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Large Subsets of Dual Banach Spaces

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We make a few remarks concerning [*S1*].

The following result of the summer of 1972 appears several times in the literature. It is explicitly stated in Lemma 1, Theorem 1, Corollary 1 and the remark preceding Theorem 3 in [S1] (see also [S2] and [S3].

Theorem. *Let X be a Banach space, $S \subseteq X^*$ such that the norm density of S is greater then the norm density of X and S the intersection of a weak* G_δ and a weak* compact set. Then there exists a compact subset K of S and a separable linear subspace Y of X such that $K|Y$ is not norm separable.*

Of course, the entire construction of [S1] (the Haar system etc.) may be made with respect to K. It is a trivial consequence of this result and the classical projective class theorems that we may make the formally weaker assumption that S is analytic (in the metrizable case) or even k-analytic in the general case. We shall go two steps further. Firstly, we recall Fremlin's [Fr] generalization of the classical projective class sets (the analytic sets, or even, the k-analytic sets). Let P be (homeomorphic to) the irrationals in the closed bounded interval I. If T is compact then a set $S \subseteq T$ is Čech analytic if there exists $C \subseteq T \times P$ such that C is the intersection of a closed subset and a G_δ subset of $T \times P$ and the projection of C is S. Since $T \times P$ is a G_δ subset of $T \times I$, we have that C is the intersection of a compact subset and a G_δ subset of $T \times I$. A topological space is Čech complete if it is homeomorphic to a G_δ subset of a compact space. Thus, S is the continuous image of the Čech complete space C. Obviously, the class of Čech analytic subsets of T contains the open sets. It is quite easy to check that the class of sets which are Čech analytic, as well as their complements, is a σ-algebra, and, hence, contains the Borel sets. Analogous to classical arguments, it can be proved that the class of Čech analytic subsets of T is stable under the Souslin operation. All of this is in the very interesting [Fr] in

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greater depth and detail, where it is also shown that Čech analytic sets are universally measurable.

Suppose that S is as in the Theorem except that we only assume that it is Čech analytic. A child's safety scissors and a little harmless glue are applied to the history lesson above. We simply increase the dimension of X by one. Consider the Banach space X as canonically embedded in $X \times \mathfrak{R}$ and let $I = [0, \delta]$ (δ can be arbitrarily small if one really wants quantitative results). Let $C \subseteq (X \times \mathfrak{R})^*$ be the intersection of a weak* compact subset and a weak* G_δ subset of $X^* \times I$ such that the projection of C is S. Trivially, the norm density of C is greater that the norm density of $X \times \mathfrak{R}$. Apply the results of [S1] to C to obtain a compact $K' \subseteq C$, Haar system, etc. In particular, there exists a separable subspace of $X \times \mathfrak{R}$, which we may assume to be of the form $Y \times \mathfrak{R}$, where Y is a separable subspace of X, such that $K'|(Y \times \mathfrak{R})$ is not norm separable. Let K be the restriction of K' to X; $K|Y$ is not norm separable and

$$K|Y \subseteq K'|Y \subseteq S|Y.$$

Apply the results of [S1] to K. That's all there is to it.

The second step, unfortunately for some readers, may actually require looking at [S1] and even at the proof of the Cantor-Bendixson theorem (see [S4]). Suppose that X is a Banach space, C is Čech complete,

$$f : C \to X^*$$

is continuous in the weak* topology and

$$\text{norm density } f(C) > \text{weight } C.$$

Then, there exists a compact subset K of C and a separable subspace Y of X such that $f(K)|Y$ is not norm separable. The proof of this fact is only an exercise in manipulating the construction in [S1] in combination with the properties of Čech complete spaces. We shall supply a few details. Let $S = f(C)$ and we may assume that S is a subset of the unit ball. Let γ be the first ordinal so that

$$\gamma > \text{weight } C$$

and choose $\varepsilon > 0$ and

$$R = \{x_\alpha^* : \alpha < \gamma\} \subseteq S$$

such that

$$\text{distance } (x_\beta^*, Z_\beta) > \varepsilon$$

where Z_β is the smallest norm closed linear subspace containing

$$\{x_\alpha^* : \alpha < \beta\}.$$

Let U be a basis of the topology of C that has the cardinality of the weight of C. Let

$$\mathbf{V} = \{U \in \mathbf{U} : \text{cardinality } (f(U) \cap R) \leqq \text{weight } C\} .$$

Since any closed subset of C is also Čech complete and

$$\text{cardinality } (\bigcup_{U \in V} f(U) \cap R) \leqq \text{weight } C$$

it is no loss of generality in assuming that $\mathbf{V} = \emptyset$; we may simply discard $\bigcup\{U \in \mathbf{V}\}$ and replace C by the complement of this set. We may assume that the cardinality of $f(U) \cap R$ is greater than the weight of C for all $U \in \mathbf{U}$; hence, the cardinality of $f(U) \cap R$ is γ. From this point on, one need only mix the ancient [S1] and the refurbished version of Cantor-Bendixson in [S4]. The relevant property is the following: for any open subset $\emptyset \neq U$ of C there exist $x \in X$, $\|x\| = 1$, and points a and b in U so that $\langle x, f(a)\rangle = 0$ and $\langle x, f(b)\rangle < \varepsilon$. The Souslin operation is superfluous.

Theorem. *Suppose that X is a Banach space and K is a weak* compact and convex subset of X^*. Suppose that the norm density of the set E of extreme points of K is greater than the norm density of X. Then there exist a norm separable linear subspace Y of X and a set $T \subseteq E$ such that $\Delta = T|Y$ is a subset of the extreme points of $K|Y$, Δ is homeomorphic to the Cantor space and Δ is discrete in the norm topology.*

If, in addition to the hypothesis above, we also assume that X is norm separable, the result follows from [S1] because E is a G_δ subset of K (this was overlooked in [S1]). Haydon [H] observed, as a consequence of the Choquet representation theorem, that there is a converse in the separable case: if E is norm separable then K is also norm separable. Actually, a consequence of our references is the following: if E is a subset of a weak (that is, a $\sigma(X^{**}, X^*)$) Lindelöf subset of X^* then K is the norm closed convex hull of E. We recall another result of Choquet (a proof can be found in [S3]). Suppose that K is a compact convex set and $\{K_n\}$ is a descending sequence of non empty closed and convex subsets of K such that each $K \setminus K_n$ is also convex. Then $\bigcap_n K_n$ contains an extreme point of K. We mix these facts with the construction in the Proposition of [S5] in order to prove this theorem. Choose a norm dense subset D of X,

$$\text{cardinality } D = \text{norm density } X = \eta$$

and for each rational r and $x \in D$ let $S(x, r)$ be the open halfspace

$$\{x^* \in X^* : x^*(x) > r\} .$$

The collection $\{S(x, r) \cap E\}$ is a basis for the weak* topology on E. There exists an $\varepsilon < 0$ such that E can not be covered by any collection of cardinality η of balls of radius ε (which we shall paraphrase as the ε norm density of E). Define

$$F = E \setminus (\bigcup S(x, r) \cap E : \varepsilon \text{ norm density of } S(x, r) \cap E \leqq \eta\}) .$$

Clearly, $F \neq \emptyset$ and if $S(x, r) \cap F \neq \emptyset$ then the ε norm density of $S(x, r) \cap F$ is greater than η. Let L be the weak* closed convex hull of F; clearly, every non empty slice $S(x, r) \cap L$ has diameter greater than ε. It follows from the results of Smulian that the Minkowski functional

$$\varrho(x) = \sup |x^*(x)|$$

has no point of Fréchet differentiability. We repeat the construction in the Proposition of [S5]. Of course, any open slice of L contains an element of F. We construct

(i) $\{x_{n,i} : n = 0, 1, 2, \ldots, 0 \leqq i < 2^n\} \subseteq X$ and $\|x_{n,i}\| = 1$ for each n and i;

(ii) rational numbers $\{r_{n,i} : n = 0, 1, 2, \ldots, 0 \leqq i < 2^n\}$;

(iii) $\{y_{n,i} : n = 0, 1, 2, \ldots, 0 \leqq i < 2^n\} \subseteq D$ and

(iv) $e^*_{n,i} \in S(y_{n,i}, r_{n,i}) \cap F \neq \emptyset$

such that

(v) $\{x^* \in K : x^*(y_{n+1,2i+j}) \geqq r_{n+1,2i+j}\} \subseteq S(x_{n,i}, r_{n,i}) \cap K$ for all n, all i, and $j = 0, 1$;

(vi)
$$\sup \{x^*(x_{n,i}) : x^* \in S(y_{n+1,2i}, r_{n+1,2i}) \cap K\} + \varepsilon/12$$
$$\leqq \inf \{x^*(x_{n,i}) : x^* \in S(y_{n+1,2i+1}, r_{n+1,2i+1}) \cap K\} \text{ and}$$

(vii)
$$S(y_{n+1,i}, r_{n+1,i}) \subseteq \bigcap_{\substack{\|x\|=1 \\ x \in Y_{n-1}}} \left\{x^* : |(x^* - e^*_{n+1,i})x \leqq \frac{1}{n+1}\right\}$$

where Y_n is the finite dimensional space spanned by

$$\{x_{p,i} : p \leqq n, 0 \leqq i < 2^p\} \cup \{y_{p,i} : p \leqq n, 0 \leqq i < 2^p\}.$$

Suppose we make an arbitrary choice of integers $\sigma : \mathrm{N} \to \mathrm{N}$ with the the properties that $\sigma(0) = 0$ and

$$\sigma(n+1) \in \{2\sigma(n), 2\sigma(n) + 1\}.$$

Then, not only is

$$\bigcap_n (S(x_{n,\sigma(n)}, r_{n,\sigma(n)}) \cap K) \neq \emptyset$$

but also contains an extreme point of K. Choose

$$x^*_\sigma \in \bigcap_n (S(x_{n,\sigma(n)}, r_{n,\sigma(n)}) \cap E)$$

and observe that if $\sigma \neq \tau$ then

$$\sup_{n,i} |(x^*_\sigma - x^*_\tau)x_{n,i}| > \varepsilon/12.$$

Let Y be the linear subspace of X spanned by $\{x_{n,i}\} \cap \{y_{n,i}\}$. Since each $x_{n,i}$ is in Y we also have that

$$R_\sigma = \bigcap_n \left(S(x_{n,\sigma(n)}, r_{n,\sigma(n)}\right) \cap (K|Y)) \neq \emptyset$$

(where the slices are taken in Y^*) contains an extreme point of $K|Y$. It is well known that

$$\text{extreme}\,(K|Y) \subseteq E|Y.$$

The variation of any $x \in Y$ over R_σ is zero; thus, R_σ is a one point set consisting of an extreme point of $K|Y$. Of course, let $T = \{x_\sigma^*\}$ be any choice of extreme points of K such that

$$x_\sigma^*|Y \in R_\sigma .$$

Clearly, $\Delta = T|Y$ has all of the desired properties. We may construct still another Cantor subspace of Δ with all of the nasty properties described in [**S1**] (or, in any of its successors). It follows, as in [**S1**], that E contains a bounded biorthogonal system of the cardinality of the continuum.

A related result can be found in [**S3**].

Theorem. *Let K be a weak* compact and convex subset of X^*. Suppose that K is not the norm closed convex hull of its extreme points. Then there exist a subset S of the extreme points of K and a separable subspace Y of X such that $S|Y$ is homeomorphic to the Cantor space and $S|Y$ is equivalent to the usual basis of $l_1(S)$.*

We take this opportunity to correct two errors in [**Fi**]. The results of Reinov, referred to in [**Fi**], are incorrect, as shown in [**RS**] by counterexample. The second error in [**Fi**] (the remark after Proposition 4.3), an error repeated in the *Reviews*, is the assertion that Corollary 1.11 of [**S2**] is incorrectly stated. Corollary 1.11 of [**S2**] is completely correct. In [**Fi**] a definition different from that of [**S2**] is used, and in [**Fi**] an elementary example is given showing that the definitions are different. The definition in [**S2**] was justified by the discussion at the top of page 511, and is, as far as we known, the first time that anything was proven about non convex sets in this connection. Using the definition of [**S2**] we have.

Theorem. *Let $K \subseteq X^*$ be weak* compact and convex. Let E be the extreme points of K. Then K has the Radon-Nikodym property if and only if $E|Y$ is norm separable for every separable subspace Y of X.*

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