^{\lambda_i^n(t)} \frac{1}{1 + x^2}\, dx \geqq \int_{e^{\varepsilon}s}^{e^{\varepsilon}s + e^{-\varepsilon}(t-s)} \frac{1}{1 + x^2}\, dx \geqq \int_s^{s + e^{-2\varepsilon}(t-s)} \frac{1}{1 + e^{2\varepsilon}x^2}\, dx =$$

$$= \int_s^t \frac{1}{1 + x^2}\, dx - (e^{2\varepsilon} - 1) \int_s^t \frac{x^2}{(1 + x^2)(1 + e^{2\varepsilon}x^2)}\, dx - \int_{s + e^{-2\varepsilon}(t-s)}^t \frac{1}{1 + e^{2\varepsilon}x^2}\, dx \geqq$$

$$\geqq \arctan t - \arctan s - (e^{2\varepsilon} - e^{-2\varepsilon})(t - s)\,.$$

Moreover,

$$\arctan t - \arctan s = \int_s^t \frac{1}{1 + x^2}\, dx \geqq \frac{1}{2}(t - s)\,.$$

Therefore, we have a lower and an upper bound

$$1 - 2(e^{2\varepsilon} - e^{-2\varepsilon}) \leqq \frac{\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s)}{\arctan t - \arctan s} \leqq 1 + 2(e^{2\varepsilon} - e^{-2\varepsilon})\,.$$

Consequently,

$$\left| \ln \left(\frac{\arctan \lambda_i^n(t) - \arctan \lambda_i^n(s)}{\arctan t - \arctan s} \right) \right| \leqq -\ln\left(1 - 2(e^{2\varepsilon} - e^{-2\varepsilon})\right)$$

if ε is small enough.

We have proved $\varrho f_n \to \varrho f$ in $D\left((0, 1)^k\right)$. Q.E.D.

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