On the Lattice Structure of Invariant Functions for Markov Operators on C(X)

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By $C(X)$ we denote the Banach lattice of all real-valued continuous functions on a compact Hausdorff space X. As usually, the Banach dual to $C(X)$ we identify with the Banach lattice $M(X)$ of all Radon measures on X. The w^*-compact and convex set of all probability (Radon) measures we denote by $P(X)$. A linear operator T acting on $C(X)$ is called Markov if T is nonnegative ($f \geqq 0 \Rightarrow Tf \geqq 0$) and $T1 = 1$. Equivalently, we can say that $\|T\| = 1$ and $T1 = 1$. For a given Markov operator T let C_T and P_T denote respectively, the set of all T-invariant functions ($f \in C_T$ iff $f \in C(X)$ and $Tf = f$) and the set of all probability T-invariant measures ($\mu \in P_T$ iff $\mu \in P(X)$ and $T^*\mu = \mu$). C_T is a closed subspace of $C(X)$ containing constant function. On the other hand, using the Markov-Kakutani fixed point theorem, we see that P_T is a non-empty, convex and w^*-compact subset of $P(X)$.

The lattice properties of the set of invariant functions are closely connected with the asymptotic behaviour of $A_n(T)$ – the Cesàro average of T ($A_n(T) = n^{-1}(I + T + \ldots + T^{n-1})$). To see this let us consider the following separation properties:

(i) C_T separates P_T, i.e. for two distinct invariant measures μ_1, μ_2 there exists an invariant function f such that $(\mu_1, f) \neq (\mu_2, f)$ (here (μ, f) denotes the canonical bilinear form corresponding to $C(X)$ and $M(X)$),

(ii) C_T separates $\operatorname{ex} P_T$ – the set of extreme points of P_T (so called ergodic measures),

(iii) A_T separates P_T where A_T is the closed algebra generated by C_T.

Sine's separation theorem says that condition (i) is equivalent to strong mean ergodicity (s.m.e.) of T, this means that the Cesàro averages converge in strong operator topology (to some Markov projection P (cf. [8])). Further, it is well known that (ii) is essential weaker than (i). On the other hand, Iwanik ([4]) has proved that (ii) is equivalent to (iii). Since $A_T = C_T$ iff C_T is a sublattice of $C(X)$, we see that in the

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case when C_T is a sublattice, T is s.m.e. provided C_T separates only ergodic measures.

To give the answer on the question when C_T is a sublattice first we consider the case when C_T has a lattice structure. We observe that in the case of s.m.e. Markov operator, C_T is always lattice with the lattice modulus $mod\, f = P|f|$, where P is the associated Markov projection, but not always a sublattice (cf. for example [9]). It is worth to notice that in general C_T even is not a lattice (cf. [5]).

Now according to [2] let $\varkappa$ be the canonical mapping from $M(X)$ onto the quotient Banach space $M(X)/C_T^{\perp}$, where $C_T^{\perp}$ is the annihilator of C_T. We note that every $f \in C_T$ defines a linear functional $(f, \cdot)$ on the quotient space by the formula

$$(f, \varkappa(\mu)) = \int f \, d\mu \, .$$

Let $Q = \varkappa(P(X))$. By continuity of $\varkappa$, Q is convex and w^*-compact. Let $A(Q)$ denote the space of all affine continuous functions on Q. Then for every $f \in C_T$, $\tilde{f}(q) = (f, q) \in A(Q)$. Our first observation is following

Proposition ([2]). *The mapping $f \to \tilde{f}$ is a linear order-preserving isometry of C_T onto the space $A(Q)$.*

To obtain characterization C_T to be a lattice we use the concept of a Bauer simplex (cf. [1]). One of the characterization of a Bauer simplex says that K is a Bauer simplex iff $A(K)$ is a lattice iff each $f \in C(\mathrm{ex}K)$ uniquely extends to an affine function $\tilde{f} \in A(K)$. Therefore we have

Corollary 1 ([2]). *C_T is a lattice iff Q is a Bauer simplex.*

Remark. A typical example of the simplex in the theory of Markov operators is the set P_T. Schaefer ([7]) has observed that for arbitrary Markov operator T, P_T is always a simplex but not necessary a Bauer simplex. Nevertheless, if T is s.m.e. then we can show that the mapping $\mu \to \varkappa(\mu)$ is an affine isomorphism between P_T and Q and therefore, in virtue of Corollary 1, since C_T is always a lattice, we see that P_T is a Bauer simplex. This result was earlier observed by Sine ([8]).

According to Sine ([8]), let $\mathscr{D}$ denote the partition of X generated by the level sets of C_T. In $\mathscr{D}$ we distinguish those elements, so called ergodic set, which support at least one invariant measure and denote this collection by $\mathscr{E}$. Take $W = \bigcup \mathscr{E}$. W is called conservative set for T.

A set $F \subset X$ is called invariant if F is closed nonempty set which satisfies the following condition

$$x \in F \Rightarrow supp\; T^*\delta_x \subset F \, .$$

Now define, so called, boundary for T as follows

$$\partial_T = cl\{x \in X \colon \varkappa(\delta_x) \in ex\; Q\} \, .$$

Theorem 1 ([2], [6]). *∂_T is invariant subset of conservative set W. If in addition C_T*

is a lattice then ∂_T *is a union of certain invariant cells in* $\mathscr{D}$. *Moreover, then we have*

$$\partial_T = \{x \in X\colon \forall f \in C_T \; mod\, f(x) = |f(x)|\} .$$

In particular, if T is s.m.e. then

$$\partial_T = \{x \in X\colon P^*\delta_x \in ex\, P_T\} = \{x \in X\colon P|f|\,(x) = |f(x)| \;\; \forall f \in C_T\}$$

(*for the rest result see also* [3] *and* [5]).

Proposition and Theorem 1 yield also another characterization

Corollary 2 ([2]). *The following are equivalent*

(i) C_T *is a lattice.*

(ii) *Every* $f \in C(\partial_T)$ *which is constant on the cells of the restricted partition* $\mathscr{D} \cap \partial_T$ *extends uniquely to some* $\bar{f} \in C_T$.

Notice that this result not answer on the basic question what the lattice modulus is. We only know from Theorem 1 that on the lattice boundary ∂_T we have $mod\, f = |f|$ and in consequence $mod\, f(x) = lim\, A_n|f|\,(x)$ on ∂_T. The next theorem explains when the above equation holds $\forall x \in X$.

Theorem 2 ([6]). *The following are equivalent*:

(i) $\forall f \in C_T\; \exists \bar{f} \in C_T\; |f| = \bar{f}$ *a.e. for every invariant measure,*

(ii) C_T *is a lattice and* ∂_T *has invariant measure one,*

(iii) C_T *is a lattice and* $\partial_T = W$,

(iv) $\forall f \in C(X)$ *which is constant on each ergodic set there exists* $\bar{f} \in C_T$ *such that* $f = \bar{f}$ *a.e. for every invariant measure,*

(v) $\forall f \in C_T$ *there exists a continuous limit* $lim\, A_n|f|\,(x)$ *which defines the lattice modulus in* C_T.

As a consequence of this theorem we get well known Sine's result (cf. [8]).

Corollary 3 ([2] *and* [5]). *If T is s.m.e. then*

$$W = \{x \in X\colon P^*\delta_x \in ex\, P_T\}$$

and each ergodic set is invariant.

Finally, we observe that from the point of view of the preceding results, the problem when C_T has a sublattice structure is trivial. Namely we have

Theorem 3 ([2] *and* [6]). C_T *is a sublattice of* $C(X)$ *iff each cell of the partition* $\mathscr{D}$ *is invariant.*

Indeed, from Theorem 1 if C_T is a sublattice then $\partial_T = X$ and each $D \in \mathscr{D}$ must be invariant. To see the converse, for every $D \in \mathscr{D}$, $f \in C_T$, $x \in D$ we have

$$T|f|\,(x) = \int |f|\, dT^*\,\delta_x = \int_D |f|\, dT^*\,\delta_x = |f(x)| \text{ since } f|_D \text{ is constant} .$$

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