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On the Absoluteness of Openly-Generated and Dugundji Spaces

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For a given partial ordering P we consider the generic extension $V[G]$ of the universe V of set theory. If X is a compact Hausdorff space in V, then, in general, X is not compact in $V[G]$. There is a natural procedure which assigns to X a compactification X_G of X in $V[G]$. The general question here is, which topological properties Q are absolute in the following sense:

X_G satisfies Q in $V[G]$ iff X satisfies Q in V.[1]

All topological spaces in this paper are assumed to be Hausdorff. We denote by $\mathrm{RO}(X)$ the regular open algebra of a topological space X. Undefined topological notions can be found in the book of Engelking [8].

1. In this section we describe X_G and discuss the behaviour of this construction under the most important operations on topological spaces. $V[G]$ is always supposed to be a generic extension of the universe V.

Let X be a compact space in V. We consider X in $V[\mathrm{G}]$ as a topological space with the topology generated by all open subsets $U \subseteq X$, $U \in V$. If S denotes the family of all continuous functions $f : X \to I$ on X into the unit segment in V, an embedding $i : X \to I^S$ is given in both V and $V[G]$ by $i(x) = (fx)_{f \in S}$ for every $x \in X$. X_G is nothing but the compactification of $i(X)$ in the Tichonov cube I^S in $V[G]$. Remark, that for each $f \in S$, there is a continuous function $f_G : X_G \to I$ in $V[G]$ with $f = f_{G|X}$.

1.1. Lemma. X_G is the smallest compactification of X in $V[G]$ such that $\mathrm{cl}_{X_G}(\mathrm{A}) \cap \cap\, \mathrm{cl}_{X_G}(B) = \emptyset$ for arbitrary disjoint nonempty closed subsets A, $B \subseteq X$, A, $B \in V$.

Proof. It is obvious, that $\mathrm{cl}_{X_G}(A) \cap \mathrm{cl}_{X_G}(B) = \emptyset$ for arbitrary disjoint nonempty closed subsets A, $B \subseteq X$, A, $B \in V$.

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[1] This notion is due to Prof. S. Fuchino (FU Berlin) and was given in his lecture on the conference „Topology and Measure" (Rostock—Warnemünde, August 1991). I would like to thank him for stimulating discussions.

A compactification bX of a completely regular space X is characterized by the family of all pairs C, D of closed subsets of X with $\mathrm{cl}_{bX}(C) \cap \mathrm{cl}_{bX}(D) = \emptyset$ (see Engelking [8], 3.5.5). Hence, it suffices to show, that for arbitrary nonempty closed subsets C, D of X in $V[G]$ with $\mathrm{cl}_{X_G}(C) \cap \mathrm{cl}_{X_G}(D) = \emptyset$ there exist closed subsets A, B of X in V, such that $C \subseteq \mathrm{cl}_{X_G}(A)$, $D \subseteq \mathrm{cl}_{X_G}(B)$ and $A \cap B = \emptyset$. If $s \in S$, let π_s denote the natural projection $\pi_s : I^S \to I$ which associates to each point x of I^S the s^{th} coordinate. A base of I^S in V as well as in $V[G]$ consists of all possible finite intersections of sets of the form $\pi_s^{-1}U$, where U is an open interval $(p, q) \subseteq I$ with rational end points. Let H denote the family of all finite unions of sets of this base in V. For $W \in H$ W' denotes the corresponding set in $V[G]$. Of course, $\mathrm{cl}(W_1) \cap \mathrm{cl}(W_2) = \emptyset$ (in V) iff $\mathrm{cl}(W_1') \cap \mathrm{cl}(W_2') = \emptyset$ (in $V[G]$ for arbitrary $W_1, W_2 \in H$.

If C, D are nonempty closed subsets of X in $V[G]$ with $\mathrm{cl}_{X_G}(C) \cap \mathrm{cl}_{X_G}(D) = \emptyset$, we find using the compactness of I^S sets $W_1, W_2 \in H$, such that $C \subseteq W_1'$, $D \subseteq W_2'$ and $\mathrm{cl}(W_1) \cap \mathrm{cl}(W_2) = \emptyset$. $A = \mathrm{cl}_X(W_1) \cap X$ and $B = \mathrm{cl}_X(W_2) \cap X$ are the desired sets in V. □

1.2. Proposition. $(X \times Y)_G = X_G \times Y_G$.

Proof. It is easy to see that there exists a natural map $(X \times Y)_G \to X_G \times Y_G$ which leaves every point of $X \times Y$ fixed.

Let A, B be disjoint nonempty closed subsets of $X \times Y$, $A, B \in V$. Using the compactness, we find open subsets $U_1 \times W_1, \ldots, U_k \times W_k$, $\bar{U}_1 \times \bar{W}_1, \ldots, \bar{U}_l \times \bar{W}_l$ of $X \times Y$ in V such that $A \subseteq \cup\{U_i \times W_i : i = 1, \ldots, k\}$, $B \subseteq \cup\{\bar{U}_j \times \bar{W}_j : j = 1, \ldots$ $\ldots, l\}$ and $\mathrm{cl}(U_i \times W_i) \cap \mathrm{cl}(\bar{U}_j \times \bar{W}_j) = \emptyset$ for every $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$. $\mathrm{cl}(U_i \times W_i) \cap \mathrm{cl}(\bar{U}_j \times \bar{W}_j) = \emptyset$ iff $\mathrm{cl}_X(U_i) \cap \mathrm{cl}_X(\bar{U}_j) = \emptyset$ or $\mathrm{cl}_Y(W_i) \cap \mathrm{cl}_Y(\bar{W}_j) = \emptyset$. If $\mathrm{cl}_X(U_i) \cap \mathrm{cl}_X(\bar{U}_j) = \emptyset$, (in V), then $\mathrm{cl}_{X_G}(U_i) \cap \mathrm{cl}_{X_G}(\bar{U}_j) = \emptyset$ (in $V[G]$) and, analogously, if $\mathrm{cl}_Y(W_i) \cap \mathrm{cl}_Y(\bar{W}_j) = \emptyset$ (in V), then $\mathrm{cl}_{Y_G}(W_i) \cap \mathrm{cl}_{Y_G}(\bar{W}_j) = \emptyset$ (in $V[G]$). Hence,

$(\cup\{\mathrm{cl}_{X_G \times Y_G}(U_i \cdot W_i) : i = 1, \ldots, k\}) \cap (\cup\{\mathrm{cl}_{X_G \times Y_G}(\bar{U}_j \cdot \bar{W}_j) : j = 1, \ldots, l\}) = \emptyset$,

i.e. $\mathrm{cl}_{X_G \times Y_G}(A) \cap \mathrm{cl}_{X_G \times Y_G}(B) = \emptyset$. By Lemma 1.1, the natural map $(X \times Y)_G \to$ $\to X_G \times Y_G$ is an homeomorphism. □

An easy consequence of the construction of X_G is, that for every continuous map $f : X \to Y$ of compact spaces there exists a unique continuous extension $f_G : X_G \to Y_G$.

1.3. Proposition. *If the compact space X is the limit of the inverse system $S =$ $= \langle X_\alpha, \pi_\alpha^\beta, A \rangle$ in V, then X_G is the limit of the inverse system $S_G = \langle X_{\alpha,G}, \pi_{\alpha,G}^\beta, A \rangle$.*

Proof. $X' = \lim S_G$ is a compactification of X in $V[G]$. Furthermore, there is a natural map of X_G onto $\varprojlim S_G$ which leaves the points of X fixed. For arbitrary disjoint nonempty closed subsets A, B of X in V we find – by means of some elementary observations – an index $\alpha \in A$ and a continuous function $f : X_\alpha \to I$ such that $A \subseteq (f \circ \pi_\alpha)^{-1}\{0\}$ and $B \subseteq (f \circ \pi_\alpha)^{-1}\{1\}$. Since f has a continuous extension over $X_{\alpha,G}$, we have $\mathrm{cl}_{X'}(A) \cap \mathrm{cl}_{X'}(B) = \emptyset$. □

An immediate consequence of 1.2 and 1.3 is the following

1.4. Proposition. *Let* $\{X_\alpha : \alpha \in A\}$ *be a family of compact spaces in V. Then* $(\Pi\{X_\alpha : \alpha \in A\})_G = \Pi\{X_{\alpha,G} : \alpha \in A\}$.

If $f_\alpha : X \to Y_\alpha$, $\alpha \in A$, is a collection of maps, the diagonal product Δf_α considered as a map from X to its image is called the interior product of the maps f_α and is denoted by $\otimes\{f_\alpha : \alpha \in A\}$.

1.5. Proposition. *Let* $f_\alpha : X \to Y_\alpha$, $\alpha \in A$, *be a collection of maps of compact spaces in V. Then* $(\otimes\{f_\alpha : \alpha \in A\})_G = \otimes\{f_{\alpha,G} : \alpha \in A\}$.

Proof. $\otimes f_\alpha$ is a map of X onto a subspace of $\Pi\{X_\alpha : \alpha \in A\}$. It is easy to see that $(\otimes f_\alpha)_G$ coincides with $\otimes f_{\alpha,G}$ for every $x \in X$. Since X is a dense subspace of X_G, they coincides on X_G. □

1.6. Proposition. *A continuous onto map* $f : X \to Y$ *of compact spaces is irreducible in V iff* f_G *is irreducible in* $V[G]$.

Proof. Suppose f_G to be irreducible. Let H be a nonempty proper closed subset of X (in V). Obviously, $H' = \mathrm{cl}_{X_G}(H) \neq X_G$ and, consequently, $f_G(H')$ is a proper closed subset of Y_G. Since Y is dense in Y_G, it follows that $Y \setminus f_G(H') \neq \emptyset$ and $f(H) \neq Y$. Hence, f is irreducible in V.

Conversely, let f be irreducible in V and let H be a proper subset of X_G in $V[G]$. By the construction of X_G, there exist continuous functions $h_1, \ldots, h_n : X \to I$ in V and rational intervals $(p_1, q_1), \ldots, (p_n, q_n) \subseteq I$ such that

$$\bar{O} = \{x \in X_G : h_{i,G}(x) \in (p_i, q_i) \text{ for every } i = 1, \ldots, n\}$$

is a nonempty open subset of X_G with $\bar{O} \cap H = \emptyset$. Fix a small positive real number ε such that

$$O = \{x \in X_G : h_i(x) \in (p_i + \varepsilon, q_i - \varepsilon) \quad \text{for every} \quad i = 1, \ldots, n\}$$

is nonempty. Then $F \setminus O$ is a proper closed subset of X in V and $H \subseteq \mathrm{cl}_{X_G}(F)$. Hence, $f_G(H) \subseteq f_G(\mathrm{cl}_{X_G}(F)) = \mathrm{cl}_{X_G}(fF) \neq Y_G$. □

A map is said to be skeletal if the preimage of the boundary of any open set in Y is nowhere dense in X (Mioduszewski, Rudolf [12]). It is easy to check that f is skeletal if and only if for every nonempty open subset U of X the set $\mathrm{Int}(\mathrm{cl}(fU))$ is nonempty.

1.7. Proposition. *A continuous onto map* $f : X \to Y$ *of compact spaces is skeletal in V iff f is skeletal in* $V[G]$.

Proof. One readily sees that f is skeletal in V if and only if it is skeletal in $V[G]$. The reason is that the topologies on X and Y in $V[G]$ are generated by sets which are elements of V. Furthermore, it is a well known fact that a map $f : X \to Y$ of completely regular spaces is skeletal if and only if for any suitable compactification bX and bY the continuous extension $bf : bX \to bY$ is skeletal. Hence, f is skeletal in $V[G]$ iff f_G is skeletal. □

1.8. Proposition. *A continuous onto map $f : X \to Y$ of compact spaces is open in V iff f_G is open in $V[G]$.*

To prove this fact we shall use the concept of representation of compact spaces as proximities on Boolean algebras (DeVries [6], Fedorchuk [9]) and the concept of absolutes of maps (Shapiro [18]). Further, we shall freely use the duality between a Boolean algebra and its space of ultrafilters.

Let $\mathscr{A}$ be an infinite Boolean algebra and δ a binary relation on $\mathscr{A}$. We write $a\delta b$ instead of $\langle a, b\rangle \in \delta$ and $a\bar{\delta}b$ instead of $\langle a, b\rangle \notin \delta$.

Definition (Fedorchuk [9]). δ is called a proximity on the Boolean algebra $\mathscr{A}$ if it satisfies the following conditions:

1. $a\delta b \leftrightarrow b\delta a$,
2. $a\bar{\delta} b$ and $a\bar{\delta} c \leftrightarrow a\bar{\delta}(b \vee c)$,
3. $a \neq 0 \to a\delta a$,
4. $a \neq 0 \to$ there exists an element $b \neq 0$ with $b\bar{\delta} -a$,
5. $a\bar{\delta} b \to$ there exists an element $c \neq 0$ such that $a\bar{\delta} -c$ and $b\bar{\delta} c$.

It is easy to check that $a\bar{\delta} -b$ implies $a \leqq b$. We shall write $a \ll b$ for $a\bar{\delta} -b$. On every Boolean algebra $\mathscr{A}$ a minimal proximity δ_0 is given by $a\bar{\delta}_0 b \leftrightarrow a \wedge b = 0$. A filter ξ on $\mathscr{A}$ is said to be a δ-filter if for every $a \in \xi$ there exists a $b \in \xi$ such that $b \ll a$. $S(\mathscr{A}, \delta)$ denotes the family of all maximal δ-filters. Set $O_\delta(a) = \{\xi \in S(\mathscr{A}, \delta) : : a \in \xi\}$. The family of all these sets induces a topology on $S(\mathscr{A}, \delta)$ and we have the following

1.9. Fact (DeVries [6]). *$S(\mathscr{A}, \delta)$ is a compact space. $O_\delta(a)$ is a regular open subset of $S(\mathscr{A}, \delta)$ for every $a \in \mathscr{A}$ and the map which assigns to every $a \in \mathscr{A}$ $O_\delta(a)$ is an isomorphism of $\mathscr{A}$ onto the corresponding subalgebra of the regular open algebra of $S(\mathscr{A}, \delta)$. Furthermore: $a\bar{\delta}b \leftrightarrow \mathrm{cl}(O(a)) \cap \mathrm{cl}(O(b)) = \emptyset$ for all $a, b \in \mathscr{A}$.*

Sketch of the proof. $S(\mathscr{A}, \delta_0)$ is nothing but the Stone space of the Boolean algebra $\mathscr{A}$, i.e. the space of all maximal filters. We write $S(\mathscr{A})$ for $S(\mathscr{A}, \delta_0)$ and $O(a)$ for $O_\delta(a)$. An equivalence relation is defined on $S(\mathscr{A})$ by

$$\xi\delta\eta \leftrightarrow a\delta b \quad \text{for every} \quad a \in \xi \quad \text{and} \quad b \in \eta .$$

Every equivalence class σ is a closed subspace of $S(\mathscr{A})$ and $\cap\sigma$ is a maximal δ-filter. The corresponding map $\pi : S(\mathscr{A}) \to S(\mathscr{A}, \delta)$ is continuous and irreducible, where $\pi^\# O(a) = O_\delta(a)$ and $\pi O(a) = \mathrm{cl}(O_\delta(a))$ for every $a \in \mathscr{A}$.
On the other hand, it is easy to prove the converse assertion.

1.10. Fact (DeVries [6]). *Let X be a compact Hausdorff space and $\mathscr{A}$ a sub-*

algebra of the regular open algebra of X and a base of the topology on X. Then:
a) $U\bar{\delta}_X V \leftrightarrow \mathrm{cl}(U) \cap \mathrm{cl}(V) = \emptyset$ *defines a proximity* δ_X *on* $\mathscr{A}$.
b) $\xi_x = \{U \in \mathscr{A} : x \in U\}$ *is a maximal* δ_X*-filter for every point* $x \in X$.
c) The map $X \to S(\mathscr{A}, \delta)$ *which relates each point* $x \in X$ *to* ξ_x *is a homeomorphism.*

The reason why we need the representation of compact spaces as proximities on Boolean algebras can be found in the following

1.11. Proposition. *Let X be a compact space in V, $\mathscr{A} = \mathrm{RO}(X)$ (in V) and δ_X the natural proximity on $\mathscr{A}$. Then X_G is homeomorphic to the space of all maximal δ_X-filters $S(\mathscr{A}, \delta)$ in $V[G]$.*

Proof. Remark that for every $x \in X$ $\xi_x = \{U \in \mathscr{A} : x \in U\}$ is a maximal δ_x-filter on $\mathscr{A}$ in $V[G]$. Hence, $X' = S(\mathscr{A}, \delta)$ is a compactification of X (in $V[G]$) and we have to prove that X_G and X' are equivalent compactifications. If $U\bar{\delta}_X V$ for $U, V \in \mathscr{A}$, there is a continuous function $f : X \to I$ in V such that $U \subseteq f^{-1}\{0\}$ and $V \subseteq f^{-1}\{1\}$. Since f has a continuous extension over X_G, it follows that $\mathrm{cl}_{X_G}(U) \cap \mathrm{cl}_{X_G}(V) = \emptyset$. Therefore $X' \leqq X_G$.

To prove that $X_G \leqq X'$ we check that every continuous function $f : X \to I$ in V has a continuous extension on X'. Let $\xi \in X'$ be a maximal δ_X-filter on $\mathscr{A}$ in $V[G]$. Set $F_\xi = \cap\{\mathrm{cl}(fU) : U \in \xi\}$. F_ξ is a nonempty closed subset of the unit segment. If $|F_\xi| > 1$, there exist rational intervals (p_1, q_1) and (p_2, q_2) such that $[p_1, q_1] \cap$ $\cap [p_2, q_2] = \emptyset$, $(p_1, q_1) \cap F_\xi \neq \emptyset$ and $(p_2, q_2) \cap F_\xi \neq \emptyset$.

$O_1 = \{x \in X : f(x) \in (p_1, q_1)\}$ and $O_2 = \{x \in X : f(x) \in (p_2, q_2)\}$ are nonempty open subsets of X. Obviously, $\mathrm{cl}_X(O_1) \cap \mathrm{cl}_X(O_2) = \emptyset$. For every $W \in \xi$ we have $W \cap O_1 \neq \emptyset$ and $W \cap O_2 \neq \emptyset$. This contradicts to the maximality of ξ. Hence, $|F_\xi| = 1$ and we define the extension of f by $f'(\xi) = t$ iff $F_\xi = \{t\}$. Applying similar arguments as above, it is easy to verify that f' is continuous. □

One can easily check the following

1.12. Lemma. *Let $f : X \to Y$ be a continuous onto map of compact spaces. Then f is open if and only if $f^{-1}\,\mathrm{cl}(U) = \mathrm{cl}(f^{-1}U)$ for all open subsets U of X.*

Now, we are going to prove Proposition 1.8. The proof is broken up into four steps.

Step 1. By Proposition 1.7, we may assume that f is skeletal. Set $\mathscr{A} = \mathrm{RO}(X)$ and $\mathscr{B} = \mathrm{RO}(Y)$. Let π_X and π_Y denote the canonical maps of $pX = S(\mathscr{A})$ and $pY = S(\mathscr{B})$ onto X and Y, respectively. The absolute of f is a continuous map $pf : pX \to pY$ such that the corresponding diagramm is commutative, i.e. $pf \circ \pi_X =$ $= \pi_Y \circ f$. Since f is skeletal, pf is unique (Shapiro [18]) and is open (Shapiro, Ponomarev [15], Bandlow [1]). Let $h : \mathscr{B} \to \mathscr{A}$ denote the dual embedding, i.e. $h(W) = U$ if $pf^{-1}O(W) = O(U)$ for all $W \in \mathscr{A}$ and $U \in \mathscr{B}$. Since pf is open, one can define a projection map $r : \mathscr{A} \to \mathscr{B}$ by $r(U) = W$ if $pfO(U) = O(W)$. One readily sees that $r(hW) = W$ for every $W \in \mathscr{B}$. Remark that $r(U) = W$ iff $W = \mathrm{Intcl}(fU)$.

Step 2. Now, let us assume that f is open. By Lemma 1.12, δ_X is a continuation of δ_Y in the sense that

$$hW_1\delta_Y hW_2 \quad \text{iff} \quad W_1\delta_X W_2 \quad \text{for all} \quad W_1, W_2 \in \mathscr{B}\,. \tag{1}$$

Furthermore,

$$\text{if} \quad U_1 \ll U_2 \quad \text{for} \quad U_1, U_2 \in \mathscr{A}\,, \quad \text{then} \quad rU_1 \ll rU_2\,. \tag{2}$$

Indeed, if $\mathrm{cl}(U_1) \subseteq U_2$, then $\mathrm{cl}(\mathrm{Int} f[U_1]) = \mathrm{cl}(fU_1) \subseteq fU_2 \subseteq \mathrm{Intcl}(fU_2)$.

Step 3. We prove now that h, r and the properties (1) and (2) of δ_X and δ_Y are sufficient for the openness of f.

Suppose h and r are given as above and the properties (1) and (2) are satisfied for proximities δ_X and δ_Y on $\mathscr{A}$ and $\mathscr{B}$, respectively.

At first, we claim that $\{rU : U \in \xi\}$ is a maximal δ_Y-filter for every maximal δ_X-filter $\xi \in S(\mathscr{A}, \delta_X)$. It is easy to see that $\{rU : U \in \xi\}$ is a filter. From property (1) it follows that $\{rU : U \in \xi\}$ is a δ_Y-filter. To prove the maximality, let W be an element of $\mathscr{B}$ with $W \notin \{rU : U \in \xi\}$. If $V \in \mathscr{B}$ satisfies $V \ll W$, i.e. $V\bar{\delta}_Y -W$, then $hV\bar{\delta}_X -hW$ and $hV \ll hW$. Since $hW \notin \xi$, there exists a $U \in \xi$ such that $U\bar{\delta}_X hV$, i.e. $U \ll -hV$, $rU \ll -V$ and $rU\bar{\delta}_Y V$. Hence, there cannot be a δ_Y-filter containing $\{W\} \cup \{rU : U \in \xi\}$ for every $\xi \in S(\mathscr{A}, \delta_X)$. Of course, $f(\xi) = \{rU : U \in \xi\}$ for every $\xi \in S(\mathscr{A}, \xi_X)$.

Next, we claim that $f\,\mathrm{cl}(O_{\delta_X}(U)) = \mathrm{cl}(O_{\delta_Y}(rU))$ for each $U \in \mathscr{A}$. Obviously, $fO_{\delta_X}(U) \subseteq O_{\delta_Y}(rU)$ and, consequently, $f\,\mathrm{cl}(O_{\delta_X}(U)) \subseteq \mathrm{cl}(O_{\delta_Y}(rU))$. If $f\,\mathrm{cl}(O_{\delta_X}(U)) \neq$ $\neq \mathrm{cl}(O_{\delta_Y}(rU))$, there is a $W \in \mathscr{B}$ such that $W \leqq rU$ and $O_{\delta_Y}(W) \cap f(O_{\delta_X}(U)) = \emptyset$. From $W \leqq rU$ it follows that $hW \wedge U > 0$. If $\xi \in O_{\delta_X}(hW \wedge U)$, then $U \in \xi$ and $r(hW) = W \in f(\xi)$. Hence, $f(\xi) \in O_{\delta_Y}(W) \cap f(O_{\delta_X}(U))$; a contradiction.

Our goal now is to prove that f is an open map. To this end, we check that $f(\xi) \in$ $\in \mathrm{Int}(fO_{\delta_X}(U))$ for every $U \in \mathscr{A}$ and $\xi \in O_{\delta_X}(U)$. Fix a $V \in \xi$ with $V \ll U$. Then $rV \ll rU$ and $\mathrm{cl}(O_{\delta_Y}(rV)) \subseteq O_{\delta_Y}(rU)$. Consequently, $f(\xi) \in P_{\delta_Y}(rV) \subseteq \mathrm{cl}(O_{\delta_Y}(rV)) =$ $= f\,\mathrm{cl}(O_{\delta_X}(V) \subseteq fO_{\delta_X}(U)$.

Step 4. We are now ready to prove Proposition 1.8.

If f is open in V, we have h, r, δ_X on $\mathscr{A}$ and δ_Y on $\mathscr{B}$ satisfying the corresponding properties. h, r, δ_X and δ_Y have the same in $V[G]$. f_G is defined by

$$f_G(\xi) = \{rU : U \in \xi\} \quad \text{for every} \quad \xi \in X_G = S(\mathscr{A}, \delta_X)\,.$$

Repeating what we have done in Step 3, we can check that f_G is open.

Now, let f_G be open in $V[G]$. Let $\mathscr{A}'$ and $\mathscr{B}'$ denote the regular open algebras of X_G and Y_G (respectively). $\mathscr{A}$ is a dense subalgebra of $\mathscr{A}'$ and $\mathscr{B}$ a dense subalgebra of $\mathscr{B}'$. By Step 1, we have an embedding $h' : \mathscr{B}' \to \mathscr{A}'$ and a projection $r' : \mathscr{A}' \to \mathscr{B}'$ corresponding to f_G. Since, by Proposition 1.7, f is skeletal, we have $h : \mathscr{B} \to \mathscr{A}$ and $r : \mathscr{A} \to \mathscr{B}$ in V as well as in $V[G]$. From the uniqueness of the absolute of f_G it follows that $h'|_{\mathscr{B}} = h$ and $r'|_{\mathscr{A}} = r$. Hence, properties (1) and (2) hold for δ_X and δ_Y on $\mathscr{A}$ and $\mathscr{B}$, respectively, in $V[G]$. They are also fulfilled in V. This, by Step 3, implies the openness of f in V.

2. All classes of compact spaces under consideration in this paper may be characterized by elementary substructures. We refer the reader to J. Baumgartner [4] for a good introduction to elementary substructures and to A. Dow [7] for an introduction to their applications to topology. A discussion of the general construction, which is a good tool for characterizing classes of topological spaces by means of elementary substructures, can be found in Bandlow [2].

Let X be a compact Hausdorff space, θ a sufficiently large regular uncountable cardinal and $\mathcal{M}$ a suitable elementary substructure of $\mathcal{H}_\theta^V$.

$\phi_{\mathcal{M}}^X$ denotes the interior product of all continuous functions $f : X \to I$ which are elements of $\mathcal{M}$, i.e.

$$\phi_{\mathcal{M}}^X = \otimes(C(X, I) \cap \mathcal{M}) . \quad \text{Set } X(\mathcal{M}) = \phi_{\mathcal{M}}^X(X) .$$

2.2. Remark. Let $\dot{\mathcal{N}}$ be an elementary usbstructure of $\mathcal{H}_\theta^{V[G]}$ in $V[G]$ such that $\dot{\mathcal{N}} \cap \mathcal{H}_\theta^V = \mathcal{M}$. We claim that $\phi_{\dot{\mathcal{N}}}^{X_G} = \phi_{\mathcal{M},G}^X$.

It can be proved: if Y is a closed subspace of a compact space Z and $\mathcal{M}$ a suitable elementary substructure, then $\phi_{\mathcal{M}}^Y = \phi_{\mathcal{M}|Y}^Z$ (see Bandlow [2]). For the Tichonov cube I^A $\phi_{\mathcal{M}}^{I^A}$ is nothing but the projection map $\pi_{A \cap \mathcal{M}}$.

X_G is a subspace of the Tichonov cube I^S where S denotes the family of all continuous functions $f : X \to I$ in V. Hence, $\phi_{\dot{\mathcal{N}}}^{X_G}$ is the interior product of all functions f_G, $f \in S \cap \dot{\mathcal{N}}$. By Proposition 1.5, $\phi_{\mathcal{M},G}^X$ is the interior product of all maps f_G, $f \in S \cap \mathcal{M}$. $S \in \mathcal{H}_\theta^V$ implies $S \subset \mathcal{H}_\theta^V$ and, consequently, $S \cap \dot{\mathcal{N}} = S \cap \mathcal{M}$. This proves the assertion.

2.2. Fact (see, for example, Devlin [5]).

Let $\mathscr{A}$ and $\mathscr{B}$ be uncountable sets, $\mathscr{A} \subseteq \mathscr{B}$.

a) *If $C \subseteq [\mathscr{B}]^\omega$ is closed and unbounded, then $\{X \cap \mathscr{A} : X \in C\}$ contains a closed unbounded subfamily of $[\mathscr{A}]^\omega$.*

b) *If $D \subseteq [\mathscr{A}]^\omega$ is closed and unbounded, then $\{X \in [\mathscr{B}]^\omega : X \cap \mathscr{A} \in C\}$ is a closed unbounded subfamily of $[\mathscr{B}]^\omega$.*

It is easy to see that the intersection of two closed unbounded subfamilies of $[\mathscr{A}]^\omega$ remains a closed unbounded subfamily for every infinite set $\mathscr{A}$.

Another very useful fact concerns families of countable elementary substructures:

The family of all countable elementary substructures of an infinite set $\mathscr{A}$ forms a closed unbounded subfamily of $[\mathscr{A}]^\omega$.

3. In this section we consider the class of all openly-generated compact spaces. There are several ways of introducing this class: by k-metrik (Scepin [16]), by a special kind of embedding in the Tichonov cube (Shirokov [20]) or as limit spaces of sigma-spectra with open projection maps (Scepin [17]). It is not difficult to show that the representation of openly-generated compact spaces as limits of open sigma--spectra is equivalent to the following characterization (see Bandlow [2]):

A compact space X is openly-generated if and only if $\phi_{\mathcal{M}}^X$ is an open map for each countable elementary substructure $\mathcal{M}$ from a closed unbounded subfamily of $[\mathcal{H}_\theta^V]^\omega$ where θ is a sufficiently large regular uncountable cardinal.

3. Theorem. *Let X be a compact Hausdorff space, P a proper partial ordering and G a P-generic set over V.*

Then X is openly-generated in V if and only if X_G is openly-generated in $V[G]$.

Proof. Suppose that X is openly-generated. Let θ be a sufficiently large regular uncountable cardinal and let D be a closed unbounded family of countable elementary substructures of $\mathcal{H}_\theta^V$ such that $\phi_{\mathcal{M}}^X$ is open for each $\mathcal{M} \in D$. By 2.2.b), there exists a closed unbounded family $\dot{C}$ of countable elementary substructures of $\mathcal{H}_\theta^{V[G]}$ such that $\mathcal{N} \cap \mathcal{H}_\theta^V \in D$ for each $\mathcal{N} \in \dot{C}$. Hence, by 2.1, $\phi_{\mathcal{N}}^{X_G}$ is open for all $\mathcal{N} \in \dot{C}$. Thus X_G is openly-generated. Conversely, suppose X_G is openly-generated. Assume, on the contrary, that the set S of all countable elementary substructures $\mathcal{M}$ of $\mathcal{H}_\theta^V$ such that $\phi_{\mathcal{M}}^X$ is not open is stationary in $[\mathcal{H}_\theta^V]^\omega$ in V. Since P is proper, S remains stationary in $[\mathcal{H}_\theta^V]^\omega$ in $V[G]$. Let $\dot{C}$ be a closed unbounded family of countable elementary substructures of $\mathcal{H}_\theta^{V[G]}$ such that $\phi_{\mathcal{N}}^{X_G}$ is open for each $\mathcal{N} \in \dot{C}$. By 2.2.a), there exists a $\mathcal{N} \in \dot{C}$ such that $\mathcal{N} \cap \mathcal{H}_\theta^V = \mathcal{M} \in S$. Hence, $\phi_{\mathcal{M},G}^X$ is open; a contradiction to Proposition 1.8. □

4. An interesting subclass of the dyadic compacta is the class of Dugundji spaces introduced by Pełczynski [14]. Haydon proved that the notions of Dugundji spaces and absolute extensors in dimension zero are equivalent. A Boolean space is Dugundji iff its dual algebra is projective (Koppelberg [11]). Remark that every Dugundji space is openly-generated (Scepin [16]). We need the following characterizations of Dugundji spaces:

(1) A compact Hausdorff space X is Dugundji iff X is the inverse limit of a continuous inverse system $\langle X_\alpha, \pi_\alpha^\beta, \varrho \rangle$ where $|X_0| = 1$ and each $\pi_\alpha^{\alpha+1}$ has weight $\leq \omega$ and is open (Haydon [10], Scepin [16]).[1]

(2) A compact Hausdorff space X is Dugundji iff for every embedding $i: X \to I^\tau$ in the Tichonov cube there is an assignment $e_i: T(X) \to T(I^\tau)$ such that:
   a) $i(U) = e_i(U) \cap i(X)$ for each open subset U of X,
   b) $e_i(U_1 \cap U_2) = e_i(U_1) \cap e_i(U_2)$ and
   c) $e_i(U_1) \cap e_i(U_2) = \emptyset$, if $U_1 \cap U_2 = \emptyset$, for arbitrary $U_1, U_2 \in T(X)$ (Shirokov [20]).

Inverse systems described in (1) are called Haydon-spectra. For our purpose the following characterization of Dugundji spaces by elementary substructures is convenient to use.

4.1. Proposition. *A compact Hausdorff space X is Dugundji iff there is a closed unbounded family D of countable elementary substructures of $\mathcal{H}_\theta^V$, where θ is*

[1] The weight of a continuous map $f: X \to Y$ is defined to be the minimal cardinality of the system γ of open cozero subsets of X such that $\gamma \cup \{f^{-1}U: U \text{ is open in } Y\}$ is a subbase for the topology in X (Pasynkov [13]).

a sufficiently large regular uncountable cardinal, such that for every set $T \subseteq D$ the interior product $\otimes\{\phi^X_{\mathcal{M}} : \mathcal{M} \in T\}$ is an open map.

Proof. To prove sufficiency we fix a transfinite sequence $\mathcal{M} \in D$, $\alpha < \varrho$, such that for every pair of distinct points $x, y \in X$ there is an $\alpha < \varrho$ and a continuous function $f : X \to I$, $f \in \mathcal{M}_\alpha$, with $f(x) \neq f(y)$. Set

$$\pi^\alpha = \otimes\{\phi^X_{\mathcal{M}_\gamma} : \gamma < \alpha\} \quad \text{and} \quad X_\alpha = \pi^\alpha(X) .$$

For $\alpha < \beta$ there is a canonical map $\pi^\beta_\alpha : X_\beta \to X_\alpha$. Since π^α is open, π^β_α is open too. Remark that, if $V \in \mathcal{M}_\alpha$ is a cozero subset of X, then $(\pi^\alpha)^{-1}\,\pi^\alpha(V) = V$ and $\pi^\alpha(V)$ is a cozero subset of X_α. Set

$$\gamma_\alpha = \{\pi^{\alpha+1}(V) : V \in \mathcal{M}_{\alpha+1} \setminus \mathcal{M}_\alpha \text{ and } V \text{ is a cozero subset of } X\} .$$

$\gamma_\alpha \cup \{\pi^{\alpha+1}_\alpha)^{-1}\, U : U$ is an open subset of $X_\alpha\}$ is a subbase of $X_{\alpha+1}$. To prove necessity we use Shirokov's characterization. Let $i : X \to I^\tau$ and $e_i : T(X) \to T(I^\tau)$ be as described in (2). As usual, π_B denotes the projection $I^\tau \to I^B$ for each set $B \subseteq \tau$. Let $\mathcal{K}$ denote the family of all e_i-admissable subsets of τ (see Shirokov [20]). $\mathcal{K}$ satisfies the following properties:

a) For each infinite set $B \subseteq \tau$ there is a $B' \in \mathcal{K}$ such that $B \subseteq B'$ and $|B| = |B'|$.

b) $\cup\sigma \in \mathcal{K}$ for every $\sigma \subseteq \mathcal{K}$.

c) $\pi_{B|X} : X \to \pi_B(X)$ is open for each $B \in \mathcal{K}$.

If $\alpha \in \tau \cap \mathcal{M}$, then there exists a countable $B \in \mathcal{K}$ with $\alpha \in B$. We may assume that $B \in \mathcal{M}$. Since B is countable, this implies that $B \subseteq \mathcal{M}$. Hence, by property b), $\tau \cap \mathcal{M} \in \mathcal{K}$ for every elementary substructure and $B = \cup\{\mathcal{M} : \mathcal{M} \in T\} \in \mathcal{K}$ for every set T of elementary substructures. □

4.2. Lemma. *Let $\mathbb{P}$ be a ccc partial ordering, G a $\mathbb{P}$-generic set over V and $\dot{D}$ a name of an element of $V[G]$. If $\mathcal{A}$ is an infinite set in V such that $p \Vdash$ "$\dot{D} \subseteq [\mathcal{A}]^\omega$ is closed and unbounded" where $p \in \mathbb{P}$, then there exists a closed unbounded family, $C \subseteq [\mathcal{A}]^\omega$ in V such that $p \Vdash \mathcal{M} \in \dot{D}$ for each $\mathcal{M} \in C$.*

Proof. Instead of $\dot{D}$ we consider the name of a function $\dot{f} : [\mathcal{A}]^\omega \to \mathcal{A}$ in $V[G]$ such that

$$p \Vdash \{X \in [\dot{D}]^\omega : X \text{ is closed under } \dot{f}\} \subseteq \dot{D}$$

(see Baumgartner [4], 1.4). Let θ be a sufficiently large regular uncountable cardinal and let $\mathcal{N}$ be a countable elementary substructure of $\mathcal{H}^V_\theta$ with $\dot{f} \in \mathcal{N}$. Set $X = \mathcal{N} \cap$ $\cap\, \mathcal{A}$. We claim that $p \Vdash$ "X is closed under $\dot{f}$".

$p \Vdash (\exists y \in \mathcal{A})\,(\dot{f}(x_1, \ldots, x_n) = y)$ for arbitrary $x_1, \ldots, x_n \in X$. Since $\mathbb{P}$ satisfies ccc, there exist a countable set $\{p_n : n = 1, 2, \ldots\} \subseteq \mathbb{P}$, which is predense below p, and $y_n \in \mathcal{A}$, $n = 1, 2, \ldots$, such that $p_n \Vdash \dot{f}(x_1, \ldots, x_n) = y_n$ for each $n = 1, 2, \ldots$. Since $x_1, \ldots, x_n, \mathcal{A}, f, \mathbb{P}$ and p are elements of $\mathcal{N}$, we may assume that $\{y_n : n =$ $= 1, 2, \ldots\} \in \mathcal{N}$. Then $y_n \in \mathcal{N}$ for each n and, consequently,

$$p \Vdash (\exists y \in X)\,(\dot{f}(x_1, \ldots, x_n) = y) .$$

C is a closed unbounded family $\subseteq [\mathscr{A}]^\omega$ such that for each $\mathscr{M} \in C$ there is a suitable countable elementary substructure $\mathscr{N}$ of $\mathscr{H}_\theta^V$ with $\mathscr{M} = \mathscr{N} \cap \mathscr{A}$ (see Fact 2.2.a). □

4.3. Theorem (Fuchino for Boolean spaces).

Let X be a compact Hausdorff space, $\mathbb{P}$ a ccc partial ordering and G a $\mathbb{P}$-generic set over V. Then X is a Dugundji space in V if and only if X_G is a Dugundji space in $V[G]$.

Proof. Suppose X is the limit space of a Haydon-spectrum $\langle X_\alpha, \pi_\alpha^\beta, \varrho\rangle$ in V. Then, by Proposition 1.3 and 1.8, X_G is the limit space of the Haydon-spectrum $\langle X_{\alpha,G}, \pi_{\alpha,G}^\beta, \varrho\rangle$ in $V[G]$.

To prove the reverse direction, let θ be a sufficiently large regular uncountable cardinal and let $\dot{C}$ be a closed unbounded family of countable elementary substructures of $\mathscr{H}_\theta^{V[G]}$ such that $\otimes\{\phi_{\dot{\mathscr{N}}}^{X_G} : \dot{\mathscr{N}} \in \dot{T}\}$ is open for every set $\dot{T} \subseteq \dot{C}$ in $V[G]$. Let $\dot{C}_0 \subseteq [\mathscr{H}_\theta^V]^\omega$ be a closed unbounded family in $V[G]$ such that for each $\dot{\mathscr{M}} \in \dot{C}_0$ there exists an $\dot{\mathscr{N}} \in \dot{C}$ with $\dot{\mathscr{N}} \cap \mathscr{H}_\theta^V = \dot{\mathscr{M}}$ (see Fact 2.2.a). By Lemma 4.2, there is a closed unbounded family $D \subseteq [\mathscr{H}_\theta^V]^\omega$ in V such that $\mathscr{M} \in \dot{C}_0$ for each $\mathscr{M} \in D$. We may assume that all $\mathscr{M} \in D$ are countable elementary substructures of $\mathscr{H}_\theta^V$. Hence, by 2.1., for each $\mathscr{M} \in D$ there is an $\dot{\mathscr{N}} \in \dot{C}$ such that $\phi_{\mathscr{M},G}^X = \phi_{\dot{\mathscr{N}}}^{X_G}$.

Apply now Propositions 1.5 and 1.8 to complete the proof. □

5. Two regular spaces are said to be coabsolute if their absolutes are homeomorphic.

5.1. Theorem (Fuchino for Boolean space).

Let X be a compact Hausdorff space, $\mathbb{P}$ a ccc partial ordering and G a $\mathbb{P}$-generic set over V. Then X is coabsolute to a Dugundji space in V if and only if X_G coabsolute to a Dugundji space in $V[G]$.

The proof repeats that of the previous theorem where the characterization of Dungundji spaces is replaced by a characterization of spaces coabsolute to Dugundji spaces. We need the following result of Shapiro [19], Corollary 1:

> A compact Hausdorff space X is coabsolute to a Dugundji space iff X is the inverse limit of a continuous inverse system $\langle x_\alpha, \pi_\alpha^\beta, \varrho\rangle$ where $|X_0| = 1$ and each $\pi_\alpha^{\alpha+1}$ has weight $\leq\omega$ and is skeletal.

5.2. Proposition. *A compact Hausdorff space X is coabsolute to a Dugundji space iff there is a closed unbounded family D of countable elementary substructures of $\mathscr{H}_\theta^V$, where θ is a sufficiently large regular uncountable cardinal, such that for every set $T \subseteq D$ the interior product $\otimes\{\phi_{\mathscr{M}}^X : \mathscr{M} \in T\}$ is a skeletal map.*

To prove this Proposition we need two lemmas.

5.3. Lemma. *Let $f: X \to Y$, $f': X' \to Y'$, $g: X \to X'$ and $g': Y \to Y'$ be continuous onto maps of compact spaces that $f' \circ g = g' \circ f$. If f and g are irreducible, then f' is skeletal iff g' is skeletal.*

Proof. f' and g' have the same absolute $pf' = pg'$. The assertion follows from the fact that a continuous onto map is skeletal iff the absolute is open (see [15], [1]). $\square$

Let $f : X \to Y$ be an irreducible map of compact spaces. Suppose X satisfies ccc and let T be a family of suitable countable elementary substructures of $\mathscr{H}_\theta$ where θ is a sufficiently large regular uncountable cardinal. Put $\phi' = \otimes\{\phi^X_{\mathscr{M}} : \mathscr{M} \in T\}$, $X' = \phi'(X)$, $\psi' = \otimes\{\phi^Y_{\mathscr{M}} : \mathscr{M} \in T\}$ and $Y' = \psi'(Y)$. There is a map $f' : X' \to Y'$ such that $f' \circ \phi' = \psi' \circ f$.

5.4. Lemma *f' is irreducible.*

Proof. We regard X' as a subspace of $\Pi\{X(\mathscr{M}) : \mathscr{M} \in T\}$ and Y' as a subspace of $\Pi\{Y(\mathscr{M}) : \mathscr{M} \in T\}$. Fix $\mathscr{M}_1, \ldots, \mathscr{M}_n \in T$ and cozero subsets $V_1, \ldots, V_n$ of X in $\mathscr{M}_1, \ldots, \mathscr{M}_n$, respectively, such that $\cap\{V_i : i = 1, 2, \ldots\} \neq \emptyset$.

Remark that $(\phi^X_{\mathscr{M}})^{-1}\, \phi^X_{\mathscr{M}}(W) = W$ for each cozero subset W of X, $W \in \mathscr{M}$. Hence, $\mathcal{O} = \{(x_t)_T \in X' : (\phi^X_{\mathscr{M}_i})^{-1}\,(x_t) \in V_i \text{ for } i = 1, 2, \ldots\}$ is a nonempty open subset of X' and we have to prove that $f'^{\#}(\mathcal{O}) \neq \emptyset$.

Let σ_i be maximal families of pairwise disjoint cozero subsets of Y' such that $\mathrm{cl}(U) \subseteq f^{\#}V_i$ for each $U \in \sigma_i$, $i = 1, \ldots, n$. Since Y satisfies ccc, the σ_i are countable. We may assume that $\sigma_i \in \mathscr{M}_i$ and, consequently, $\sigma_i \subset \mathscr{M}_i$ for each i. It is obvious, that $\cup\sigma_i$ is dense in $f^{\#}V_i$ for each i. Since $f^{\#}V_1 \cap \ldots \cap f^{\#}V_n \neq \emptyset$, there exist $W_1 \in \sigma_1$, $\ldots$, $W_n \in \sigma_n$ such that $W_1 \cap \ldots \cap W_n \neq \emptyset$. Take a point $y \in W_1 \cap \ldots \cap W_n$ and set $y' = \psi'(y)$. We claim that $f'^{-1}(y) \subseteq \mathcal{O}$.

If $x \in X$ and $f'(\phi'x)) = y' = \psi'(fx)$, then $fx \in W_1 \cap \ldots \cap W_n$. From $fx \in W_i$ it follows that $x \in V_i$ for each i. Hence, $x \in V_1 \cap \ldots \cap V_n$ and $\phi'(x) \in \mathcal{O}$. $\square$

Proof of Proposition 5.2.

If X is coabsolute to a Dugundji space, there exist irreducible onto maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ where Y is a Dugundji space. The conclusion now follows from Proposition 4.2 and Lemma 5.4.

To prove the converse implication we apply Shapiro's result and similar arguments as in the proof of Proposition 4.1. $\square$

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