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A Note on Banach Spaces and Subdifferentials of Convex Functions

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A characterization of the reflexivity of Banach spaces by means of bilinear forms is given and subdifferenials of convex functions are studied in connection with the properties of Banach spaces.

Introduction

We characterize the reflexivity of Banach spaces by means of the so-called Lax--Milgram property of bilinear forms and the sequential weak completeness of Banach spaces. A necessary and sufficient condition for the fact that a point $u_0^{**} \in X^{**}$ belongs to the range $R(J^*)$ of the duality mapping J^* defined on X^* is derived. In particular, we have that $u_0^{**} \in J^*(u_0^*)$ for some u_0^* of the unit sphere in X^* if and only if there exists a weak Cauchy sequence $(u_n) \subset X$ such that $\hat{u}_n \to u_0^{**}$ weakly* in X^{**} and $\langle u_0^*, u_n \rangle \to 1$.

Recall that the duality mapping $J : X \to 2^{X^*}$ which is a subdifferential of the convex function $1/2\|u\|^2$, $u \in X$, plays an important role in the theory of monotone and accretive operators, the fixed point theory and the solvability of the operator equations and the geometry of Banach spaces.

The proof of the last main result depends on the properties of the subdifferentials of convex functions and the conjugate functions.

If X is a dual Banach space (i.e. $X = Z^*$ for some Banach space Z), $M \subset X^*$ an open convex subset, $\hat{u}_0 \in M$, where $\hat{u}_0$ is a canonical image of $u_0 \in Z$ in $X^* = Z^{**}$, and $f : M \to R$ a weak* lower semicontinuous convex functional having the Gâteaux derivative $f'(\hat{u}_0)$ at $\hat{u}_0$, then $f'(\hat{u}_0)$ is a weak* continuous linear functional on X^*, i.e. $f'(\hat{u}_0) \in \hat{X}$. In particular, if X is a dual Banach space and X^* is smooth at the point $\hat{u}_0$ of the unit sphere of X^*, where $u_0 \in Z$, then $J^*(\hat{u}_0) \in \hat{X}$, where J^* denotes the duality mapping on X^*.

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Definitions and notations

Let X be a real normed linear space, X^* and X^{**} its dual and bidual, respectively, $\langle , \rangle$ the pairing between X and X^*, $B_1(0)$, $B_1^*(0)$, $B_1^{**}(0)$ the unit closed ball and $S_1(0)$, $S_1^*(0)$, $S_1^{**}(0)$ its unit sphere in X, X^*, X^{**}, respectively. By R, R_+, C we denote the set of all real, nonnegative and complex numbers, respectively. By $\tau : X \to X^{**}$ we mean the canonical mapping, while $\hat{A}$ denotes the image of $A \subset X$ under τ in X^{**}. Recall that the duality mapping $J : X \to 2^{X^*}$ is defined by $J(u) = \{u^* \in X^* : \langle u^*, u\rangle = \|u\|^2, \|u^*\| = \|u\|\}$, $u \in X$. For each $u \in X$, the set $J(u)$ is nonempty convex and weak* compact in X^*. Let $T : X \to 2^{X^*}$ be a multivalued mapping (2^{X^*} denotes the system of all subsets of X^*), $D(T) = \{u \in X : T(u) \neq \emptyset\}$ its domain, $G(T) = \{(u, u^*) \in X \times X^*, u \in D(T), u^* \in T(u)\}$ its graph in the space $X \times X^*$. A mapping $T : X \to 2^{X^*}$ is said to be: (i) monotone [26], if for each $u, v \in D(T)$ and each $u^* \in T(u)$ and $v^* \in T(v)$ there is $\langle u^* - v^*, u - v\rangle \geqq 0$; (ii) maximal monotone [26], if T is monotone and if $(u, u^*) \in X \times X^*$ is a given element such that $\langle u^* - v^*, u - v\rangle \geqq 0$ for each $(v, v^*) \in G(T)$, then $(u, u^*) \in G(T)$; (iii) locally bounded at $u_0 \in D(T)$, if there exists a neighborhood V of u_0 such that $T(D(T) \cap V)$ is bounded in X^*. Let $M \subset X$ be a convex nonempty open subset, $f : M \to R$ a continuous convex function. A subdifferential map ∂f of f is defined [23] by $M \ni u \to \partial f(u) = \{u^* \in X^* : \langle u^*, v - u\rangle \leqq f(v) - f(u)$ for each $v \in M\} \subset X^*$. Note that ∂f is maximal monotone on M and $J(u) = \frac{1}{2}\partial(\|u\|^2) = \|u\| \, \partial\|u\|$, $u \in X$ (see [26]). We shall use the notion of Giles [10] for rotund (i.e. strictly convex) spaces, convex functions, Gâteaux and Fréchet differentials and derivatives. Recall that X is said to be: (i) smooth (Fréchet smooth), if the norm of X is Gâteaux (Fréchet) differentiable on $S_1(0)$; (ii) sequentially weakly complete, if each weakly Cauchy sequence (x_n) has a weak limit point in X; (iii) a dual Banach space, if there is a Banach space Z such that $X = Z^*$ (in the sense of the topology and the norm). A sequence $(x_n) \subset X$ is a weak Cauchy one, if $\langle x^*, x_n - x_m\rangle \to 0$ as $n, m \to \infty$, for each $x^* \in X^*$, i.e. $\lim_{n\to\infty} \langle x^*, x_n\rangle$ exists for each $x^* \in X^*$. Recall that l_1, L_1 are sequentially weakly complete [30], while the Orlicz space L_Φ is sequentially weakly complete if Φ satisfies Δ_2-condition ([25]). Recall that a real Banach space X is said to be weakly compact generated (WCG), if there exists a weakly compact set $K \subset X$ such that $X = \overline{\operatorname{span}} K$. This class of spaces includes separable and reflexive spaces, furthermore $c_0(\Gamma)$, $L_1(\mu)$ for finite μ, $C(K)$ spaces, where K is a weakly compact subset of a Banach space, etc. A Banach space X is said to be a weak Asplund, if each convex continuous function f on X is Gâteaux differentiable on a G_δ dense subset of X. By $R(T)$ we denote a range of a mapping T and the symbols $\sigma(X, X^*)$, $\sigma(X^*, X)$ are used for the weak and weak* topology on X and X^*, respectively.

Results

Lemma 1 ([8]), *Let X be a Banach space, $G \subseteq X$ an open subset, $T : G \to X^*$ a lipschitzian mapping such that $\langle Tx - Ty, x - y\rangle \geqq c\|x - y\|^2$ for some $c > 0$ and each $x, y \in G$. Then X is reflective.*

In view of Lemma 1, the assumption of reflexivity of X is included in the hypotheses of the following

Theorem (Lax-Milgram). *Let X be a Banach space, $B: X \times X \to C$ a bounded bilinear form such that B is coercive, i.e. $|B(u, u)| \geqq c\|u\|^2$ for some $c > 0$ and each $u \in X$. Then for each $u^* \in X^*$ there exists a unique point $u_0 \in X$ such that $\langle u^*, u\rangle = B(u, u_0)$ for each $u \in X$.*

Hayden [13] proved the following assertion: Let X and Y be Banach spaces, $B: X \times Y \to C$ a bounded nondegenerate bilinear form. Assume that for each $v^* \in Y^*$ there is a unique point $u_0 \in X$ such that $\langle v^*, v\rangle = B(u_0, v)$ for each $v \in Y$ and for each $u^* \in X^*$ there exists a unique point $v_0 \in Y$ such that $\langle u^*, u\rangle = B(u, v_0)$ for each $u \in X$. Then X and Y are reflexive. If X is a Banach space which admits a totally nonsingular bilinear form, then X is reflexive [28].

Definition 1 ([18]) *Let X, Y be normed linear spaces. We shall say that X has the Lax-Milgram property (LMP) with respect to Y, if there exists a nondegenerate bilinear form $B: X \times Y \to C$ with the following property: For a given linear closed separable subspace F of X there exists a closed separable subspace P of Y such that B is bounded on $F \times P$ and for each $u^* \in F^*$ there exists a unique point $v_P \in P$ such that $\langle u^*, u\rangle = B(u, v_P)$ for each $u \in F$.*

Similarly, we shall say that Y has the (LMP) with respect to X, if there is a nondegenerate bilinear form $B: X \times Y \to C$ with the following property: For a given linear closed separable subspace V of Y there exists a closed separable linear subspace E of X such that B is bounded on $E \times V$ and for each $v^* \in V^*$ there exists a unique element u_E of E such that $\langle v^*, v\rangle = B(u_E, v)$ for each $v \in V$.

The following result improves the corresponding one in [18]. First of all, we need the following simple

Lemma 2. *Let X be a normed linear space. Then X is sequentially weakly complete if and only if each closed linear separable subspace M of X is sequentially weakly complete.*

Theorem 1. *Let X, Y be normed linear spaces. Then:*

(i) *If X has the (LMP) with respect to Y, then X is reflexive if and only if X is sequentially weakly complete.*

(ii) *If X, Y are complete and X has the (LMP) with respect to Y and Y has the (LMP) with respect to X, then X and Y are reflexive.*

Proof. Let (u_n) be a bounded sequence of X. Put $F = \overline{\operatorname{span}}\{(u_n)\}$. Then F is a closed separable subspace of X. By our hypothesis there exists a separable closed linear subspace P of Y and a nondegenerate bilinear form $B: X \times Y \to C$ such that B is bounded on $F \times P$, i.e. $|B(u, v)| \leqq M_{F,P}\|u\| \cdot \|v\|$ for each $u \in F$ and $v \in P$ and some constant $M_{F,P} > 0$ and the representation of the elements $u^* \in F^*$ by means of B and the unique points $v_P \in P$ are valid. Let (v_n) be a dense sequence in P. Define ([28]) $(u_n^*) \subset F^*$ by $\langle u_n^*, u\rangle = B(u, v_n)$ for each $u \in F$ and n $(n = 1, 2, \ldots)$.

Clearly, $u_n^* \in F^*$ for each n. We claim that the sequence (u_n^*) is dense in F^* and that P is isomorphic with F^*. Define a linear continuous mapping $A : P \to F^*$ by $\langle u, Av \rangle = B(u, v)$ for each fixed $v \in P$ and each $u \in F$. Since B is nondegenerate, we have that A is one to one and by our assumption, onto F^*. Hence by the open mapping theorem A^{-1} is continuous. As $u_n^* = A(v_n)$ for each n and (v_n) is dense in P, we conclude that (u_n^*) is dense in F^*.

Now $(\langle u_i^*, u_n \rangle)_{n=1}^\infty$ is a bounded sequence for each fixed i $(i = 1, 2, \ldots)$. By the diagonal process, one can extract a subsequence $(\bar{u}_k)$ of (u_n) such that $(\langle u_i^*, \bar{u}_k \rangle)_{k=1}^\infty$ is convergent for each i. In view of the density of (u_n^*) in F^*, we have that $\langle u^*, \bar{u}_k \rangle$ is convergent for each $u^* \in F^*$. Hence $(\bar{u}_k)$ is a weak Cauchy sequence in F with respect to F^*. Assume that X is sequentially weakly complete. By Lemma 2, F is sequentially weakly complete and therefore $(\bar{u}_k)$ has a weak limit point u_0 in F, i.e. $\bar{u}_k \to u_0$ weakly in F. Since $\sigma(X, X^*)|_F = \sigma(F, F^*)$, we have that $\bar{u}_k \to u_0$ weakly in X, which implies that X is reflexive, it is well-known that X is sequentially weakly complete. (ii) It is sufficient to apply the Hayden ([13]) result to separable closed subspaces of X and Y, respectively, and use the fact that if each closed separable linear subspace of a given Banach space is reflexive, then this space is reflexive.

Note that in the proof of the first assertion of Theorem 1, we need not assume in Definition 1 that a separable linear subspace P of Y is closed (or that a separable linear subspace E of X is closed, compare [18]).

Lemma 3 ([22]). *Let X be a Banach space, Y a norm closed subspace, of X. Then $X^{**}/\hat{X}$ is separable if and only if $Y^{**}/\hat{Y}$ and $(X/Y)^{**}/(X/Y)$ are separable. Moreover, X^{**} is separable if and only if X and $X^{**}/\hat{X}$ are both separable.*

Lemma 4. *Let X be a Banach space such that the closed unit ball $B_1^{**}(0)$ of X^{**} is sequentially weakly* compact. Then X is reflexive if and only if X is sequentially weakly complete.*

Proof. Assume that X is sequentially weakly complete and $B_1^{**}(0)$ is sequentially weakly* compact. Let $(u_n) \subset B_1(0)$ be arbitrary. By our hypothesis, there exists a subsequence $(\hat{u}_{n_k})$ of $(\hat{u}_n)$ such that $\hat{u}_{n_k} \to u_0^{**}$ weakly* in X^{**} and $u_0^{**} \in B_1^{**}(0)$. Hence (u_{n_k}) is a weak Cauchy sequence in X. According to our assumption $(u_{n_k}) \to u_0$ weakly in X for some $u_0 \in X$ and clearly $u_0 \in B_1(0)$. The converse assertion is clear.

Proposition 1. *Let X be a Banach space such that one of the following three conditions is satisfied:*

(i) *$X^{**}/\hat{X}$ is separable;*
(ii) *X^* is a weak Asplund;*
(iii) *X is separable and X contains no isomorphic copy of l_1.*

Then X is reflexive if and only if X is sequentially weakly complete.

Proof. (i) We prove that if $X^{**}/\hat{X}$ is separable, then the closed unit ball $B_1^{**}(0)$ is sequentially weakly* compact. Suppose that $(u_n^{**}) \subset B_1^{**}(0)$. Since $X^{**}/\hat{X}$ is separable, then the sequential $\sigma(X^{**}, X^*)$-closure of $\widehat{B_1(0)}$ in X^{**} is equal to $B_1^{**}(0)$ (see [22]). Hence for each fixed n $(n = 1, 2, \ldots)$ there exists a sequence $(u_{n_k})_{k=1}^{\infty}$ in $B_1(0)$ such that $\hat{u}_{n_k} \to u_n^{**}$ in the $\sigma(X^{**}, X^*)$-topology of X^{**} as $k \to \infty$. Set $Y = \overline{\operatorname{span}}\left(\bigcup_{n,k=1}^{\infty} \{u_{n_k}\}\right)$, then Y is a closed separable Banach space. By our hypothesis $X^{**}/\hat{Y}$ is separable and Lemma 3 implies that Y^{**} is separable and therefore Y^* is separable. Hence the $\sigma(Y^{**}, Y^*)$-topology is metrizable on $\tilde{B}_1^{**}(0) = \{u^{**} \in Y^{**} : \|u^{**}\| \leqq 1\}$. From the sequence $(u_n^{**}) \subset \tilde{B}_1^{**}(0)$ one can extract a subsequence, say $(u_{n_j}^{**})$, such that $u_{n_j}^{**} \to u_0^{**}$ in the $\sigma(Y^{**}, Y^*)$-topology of Y^{**} and $u_0^{**} \in \tilde{B}_1^{**}(0) \subset B_1^{**}(0)$. Since $\sigma(Y^{**}, Y^*) = \sigma(X^{**}, X^*)|\, Y^{**}$, we have that $B_1^{**}(0)$ is sequentially $\sigma(X^{**}, X^*)$-compact. Now (i) follows at once from Lemma 4. (ii) If X^* is a weak Asplund space, by the Stegall result [31] we have that $B_1^{**}(0)$ is sequentially weakly* compact. Lemma 4 gives at once the result. (iii) follows at once from the Rosenthal and Odell theorem (see [5]) and Lemma 4.

Remark 1. Let X be a Banach space such that one of the following three conditions is satisfied:

(i) X is quasi-reflexive i.e. $\dim X^{**}/\hat{X} < \infty$;

(ii) X^* is (WCG);

(iii) X^* admits an equivalent smooth norm.

Then X is reflexive if and only if X is sequentially weakly complete.

Indeed, if X is quasi-reflexive, then $X^{**}/\hat{X}$ is separable. If X^* is (WCG), according to the Amir and Lindenstrauss result (see [4]) X^* admits a smooth and rotund equivalent norm such that its dual norm on X^{**} is rotund. Hence X^* is a weak Asplund space (see [1] or [17]). (iii) From the general result of Preiss, Phelps and Namioka [27], it follows that X^* is a weak Asplund space (or use the Hagler and Sullivan result [12] that if X satisfies (iii), then $B_1^{**}(0)$ is sequentially weakly* compact). Proposition 1 (ii) (or Lemma 4) gives at once the result.

Let us remark that the result of Remark 1 (i) was proved by Civin and Yood [6] using a different method, while the assertion of Remark 1 (iii) can be proved at once on the base of the Šmulian characterization of differentiability of the norm in X^* and the James characterization of reflexivity. Cudia [2, Theorem 5.4] proved that if a Banach space X is weakly k-rotund, then X is reflexive if and only if X is sequentially weakly complete.

If X is a Banach space, then $X^{**}/\hat{X}$ is separable if and only if X has a norm closed subspace Y such that Y^{**} is separable and X/Y is reflexive. A Banach space X is quasi-reflexive of order $\leqq n$ (i.e. $\dim X^{**}/\hat{X} \leqq n$) if and only if $X^{**}/\hat{X}$ is separable and X has the property P_n that every norm-closed subspace Y of X^* has codim $Y \leqq n$ in its $\sigma(X^*, X)$-sequential closure $K_X(Y)$ in X^* (see McWilliams [22]). If X is a Banach space and $X^{**}/\hat{X}$ is separable, then X is a direct sum $X = A \oplus B$ of the closed linear subspaces A and B, where A is separable and B is reflexive. A similar result is valid, if X is a Fréchet space and the bidual X^{**} is provided by the strong topology $\beta(X^{**}, X^*)$ (McWilliams [22] and Valdivia [32]).

Note that Orlicz spaces containing an isomorphic copy of l_1 were characterized for non-anatomic and purely anatomic measures by Hudzik [14].

Smith [29] pointed out that there exists a nonreflexive Banach space X such that X^{**} is rotund and smooth (and hence both X and X^* are rotund and smooth) and X^{***} is rotund. However, smoothness of X^{***} implies reflexivity of X (Giles [11]). Note that the James space (which is quasi-reflexive of order one and separable with its separable dual) admits an equivalent norm such that its third conjugate space X^{***} is rotund (Smith [29]). By Proposition 1 (i), the James space is not sequentially weakly complete.

Let X be a real normed linear space, J and J^* the duality mappings in X and X^*, respectively.

Theorem 2. *Let X be a normed linear space, (u_n^*) a sequence dense in X^*, $u_0^{**} \in S_1^{**}(0)$. Then:* (i) *$u_0^{**} \in R(J^*)$, i.e. $u_0^{**} \in J^*(u_0^*)$ for some $u_0^* \in S_1^*(0)$, if and only if the following condition is satisfied: There exists a weak Cauchy sequence (u_n) in X such that $\|u_n\| \leqq 1 + 1/n$ for each n, $\langle u_0^{**}, u_i^* \rangle = \langle u_i^*, u_n \rangle$ for each fixed n $(n = 1, 2, \ldots)$ and each $i = 1, 2, \ldots, n$, $\langle u_0^*, u_n \rangle \to 1$, $\|u_n\| \to 1$ and $\hat{u}_n \to u_0^{**}$ weakly* in X^{**}.*

(ii) *If, in addition, X^* is smooth at u_0^* and X is sequentially weakly complete, then $\langle u_0^* u_0 \rangle = 1$, $u_0 \in S_1(0)$ and $u_0^{**} = J^*(u_0^*) = \hat{u}_0$.*

Proof. Let (u_n^*) be dense in X^*, $u_0^{**} \in S_1^{**}(0)$, $u_0^{**} \in R(J^*)$, i.e. $u_0^{**} \in J^*(u_0^*)$ for some $u_0^* \in S_1^*(0)$. According to the Helly theorem, there exists a sequence $(u_n) \subset X$ such that $\|u_n\| \leqq \|u_0^{**}\| + 1/n = 1 + 1/n$ for each n and $\langle u_0^{**}, u_i^* \rangle = \langle u_i^*, u_n \rangle$ for each fixed n $(n = 1, 2, \ldots)$ and each $i = 1, 2, \ldots, n$. Define $u_n^{**} \in X^{**}$ by $u_n^{**} = \hat{u}_n$, i.e. $\langle u_n^{**}, u^* \rangle = \langle u^*, u_n \rangle$ for every $u^* \in X^*$ and each fixed n. Then (u_n^{**}) is uniformly bounded in X^{**} and (u_n^{**}) converges pointwise for each fixed u_i^* $(i = 1, 2, \ldots)$, since $\langle u_n^{**}, u_i^* \rangle = \langle u_i^*, u_n \rangle = \langle u_0^{**}, u_i^* \rangle$ for each n and each $i = 1, 2, \ldots, n$. By the Banach-Steinhaus theorem (u_n^{**}) converges weakly* in X^{**} to a point u_0^{**}. Since $\lim_{n\to\infty} \langle u_n^{**}, u^* \rangle = \langle u_0^{**}, u^* \rangle = \lim_{n\to\infty} \langle u^*, u_n \rangle = \lim_{n\to\infty} \langle \hat{u}_n, u^* \rangle$ for each $u^* \in X^*$, we conclude that $\langle u_0^*, u_n \rangle \to 1$, $\hat{u}_n \to u_0^{**}$ weakly* in X^{**} and (u_n) is a weak Cauchy sequence in X. We have that $1 = \lim_{n\to\infty} \langle u_0^*, u_n \rangle \leqq \liminf_{n\to\infty} \|u_n\| \leqq \limsup_{n\to\infty} \|u_n\| \leqq \leqq \lim_{n\to\infty} (1 + 1/n) = 1$ and therefore $\|u_n\| \to 1$. Conversely, if $\hat{u}_n \to u_0^{**}$ weakly* in X^{**} and $\langle u_0^*, u_n \rangle \to 1$ for some sequence $(u_n) \subset X$ and some $u_0^* \in S_1^*(0)$, then $1 = = \lim_{n\to\infty} \langle u_0^*, u_n \rangle = \lim_{n\to\infty} \langle u_0^*, \hat{u}_n \rangle = \langle u_0^*, u_0^{**} \rangle$, which gives that $u_0^{**} \in J^*(u_0^*)$ and $u_0^{**} \in R(J^*)$. Let us assume (ii). We have that $u_n \to u_0$ weakly for some $u_0 \in X$. Hence $\langle u_0^*, u_0 \rangle = 1$ and $\|u_0\| \geqq 1$. On the other hand, we have that $\|u_0\| \leqq \liminf_{n\to\infty}$.

$\cdot \|u_n\| = \lim_{n\to\infty} \|u_n\| = 1$. Hence $u_0 \in S_1(0)$ and $u_0^* \in J(u_0)$. Since X^* is smooth at u_0^*, we have that $\hat{u}_0 = J^*(u_0^*) = u_0^{**}$, which proves our theorem.

Corollary 1. *Let X be a normed linear space, $u_0^{**} \in S_1^{**}(0)$. Then $u_0^{**} \in J^*(u_0^*)$ for some $u_0^* \in S_1^*(0)$ if and only if there exists a weak Cauchy sequence $(u_n) \subset X$. such that $\hat{u}_n \to u_0^{**}$ weakly* in X^{**} and $\langle u_0^*, u_n\rangle \to 1$ as $n \to \infty$.*

Recall that by the Bishop-Phelps theorem each Banach space X is subreflexive, i.e the set of all linear continuous functionals of X^* which attain their norm on $S_1(0)$ is norm dense in X^*, which implies that $\overline{R(J)} = X^*$ and $\overline{R(J^*)} = X^{**}$. By the James theorem, both $R(J)$ and $R(J^*)$ are not closed in a nonreflexive Banach space X. Using the properties of the duality mapping reflexivity of Banach spaces was studied in [3], [18], [19] and [24]. If X is a normed linear space, then X is reflexive if an only if $R(J) = X^*$ and $R(J^*) = X^{**}$ (De Prima and Petryshyn [3]). If X is a Banach space, $J : X \to 2^{X^*}$ a duality mapping, $F = \operatorname{span} J(X)$, then X is reflexive if and only if the closed unit ball of X is $\sigma(X, F)$-compact (Laursen [24], see also [20] for another proof of this result). If X is a real Banach space, $\varphi : X \to R \cap \{+\infty\}$ a lower semicontinuous convex function such that intdom $\varphi \neq \emptyset$, then the following are equivalent; (i) $R(\partial\varphi) = X^*$; (ii) X is reflexive and for all $u^* \in X^*$, the function $u^* - \varphi$ is bounded above (Fitzpatrick, Calvert [8]).

Let X be a normed linear space, J, J^* duality mappings on X and X^*, respectively. Then (see [3]) $u_0^* \in J(u_0)$ for some $u_0 \in X$ if and only if $\hat{u}_0 \in J^*(u_0^*)$.

Propositon 2. *Let X be a Banach space (a sequentially weakly complete Banach space) such that X^* is Fréchet smooth (X^* is smooth) at some point $u_0^* \in S_1^*(0)$. Then there exist sequences $(u_n) \subset S_1(0)$, $(u_n^*) \subset S_1^*(0)$ and a point $u_0 \in S_1(0)$ such that $u_n^* \to u_0^*$, $\langle u_n^*, u_n\rangle = 1$, $\langle u_0^*, u_0\rangle = 1$, $u_n \to u_0$ in the norm of X ($u_n \to u_0$ weakly in X) and $J^*(u_0^*) \in \hat{X}$.*

Proof. By the Bishop-Phelps theorem for a given $u_0^* \in S_1(0)$ there exist sequences $(u_n) \subset S_1(0)$, $(u_n^*) \subset S_1^*(0)$ such that $u_n^* \to u_0^*$ in the norm of X and $\langle u_n^*, u_n\rangle = 1$ for each n. Then $|\langle u_0^*, u_n\rangle - 1| = \langle u_0^*, u_n\rangle - \langle u_n^*, u_n\rangle| \leqq \|u_0^* - u_n^*\|$. Hence $\langle u_0^*, u_n\rangle \to 1$ as $n \to \infty$. Since X^* is Fréchet smooth (X^* is smooth) at u_0^*, according to the Šmulian theorem, we conclude that (u_n) is a Cauchy (a weak Cauchy) sequence in X. By our hypothesis, there exists a point $u_0 \in X$ such that $u_n \to u_0$ in the norm of X ($u_n \to u_0$ weakly in X). Clearly $\langle u_0^*, u_0\rangle = 1$ and $u_0 \in S_1(0)$, which implies that $u_0^* \in J(u_0)$. Since X^* is smooth at u_0^*, we have that J^* is singlevalued at u_0^* and therefore $\hat{u}_0 = J^*(u_0^*)$. Hence $J^*(u_0^*) \in \hat{X}$.

Theorem 3. *Let X be a dual Banach space (i.e. $X = Z^*$ for some Banach space Z), $M \subseteq X^*$ a convex open subset, $\hat{u}_0 \in M$, where $\hat{u}_0$ is a canonical image of $u_0 \in Z$ in X^*. Let $f : M \to R$ be a weak* lower semicontinuous convex functional having the Gâteaux derivative $f'(\hat{u}_0)$ at $\hat{u}_0$. Then:*

(i) *$f'(\hat{u}_0) \in \hat{X}$, i.e. $f'(\hat{u}_0)$ is a weak* continuous linear functional on X^*;*

(ii) *if* $(u_n^*) \in M$, $u_n^* \to \hat{u}_0$ *in the norm of* X^* *and* $\hat{x}_n \in \partial f(u_n^*)$ *for some sequence* $(x_n) \subset X$, *then* $x_n \to f'(\hat{u}_0)$ *weakly in* X.

Proof. Since f is weakly* lower semicontinuous convex and finite on M, we get that f is continuous on M. Hence $D(\partial f) = M$ and the mapping $M \ni u^* \to \partial f(u^*)$ is maximal monotone on M ([26]). As f is a weak* lower semicontinuous convex function and M equals the domain of the norm continuity of f, then, by the Phelps variant of the Bishop-Phelps theorem ([10]), the set of all $u^* \in M$, where $\partial f(\dot{u}^*) \cap \cap \hat{X} \neq \emptyset$, is norm-dense in M. Hence there exist sequences $(u_n^*) \subset M$ and $(x_n) \subset X$ such that $u_n^* \to \hat{u}_0$ in the norm of X^* and $\hat{x}_n \in \partial f(u_n^*)$, where $\hat{x}_n$ denotes the weak* continuous subgradients of the subdifferential ∂f at the points u_n^*. As $u_n^* \to u_0^*$ and ∂f is monotone on M, we have that ∂f is locally bounded at the points of M. Therefore $(\hat{x}_n)$ is bounded in X^{**} and hence there exists a subnet $(\hat{x}_{n_\alpha})$ of $(\hat{x}_n)$ and a point $x_0^{**} \in X^{**}$ such that $(\hat{x}_{n_\alpha})$ converges to x_0^{**} weakly* in X^{**}. We have that $\langle \hat{x}_{n_\alpha} - u^{**}, u_{n_\alpha}^* - u^* \rangle \geqq 0$ for each $(u^*, u^{**}) \in G(\partial f)$. Since ∂f is locally bounded on M, $(\hat{x}_{n_\alpha})$ is bounded and therefore $\langle x_0^{**} - u^{**}, \hat{u}_0 - u^* \rangle \geqq 0$ for each $(u^*, u^{**}) \in G(\partial f)$. By the maximal monotonicity of the mapping $M \ni u^* \to \partial f(u^*)$ on M we get that $x_0^{**} = \partial f(\hat{u}_0) = f'(\hat{u}_0)$. Now we prove that $f'(\hat{u}_0) \in X$, i.e. that $f'(\hat{u}_0) = \{\hat{x}_0\}$ for some $x_0 \in X$. First of all, $\hat{x}_n \in \partial f(u_n^*)$ gives that ([16]) $\langle \hat{x}_n, u_n^* \rangle = f(u_n^*) + f^*(\hat{x}_n)$ for each n, where f^* is a dual convex function defined by $f^*(u^{**}) = \sup\{\langle u^{**}, u^* \rangle - f(u^*) : u^* \in M\}$, $u^{**} \in X^{**}$. By our assumption, X is a dual Banach space, hence there exists a Banach space Z such that $X = Z^*$ in the sense of the topology and the norm. Since (x_n) is bounded, thee exists a subnet (x_{n_β}) of (x_n) and a point $x_0 \in X$ such that $x_{n_\beta} \to x_0$ in $\sigma(Z^*, Z)$-topology of X. We have that $f^*(\hat{x}_n) = \sup\{\langle \hat{x}_n, u^* \rangle - - f(u^*) : u^* \in M\} = \sup\{\langle x_n, u^* \rangle - f(u^*) : u^* \in M\} = f^*(x_n)$ for each n. Let us embed X into X^{**}. Since f^* is weakly* lower semicontinuous on X^{**} and $x_{n_\beta} \to x_0$ weakly* in X, we get that $f^*(x_0) \leqq \lim_\beta \inf f^*(x_{n_\beta})$. Furthermore, $|\langle u_{n_\beta}^*, x_{n_\beta} \rangle - - \langle \hat{u}_0, x_0 \rangle| \leqq |\langle u_{n_\beta}^* - \hat{u}_0, x_{n_\beta} \rangle| + |\langle \hat{u}_0, x_{n_\beta} - x_0 \rangle|$ and $\langle \hat{u}_0, x_{n_\beta} - x_0 \rangle = = \langle u_0, x_{n_\beta} - x_0 \rangle \to 0$, as $x_{n_\beta} \to x_0$ in the $\sigma(Z^*, Z)$-topology and $u_0 \in Z$. Since (x_{n_β}) is bounded, we get that $\langle u_{n_\beta}^*, x_{n_\beta} \rangle \to \langle \hat{u}_0, x_0 \rangle$. Now the continuity of f on M gives that $f(\hat{u}_0) + f^*(x_0) \leqq \lim_\beta f(u_{n_\beta}^*) + \lim_\beta \inf f^*(x_{n_\beta}) \leqq \lim_\beta \inf (f(u_{n_\beta}^*) + + f^*(x_{n_\beta})) = \lim_\beta \inf \langle u_{n_\beta}^*, x_{n_\beta} \rangle = \lim_\beta \langle u_{n_\beta}^*, x_{n_\beta} \rangle = \langle \hat{u}_0, x_0 \rangle$.

On the other hand, the Young-Fenchel [16] inequality gives that $\langle \hat{u}_0, x_0 \rangle \leqq \leqq f(\hat{u}_0) + f^*(x_0)$. Therefore $\langle \hat{u}_0, x_0 \rangle = f(\hat{u}_0) + f^*(x_0)$, which implies that $\hat{x}_0 = = x_0 = f'(\hat{u}_0) = x_0^{**}$, where $\hat{x}_0$ denotes the canonical image of $x_0 \in X$ in X^{**}. Hence for the whole sequence $(\hat{x}_n)$, we have that $\hat{x}_n \to x_0$ weakly* in X^{**} and therefore $x_n \to f'(\hat{u}_0)$ weakly in X, which proves (ii).

Note that the assertion (ii) of Theorem 2 follows also at once from the fact that the mapping $M \ni u^* \to \partial f(u^*)$ is norm to weak* upper semicontinuous on M, the result (i) of Theorem 2 that $\partial f(\hat{u}_0) = f'(\bar{u}_0) \in X$ and the fact that the canonical embedding $\tau : X \to X^{**}$ is a homeomorphism of $(X, \sigma(X, X^*))$ into $(X^{**}, \sigma(X^{**}, X^*))$.

Let X be a linear normed space such that X is smooth at $u_0 \in X$. Then $J(u_0) = \partial(\frac{1}{2}\|u_0\|^2) = \|u_0\| \operatorname{grad} \|u_0\|$.

Corollary 2. *Let X be a dual Banach space (i.e. $X = Z^*$ for a Banach space Z) such that X^* is smooth at $\hat{u}_0$, where $u_0 \in Z$, $\|u_0\|_Z = 1$ and $\hat{u}_0$ is a canonical image of u_0 in X^*. Then $J^*(\hat{u}_0) \in X$, where J^* denotes the duality mapping on X^*.*

Giles [9] proved that if X is a Banach space, $x \in S_1(0)$, then X^{**} is smooth at $\hat{x}$ if and only if every support mapping $S_1(0) \ni x \to x_x^* \in S_1^*(0)$ is continuous at x when X has the norm topology and X^* has the $\sigma(X^*, X^{**})$-topology. Recall that a Banach space is said to be very smooth, if there exists a support mapping $S_1(0) \ni x \to x_x^* \in S_1^*(0)$ which is norm to weak continuous on $S_1(0)$. Corollary 2 and the Giles result imply the following

Corollary 3. *Let X be a dual Banach space such that Z is very smooth. Then $\{J^*(\hat{u}) : u \in Z, \|u\|_Z = 1\} \subset X$, where J^* is the duality mapping on X^*.*

Recall that if X is a Banach space, then X is reflexive and rotund if and only if X^* is very smooth (Giles [11]).

The following useful result was proved by John and Zizler [15]. Assume that X and X^* are both (WCG). Then there exists an equivalent norm $|||\cdot|||$ on X such that: (i) $|||\cdot|||$ is locally uniformly rotund; (ii) the dual norm of $|||\cdot|||$ on X^* is locally uniformly rotund; (iii) the second dual norm of $|||\cdot|||$ on X^{**} is rotund. Since the second dual norm of $|||\cdot|||$ on X^{**} is rotund, we conclude that the dual norm of $|||\cdot|||$ on X^* is smooth.

Let $X = Z^*$ for some Banach space Z. Assume that X and Z are both (WCG). Let $|||\cdot|||$ be an equivalent norm on Z such that its dual norm on X^* is smooth. Then $\{\operatorname{grad} |||\hat{u}||| : u \in Z, |||u|||_Z = 1\} \subset \hat{X}$.

Let $G \subset R^n$ be an open subset such that mes $G < +\infty$, Φ and M-function (in the sense of [21]), $L_\Phi(G)$ an Orlicz space provided by the Orlicz norm, $B(G)$ the set of all bounded functions defined on G. Then $E_\Phi(G) \subset \tilde{L}_\Phi(G) \subset L_\Phi(G)$, where $\tilde{L}_\Phi(G)$is the Orlicz class of all measurable functions u defined on G and such that $\varrho(u, \Phi) = \int_G \Phi(u(t))\, dt < +\infty$. Let $E_\Phi(G)$ be a space defined as the closure of $B(G)$ in the Orlicz norm $\|\cdot\|_\Phi$ of $L_\Phi(G)$. Then $E_\Phi(G) \subset \tilde{L}_\Phi(G)$. Moreover, if Φ satisfies the Δ_2-condition, then $E_\Phi = \tilde{L}_\Phi = L_\Phi$. Let Φ^* be a dual function. Then $E_{\Phi^*}(G) \subset L_{\Phi^*}(G)$ and $(E_{\Phi^*})^* = L_\Phi$ and hence L_Φ is a dual Banach space. If $\Phi^*(t)$ has a continuous derivative which is monotone and positive for each $t > 0$, then the Orlicz norm of E_{Φ^*} is Fréchet differentiable on E_{Φ^*} away the origin. Hence $\{s - \operatorname{grad} \|\hat{u}\| : u \in E_{\Phi^*}, \|u\| = 1\} \subset L_\Phi$, where s-grad $\|\hat{u}\|$ denotes the Fréchet gradient of the norm at $\hat{u}$. A similar result is valid for the gradient of the norm of L_Φ under the weaker assumption that E_{Φ^*} is only very smooth.

Each dual norm, each dual function [16] and each support function of a convex closed subset of X are weak* lower semicontinuous functionals on X^*. Moreover, if X is a Banach space, M a weakly* closed linear subspace of X^*, then the pseudonorm $X^* \ni u^* \to \operatorname{dist}(u^*, M)$ is a weak* lower semicontinuous function on X^* [10].

Recall that Asplund [1] proved the following assertion: Let X be a Banach space, $f: X \to (-\infty, +\infty]$ a lower semicontinuous function such that $f \not\equiv +\infty$. If the dual function f^* defined on X^* is Fréchet differentiable at some point $u^* \in X^*$, then $(f^*)'(u^*) \in \hat{X}$.

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