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Strict Differentiability via Differentiability

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Let $(X, |\cdot|)$, $(Y, \|\cdot\|)$ be real normed linear spaces. A mapping $F: X \to Y$ is said to be strictly differentiable at $a \in X$ if there exists a continuous linear operator $A: X \to Y$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|F(y) - F(x) - A(y - x)\| \leqq \varepsilon|y - x| \tag{1}$$

whenever $|x - a| < \delta$ and $|y - a| < \delta$. In this case the operator A is called a strict derivative of F at a. Of course, A is a Frechet derivative of F at a.

The natural and useful notion of a strict derivative is very old and well-known (cf. e.g. [5] for $F: R \to R$, [1], [2], [4]).

It is well-known (see [3] or [6], p. 138) that for a continuous function $F: R \to R$ the set of points at which F is differentiable and is not strictly differentiable is of the first category.

The aim of the present note is to prove that this assertion holds for quite arbitrary possibly discontinuous mappings $F: X \to Y$.

We shall need the following essentially well-known lemma.

Lemma. Let $(X, |\cdot|)$, $(Y, \|\cdot\|)$ be real normed linear spaces and $F: X \to Y$ a mapping. Suppose that $A: X \to Y$ is a linear mapping, $c \in X$, $\varepsilon > 0$, $\delta > 0$ such that $\|F(c + h) - F(c) - A(h)\| < \varepsilon|h|$ whenever $|h| < \delta$. Then the inequalities $|x - c| < \delta$, $|y - c| < \delta$ and $|x - y| \geqq |x - c|$ imply the inequality

$$\|F(y) - F(x) - A(y - x)\| < 3\varepsilon|y - x| .$$

Proof. By the assumptions we have $\|F(x) - F(c) - A(x - c)\| < \varepsilon|x - c|$ and $\|F(y) - F(c) - A(y - c)\| < \varepsilon|y - c|$. Consequently

$$\|F(y) - F(x) - A(y - x)\| < \varepsilon(|x - c| + |y - c|) \leqq \varepsilon(|x - c| + |x - c| + |y - x|) \leqq 3\varepsilon|y - x|$$

The open ball with center $x \in X$ and radius $r > 0$ will be denoted by $U(x, r)$. Further observe that the inequality (1) from the definition of strict differentiability

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is equivalent to

$$\left\| \frac{F(y) - F(x)}{|y - x|} - A\left(\frac{y - x}{|y - x|}\right) \right\| \leqq \varepsilon . \tag{1)'$$

Theorem. Let $(X, |\cdot|)$, $(Y, \|\cdot\|)$ be real normed linear spaces and $F: X \to Y$ be a mapping. Then the set V of points $x \in X$ at which F is Frechet differentiable but is not strictly differentiable is of the first category.

Proof. Denote by $V_{n,p}$ the set of all points $a \in V$ for which

(2) $\|F(a + h) - F(a) - (F'(a))(h)\| < |h|/p$ whenever $|h| \leqq 1/n$ and

(3) for any $\delta > 0$ there exist points $x, y \in U(a, \delta)$ such that

$$\|F(y) - F(x) - (F'(a))(y - x)\| > (8/p)|y - x|.$$

It is easy to see that $V = \bigcup_{n,p=1}^{\infty} V_{n,p}$. Thus it is sufficient to prove that all sets $V_{n,p}$ are nowhere dense. Suppose on the contrary that for some fixed n, p the set $V_{n,p}$ is dense in a ball $U(a, \varrho)$, $a \in V_{n,p}$. Put $\delta = \min(\varrho/4, 1/(8n))$. By (3) we can find points $x, y \in U(a, \delta)$ such that

$$\|F(y) - F(x) - (F'(a))(y - x)\| > (8/p)|y - x| . \tag{4}$$

Since $|a - x| < \varrho/4$ and $|y - x| < \varrho/2$ we obtain $U(x, |y - x|) \subset U(a, \varrho)$; consequently we can choose a point $\tilde{a} \in U(x, |y - x|) \cap V_{n,p}$. Since $|y - x| < 1/(4n)$ and $|\tilde{a} - x| < |y - x| < 1/(4n)$, we have $|y - \tilde{a}| < 1/(2n)$. Clearly $|x - y| \geqq |x - \tilde{a}|$. On account of (2) we see that the assumptions of Lemma are satisfied for $c = \tilde{a}$, $\varepsilon = 1/p$, $\delta = 1/n$ and $A - F'(\tilde{a})$. Consequently Lemma implies

$$\|F(y) - F(x) - F'(\tilde{a})(y - x)\| < (3/p)|y - x| . \tag{5}$$

Put $v = (y - x)/\|y - x\|$ and $b = a + v/(2n)$. By (2) we obtain

$$\|F(b) - F(a) - F'(a)(b - a)\| < (1/p)|b - a| . \tag{6}$$

Clearly $|\tilde{a} - a| \leqq |\tilde{a} - x| + |x - a| < 1/(4n) + 1/(8n) = 3/(8n)$ and $|\tilde{a} - b| \leqq$ $\leqq |\tilde{a} - a| + |a - b| < 3/(8n) + 1/(2n) < 1/n$. Further $|a - b| = 1/(2n) >$ $> 3/(8n) > |\tilde{a} - a|$. Since $\tilde{a} \in V_{n,p}$ we obtain by (2) and the above inequalities that the assumptions of Lemma are satisfied for $c = \tilde{a}$, $\varepsilon - 1/p$, $\delta = 1/n$, $x = a$, $y = b$ and $A = F'(\tilde{a})$. Consequently Lemma implies

$$\|F(b) - F(a) - F'(\tilde{a})(b - a)\| < (3/p)|b - a| . \tag{7}$$

The inequalities (4) and (5) clearly imply

$$\|F'(a)(y - x) - F'(\tilde{a})(y - x)\| > (5/p)|y - x| . \tag{8}$$

On account of (6) and (7) we obtain

$$\|F'(a)(b - a) - F'(\tilde{a})(b - a)\| < (4/p)\,|b - a|\,. \tag{9}$$

Now (8) implies $\|F'(a)(v) - F'(\tilde{a})(v)\| > 5/p$ and (9) implies $\|F'(a)(v) - F'(\tilde{a})(v)\| < 4/p$. This is a contradiction which completes the proof.

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