Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 31 (1990), No. 2, 35--40

Persistent URL: http://dml.cz/dmlcz/701950

Disjointly Strictly-Singular Operators in Banach Lattices

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Received 11 March 1990

The aim of this note is to study some basic properties of the class of all disjointly strictly-singular operators defined on Banach lattices. We also present some applications in the context of Orlicz function spaces.

If X is a Banach lattice and Y is a Banach space, an operator $T: X \to Y$ is said to be *disjointly strictly-singular* if there is no disjoint sequence of non-null vectors (x_n) in X such that the restriction of T to the subspace $[x_n]$ spanned by the vectors (x_n) is an isomorphism.

This new class of operators, bigger than the class of strictly-singular operators, has been introduced recently in ([2], pp. 48), where applications to the problem of finding "non-natural" projections in lattices of measurable functions were given. More precisely, it happens that if for a Banach lattice X there exists a Riesz operator $T: X \to L^p(0, 1)$, for $p \geqq 1$, which is not disjointly strictly singular, then X contains a complemented subspace isomorphic to l^p.

Let us recall that an operator T between two Banach spaces X and Y is called *strictly-singular (or Kato)* if it fails to be an isomorphism on any infinite dimensional subspace. It is well-known that the class of all strictly singular operators from X to Y is a closed operator ideal in $L(X, Y)$, the space of all bounded operators endowed with the usual norm. (cf. [7], [8]; for other properties and extensions see eg. [1], [4], [5]).

Clearly, every strictly-singular operator is a disjointly strictly singular operator. However the converse does not hold in general:

An easy example is the inclusion operator $T: L^p(0, 1) \hookrightarrow L^q(0, 1)$ for $1 \leqq q < p$, which is disjointly strictly singular because for any sequence of disjoint functions (f_n) in $L^p(0, 1)$ we have $[f_n]_p \approx l^p$ and $[T(f_n)]_q \approx l^p$. However the operator T is not strictly singular because the restriction of T to the subspace generated by the Rademacher functions $[r_n]$ is an isomorphism $[r_n]_p \approx l^2 \approx [T(r_n)]_q$.

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Supported in part by D.G.I.C.Y.T. grant PB88-0141

We will show that, in general, the class of disjointly strictly singular operators is not an operator ideal (but it is from a lattice point of view).

When we consider Banach lattices X, having a Schauder basis of disjoint vectors, it comes out that both class of operators coincide:

Proposition 1. *Let X be a Banach lattice with a Schauder basis of disjoint vectors and Y be a Banach space. An operator $T: X \to Y$ is disjointly strictly-singular if and only if it is strictly singular.*

Proof. Assume that T is not strictly singular, so there exists a subspace Z such that $T_{|Z}$ is an isomorphism. Now, by ([6] Proposition 1.a.11) there exists a subspace $Z_1 = [x_n]$ with a basis (x_n) which is equivalent to a normalized block basis (x_n) of the disjoint basis (e_n) of X. Hence there exists $\delta > 0$ such that $\|T(\Sigma a_n x_n)\| \geqq \delta \|\Sigma a_n x_n\|$. Now using that (x'_n) is disjoint, we have $|a_k| \leqq \|\Sigma a_n x'_n\|$ for every k, hence

$$\begin{aligned}
\|T(\Sigma a_n x'_n)\| &\geqq \|T(\Sigma a_n x_n)\| - \|T(\Sigma a_n x_n - \Sigma a_n x'_n)\| \geqq \\
&\geqq \delta\|\Sigma a_n x_n\| - \|T\| \, \|\Sigma a_n x'_n\| \, (\Sigma \|x_n - x'_n\|) \geqq \\
&\geqq \frac{\delta}{K} \|\Sigma a_n x'_n\| - \|T\| \, \varepsilon \|\Sigma a_n x'_n\| \\
&\geqq \left(\frac{\delta}{K} - \varepsilon\|T\|\right) \|\Sigma a_n x'_n\|
\end{aligned}$$

where K is the constant of the equivalence between (x_n) and (x'_n) and ε is taken sufficiently small. So $T_{|[x'_n]}$ is an isomorphism and T is not disjointly strictly-singular.

q.e.d.

Example. For operators defined on separable modular (or Lorentz) sequence spaces to be disjointly strictly singular is the same as to be strictly singular.

Proposition 2. *Let S and T be operators from a Banach lattice X to a Banach space Y. If S and T are disjointly strictly-singular then $S + T$ is disjointly strictly singular.*

Proof. We assume that $S + T$ is not a disjointly strictly singular operator. Thus, there exists a sequence of disjoint vectors (x_n) such that $S + T_{|[x_n]}$ is an isomorphism, i.e. there exists a constant $K > 0$ such that

$$\|(S + T)(x)\| \geqq K\|x\|$$

for every $x \in [x_n]$. (We can assume w.l.o.g. that $K > \frac{1}{9}$).

Since T is disjointly strictly singular, $T_{|[x_n]}$ is not an isomorphism and we can build a block basis (u_n) of (x_n) verifying that

$$(*) \qquad \|T(u_n)\| \leqq \frac{1}{10^n} \|u_n\|, \quad n \in \mathbb{N}.$$

Indeed, there exists $y_1 = \sum_{n=1}^{\infty} a_{1,n}x_n$ with $\|y_1\| = 1$ and $\|T(y_1)\| \leqq 1/(4\,.\,10)$. Now, take $p_1 \in \mathbb{N}$ big enough such that $u_1 = \sum_{n=1}^{p_1} a_{1,n}x_n$ verifies $\|y_1 - u_1\| < 1/(4\,.\,10\|T\|)$. Hence $\|u_1\| > \frac{1}{2}$ and

$$\|T(u_1)\| \leqq \|T(u_1 - y_1)\| + \|T(y_1)\| \leqq \frac{\|u_1\|}{10}.$$

Since $T_{|[x_n]_{n > p_1}}$ is not an isomorphism, we can repeat the process obtaining, by induction, the sequence (u_n) wanted.

Now, let us consider the closed span $[u_n]$. If $v \in [u_n]$, $v = \sum_{k=1}^{\infty} \lambda_k u_k$, we have $\|\lambda_n u_n\| \leqq \leqq \|v\|$ and

$$\|Tv\| \leqq \sum_n \|T(\lambda_n u_n)\| \leqq \sum_n |\lambda_n| \frac{\|u_n\|}{10^n}$$

$$\leqq \|v\|/9\,.$$

This implies that for $v \in [u_n]$,

$$\|S(v)\| \geqq K\|v\| - \tfrac{1}{9}\|v\|\,.$$

So S is an isomorphism on $[u_n]$, which is a contradiction. q.e.d.

Disjointly strictly-singular operators are also stable with respect to the composition on the left:

Proposition 3. *Let X be a Banach lattice, Y and Z be Banach spaces. If $T: X \to Y$ is disjointly strictly-singular and $S: Y \to Z$ is a bounded operator, then $S \circ T$ is a disjointly strictly-singular operator.*

The verification is straightforward.

In general disjointly strictly singular operators are not stable with respect to the composition on the right with bounded operators:

Example. Let S be the canonic inclusion $L^2(0,1) \hookrightarrow L^1(0,1)$ and let T be the bounded operator $T: l^2 \to L^2(0,1)$ defined by $T(e_n) = r_n$, where (r_n) is the Rademacher function sequence and (e_n) is the canonic basis of l^2.

The operator S is disjointly strictly singular since for every disjoint function sequence (f_n), $[f_n]_2 \approx l^2$ and $[S(f_n)]_1 \approx l^1$. However the composition operator $S \circ T$ is not disjointly strictly singular since, by Khinchine inequality, there exist constants A_1 and $A_2 > 0$ such that

$$A_1\|\Sigma a_n e_n\|_2 \leqq \|\Sigma a_n(S \circ T)(e_n)\|_1 \leqq A_2\|\Sigma a_n e_n\|_2\,.$$

Hence the restriction of $S \circ T$ to $[e_n]$ is an isomorphism.

Proposition 4. *Let X and Y be Banach lattices and $T: X \to Y$ a Riesz operator. If $S: Y \to Z$ is a disjointly strictly singular operator for Z a Banach space, then $S \circ T$ is disjointly strictly singular.*

Proof. Assume that $(S \circ T)$ is not disjointly strictly singular. So there exists a disjoint vector sequence (x_n) in X and $K > 0$ such that

$$\|S \circ T(x)\| \geqq K\|x\| \geqq \frac{K}{\|T\|} \|T(x)\| \quad \text{for} \quad x \in [x_n] .$$

Now, if (y_n) denotes the sequence of disjoint vectors $(T(x_n))$ in Y, we deduce $\|S(y)\| \geqq \geqq (K/\|T\|) \|y\|$ for every $y \in [y_n]$. So S is not disjointly strictly singular. q.e.d.

If $D.S.(X, Y)$ means the class of all disjointly strictly-singular operators from a Banach lattice X to a Banach space Y, we have the following:

Proposition 5. *The class $D.S.(X, Y)$ is closed in $L(X, Y)$.*

Proof. Reasoning in a standard form, suppose that $(T_n) \subset D.S.$ converges to T and T is not disjointly strictly-singular. So there exists a disjoint vector sequence (x_n) in X and a constant $K > 0$ such that $\|T(x)\| \geqq K\|x\|$ for every $x \in [x_n]$. Now, there exists $n_0 \in \mathbb{N}$ such that $\|T_n - T\| \leqq K/2$ for $n \geqq n_0$. Hence

$$\|T_{n_0}(x)\| \geqq \|T(x)\| - \|(T - T_{n_0})(x)\| \geqq \frac{K}{2} \|x\|$$

for $x \in [x_n]$, so T_{n_0} is not disjointly strictly singular, which is a contradiction. q.e.d.

We pass now to study disjointly strictly singular operators in the context of Orlicz functions spaces. If α_F^∞, β_F^∞ denote the associated indices to an Orlicz function space $L^F(0, 1)$, we have the following result given in ([3] Proposition 3):

Proposition 6. *If T is a Riesz operator $T: L^F(0, 1) \to L^G(0, 1)$ and $[\alpha_F^\infty, \beta_F^\infty] \cap \cap [\alpha_G^\infty, \beta_G^\infty] = \emptyset$, then T is disjointly strictly singular.*

In the special case of the operator T be the inclusion operator $L^F(0, 1) \hookrightarrow L^G(0, 1)$, a characterization of the disjointly strict singularity was obtained in ([2], pp. 51).

Proposition 7. *Suppose $L^F(0, 1) \hookrightarrow L^G(0, 1)$. The following conditions are equivalent:*

(1) *The inclusion operator $T: L^F(0, 1) \hookrightarrow L^G(0, 1)$ is disjointly strictly singular.*

(2) *For any $K > 0$, there exist $y_1 < y_2 < \ldots < y_n < 1$ and $c_1, \ldots, c_n > 0$ such that*

$$\sum_{i=1}^{n} c_i F(ty_i) \geqq K \sum_{i=1}^{n} c_i G(ty_i) \quad (t \geqq 1) .$$

(3) *For any $K > 0$ there exist $1 \leqq x_1 < x_2 < \ldots < x_n$ and $c_1, \ldots, c_n > 0$ such that*

$$\sum_{i=1}^{n} c_i F(tx_i) \geqq K \sum_{i=1}^{n} c_i G(tx_i) \quad (t \geqq 1)$$

(4) *For any $K > 0$ there exists $a > 1$ and a positive Borel measure μ with support contained in $[1, a]$ such that*

$$\int F(tx)\, \mathrm{d}\mu(x) \geqq K \int G(tx)\, \mathrm{d}\mu(x) \quad (t \geqq 1) .$$

We present now a suitable analytic criterium for the inclusion operator $L^p(0,1) \hookrightarrow L^F(0,1)$ be disjointly strictly singular:

Proposition 8. *The inclusion operator* $L^p(0,1) \hookrightarrow L^F(0,1)$ *is disjointly strictly-singular if and only if*

$$\limsup_{a\to\infty} \sup_{s\geqq 1} \frac{1}{\log a} \int_1^a \frac{F(su)}{s^p u^{p+1}}\, du = 0 . \tag{$*$}$$

Proof. Suppose that $L^p(0,1) \hookrightarrow L^F(0,1)$ is disjointly strictly singular. Then, using the above Proposition 7(2), for any constant $K > 0$, there exist $y_1 < y_2 < \ldots < y_n \leqq \leqq 1$ and $c_1, c_2, \ldots, c_n > 0$ such that

$$\sum_{i=1}^n c_i (sty_i)^p \geqq K \sum_{i=1}^n c_i F(sty_i)^p \quad (s, t \geqq 1) .$$

For $a \geqq 1/y_1$,

$$\int_1^{a^2} \sum_{i=1}^n c_i \frac{(sty_i)^p}{t^{p+1}}\, dt \geqq K \sum_{i=1}^n \int_1^{a^2} c_i \frac{F(sty_i)}{t^{p+1}}\, dt . \tag{$+$}$$

Now

$$\sum_{i=1}^n \int_1^{a^2} c_i \frac{(sty_i)^p}{t^{p+1}}\, dt = \left(\sum_{i=1}^n c_i y_i^p\right) s^p\, 2 \log a$$

and

$$\sum_{i=1}^n c_i \int_1^{a^2} \frac{F(sty_i)}{t^{p+1}}\, dt = \sum_{i=1}^n c_i y_i^p \int_{y_i}^{a^2 y_i} \frac{F(su)}{u^{p+1}}\, du \geqq \left(\sum_{i=1}^n c_i y_i^p\right)\left(\int_1^a \frac{F(su)}{u^{p+1}}\, du\right).$$

Then, from $(+)$ we get

$$\frac{1}{s^p \log a} \int_1^a \frac{F(su)}{u^{p+1}}\, du \leqq \frac{2}{K}$$

for $s \geqq 1$ and $a > 1/y_1$.

Assume now that $(*)$ holds. Then, for any $K > 0$ there exists $a > 1$ such that, (for $s \geqq 1$)

$$\int_1^a \frac{F(su)}{u^{p+1}}\, du \leqq K s^p \log a = K \int_1^a \frac{(su)^p}{u^{p+1}}\, du .$$

Then, by Proposition 7.(4), we get that T is disjointly strictly singular. q.e.d.

Example. If F_p denotes the Orlicz function $x^p/\log(1 + x)$, for $p > 1$, then the inclusion operator $L^p(0,1) \hookrightarrow L^{F_p}(0,1)$ is a disjointly strictly-singular operator, since the condition $(*)$ is verified. (Notice that the indices $\alpha_F^\infty = \beta_F^\infty = p$, hence the converse of Proposition 6 does not hold).

Finally, let us mention that a similar characterization of when the inclusion $L^F(0,1) \hookrightarrow L^p(0,1)$ is a disjointly strictly-singular operator has been obtained in ([2] Proposition 3.3), which is used to find Orlicz spaces $L^F(0,1)$ containing "singular"

l^p-complemented copies for $p > 1$, that is, $L^F(0, 1)$ has a l^p-complemented subspace and it does not exist any sequence of mutually disjoint characteristic functions (χ_{A_n}) spanning an l^p-subspace.

References

[1] Drewnowski, On minimal topological linear spaces and strictly singular operators. Comm. Math. Tom. spec. in hom. L. Orlicz II (1979), 89—106.

[2] Hernández F., Rodriguez-Salinas B., On l^p-complemented copies in Orlicz spaces II. Israel J. of Math. 68 (1989), 27—55.

[3] Hernández F., Rodrigues-Salinas B., Orlicz spaces containing singular l^p-complemented copies. Proc. II Conf. Function Spaces (Poznań) 1989 to appear in Teubner-Texte for Math.

[4] Kalton N., Compact and strictly singular operators on Orlicz spaces. Israel J. Math. 26 (1977), 126—136.

[5] Kalton N., Orlicz sequence spaces without local convexity. Math. Proc. Cambridge. Phil. Soc. 81 (1977), 253—277.

[6] Lindenstrauss J. and Tzafriri L., Classical Banach spaces I. Springer-Verlag 1977.

[7] Pietsch A., Operator ideals. VEB Berlin. 1978.

[8] Przeworska-Rolewicz D., Rolewicz S., Equations in linear spaces. P.W.N.Warsaw, 1968.

[9] Zaanen A., Riesz spaces II. North-Holland (1983).