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Integral Extension with Locally-Integral seminorm

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1. For a ring Ω of sets from an arbitrary set X and a finitely additive measure μ *on* Ω, the space of Riemann-μ-integrable-functions $R_1(\mu, \bar{R})$ were presented essentially by Loomis [8], by Dunford-Schwartz [4], and more generally by Günzler [6].

We present some results concerning analogue to the Daniell extension process, starting in this case with an arbitrary nonnegative linear functional on function vector lattices, without continuity conditions on the elementary integral.

Generalizing the definition of Aumann [1], a $q: \bar{R}^X_+ \to \bar{R}_+$ will be called an *integral seminorm* on X iff for $f, g \in \bar{R}^X_+$, $q(f + g) \leqq q(f) + q(g)$, $q(f) \leqq q(g)$ if $f \leqq g$ and $q(\alpha f) = \alpha\, q(f)$ if $\alpha \in R^+$.

In of all the following we assume a nonempty set X, B a vector lattice of real-valued functions on X, under the pointwise operations and relations $+, \alpha, =, \leqq, \vee, \wedge, |\,|$; and I a nonnegative linear functional on B, i.e. $I(f) \geqq 0$ if $0 \leqq f \in B$.

In [2] there has been generalized the process of a Daniell-Bourbaki integral (one may consult Pfeffer [9] as a specific reference). The following definitions and results of [2] are used: A preliminary extension is defined by

$$B^+ := \{f \in \bar{R}^X;\ f = \sup g,\ g \in B,\ g \leqq f\} =$$

$= \{f \in]-\infty, \infty]^X$; to each $x \in X$ there exist $h_n \in B$ with $h_n \leqq f$ and $h_n(x) \to f(x)\}$.

For any $f \in \bar{R}^X$, $I^+(f) := \sup\{I(g);\ g \leqq f,\ g \in B\}$, with $\sup \Theta = -\infty$; $B^- := -B^+$ and

$$I^-(f) := -I^+(-f) = \inf\{I(g);\ g \in B,\ f \leqq g\}.$$

Since I^+ is not additive on B^+ it is introduced the class $B_+ := \{f \in B^+; I^+(f + g) = I^+(f) + I^+(g)$ for all $g \in B^+\}$ and $B_- := -B_+$.

Now, using the class B_+ and B_-, for each $f \in \bar{R}^X$ the upper and lower integrals $\bar{I}$ and $\underline{I}$ are defined as usual:

$\bar{I}(f) := \inf\{I^+(g);\ g \in B_+,\ g \leqq f\}$, with $\inf \Theta = \infty$ and $\underline{I}(f) := -\bar{I}(-f)$.

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For any $f \in \bar{R}^X$ we have the relation

$$I^+(f) \leqq \mathbf{I}(f) \leqq \bar{I}(f) \leqq I^-(f).$$

and $\mathbf{I}$ and I^- both are integral seminorms on X. The elements of $R_{prop}(B, I) := := \{f \in R^X;\ \forall \varepsilon > 0,\ \exists h,\ g \in B,\ h \leqq f \leqq g \text{ and } I(|g - h|) < \varepsilon\}$ in [8] are called *proper-Riemann-integrable.*

$R_{prop}(B, I)$ is the closure of B with respect to the integral-seminorm I^-.

The class of I-summable functions is defined in [2] by

$$\bar{B} := \{f \in \bar{R}^X;\ \mathbf{I}(f) = \bar{I}(f) \in R\}.$$

$\bar{B}$ is a vector lattice of extended real-valued functions, with the usual convention, and $I := \bar{I} = \mathbf{I}$ is linear on $\bar{B}$. $R_{prop}(B, I) \subsetneqq \bar{B}$. $I/\bar{B}$ is the maximal extension of I/B in sense of Aumann [1] with respect to the integral-seminorm $\|\cdot\|_I := \bar{I}(|\cdot|)$, and B is $\|\ \|_I$ – dense in $\bar{B}$, i.e.:

$$f \in \bar{B} \text{ iff for any } \varepsilon > 0 \text{ there exists } g \in B \text{ such that } \bar{I}(|f - g|) < \varepsilon.$$

2. Note that to get convergence theorems in the finitely additive case, everywhere convergence or a.e. convergence is not sufficient (see, for example, Dunford-Schwartz [4]), thus we use the following appropriate local "convergence in measure" in our integral extension.

Definition 1 (T-convergence). For any $T\colon \bar{R}^X_+ \to \bar{R}$, $(f_n)_n$, $f \in \bar{R}^X$, $f_n \to f(T)$ means that for each fixed $h \in B$, with $h \geqq 0$, one has $T(|f_n - f| \wedge h) \to 0$, (where $\infty - \infty = 0$).

Recently, Günzler [7] has obtained Lebesgue's and Monotone convergence theorems for $\bar{B}$ and $\bar{I}$, using this local convergence in measure with $T = \bar{I}$.

Now, with $T = I^-$ in definition 1, we introduce the following class of *I-integrable functions*:

$$R_1(B, I) := \{f \in \bar{R}^X;\ \exists (h_n)_n \subset B,\ (h_n)_n \to f(I^-),\ I(|h_n - h_m|) \to 0 \text{ as } n, m \to \infty\}.$$

The map $I\colon R_1(B, I) \to R$, with $I(f) := \lim I(h_n)$ as $n \to \infty$, for such $(h_n)_n$, is well defined. $R_1(B, I)$ is a vector lattice of extended real-valued functions, and I is linear on it. Also, B is dense in $R_1(B, I)$ with respect to the seminorm $\|f\|_I := I(|f|)$ $(= I^+(|f|) = \mathbf{I}(|f|)$ for all $f \in R_1(B, I)$.

For any $f \in \bar{R}^X_+$ the corresponding local integral-seminorm in sense of Schäfke ([10]) is defined by

$$I_l^-(f) := \sup\{I^-(f \wedge h);\ 0 \leqq h \in B\}.$$

I_l^- is again an integral-seminorm, such that $I_l^-(f) = I(f)$ for each $f \in R_1(B, I)$, and $R_1(B, I)$ is the closure of B in $\bar{R}^X$ with respect to distance $d(f, g) := I_l^-(|f - g|)$.

In general $R_{prop}(B, I) \subsetneqq R_1(B, I)$, and

$$f \in R_{prop}(B, I) \quad \text{iff} \quad f \in R_1(B, I) \quad \text{and} \quad |f| \leqq h \in B$$

$R_1(B, I)$ is closed with respect to improper integration; and the extension process $B \to R_1(B, I)$ is iteration complete, i.e.:

$$R_1(B, I) = R_1(\tilde{B}, \tilde{I}) \quad \text{with} \quad \tilde{B} := R_1(B, I) \cap R^X, \quad I := I/\tilde{B}.$$

With the localized convergence in measure of definition 1, we obtain the usual convergence theorems for this integral (Lebesgue's bounded and monotone convergence theorems), and since $R_1(B, I) \cap B^+ \subset \bar{B}$, we generalize theorem p. 262 of [3]:

$$R_1(B, I) \subset \bar{B} + N_1(B, I)$$

where $N_1(B, I) := \{f \in R_1(B, I);\ I(|f|) = 0\}$.

All the results become true for a suitable extension of $\bar{B}$, with the localized integral-seminorm $T = \bar{I}$, which will be treated in a paper of Günzler and Díaz Carrillo.

3. We assume the space of Riemann-μ-integrable functions $R_1(\mu, \bar{R})$. If Ω is an algebra and $\mu(X) < +\infty$, $R_1(\mu, \bar{R}) = L(S, \mu)$ (= μ-integrable functions defined by Dunford-Schwartz), and for σ-ring Ω and $\mu\colon \Omega \to [0, \infty[$ σ-additive, one has $R_1(\mu, R) = L^1(\mu, R)$ (= Lebesgue-μ-integrable functions).

Now, if B_Ω denotes the set of all step-functions $S(\Omega, R)$, and $I_\mu(f) = \sum_{i=1}^{n} a_i \, \mu(A_i)$ with $n \in N$, $a_i \in R$ and $A_i \in \Omega$, for all $f \in B_\Omega$; with I_μ/B_Ω we obtain the class $R_1(B_\Omega, I_\mu)$ of all I_μ-integrable functions.

Since for any $(f_n)_n, f \in \bar{R}^X$, we have $(f_n)_n \to f(\mu)$ (μ-local-convergence of Günzler, or as special case the convergence in μ-measure of Dunford-Schwartz), iff $(f_n)_n \to$ $\to f(I_\mu^-)$, one gets that $R_1(B_\Omega, I_\mu) = R_1(\mu, \bar{R})$, and our results subsume the corresponding results for $R_1(\mu, \bar{R})$.

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