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Cartesian Subdirect Irreducibility in Graphs

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There is characterized subdirect irreducibility for cartesian products of graphs.

V práci je charakterizována subdirektní ireducibilita pro kartézské součiny grafů.

В работе характеризуется подпрямая неприводимость для декартовых произведений графов.

Introduction

Throughout this paper, the topic "graph" is used for undirected graphs without loops and multiple edges. The concept of subdirect irreducibility was introduced for algebras by G. Birkhoff in [1] and extended for concrete categories by A. Pultr and the author of this paper in [2]. Roughly speaking, the motivation for studying subdirect irreducibility is to construct general objects from simple ones using products and subobjects. For the case of graphs, it is useful to consider induced subgraphs as subobjects in order not to lose some good properties of graphs. Categorical theorems from [2] can be applied to the case of categorical (direct) products of graphs. Subdirect irreducibility with respect to categorical products is studied e.g. in [4] and [5]. Categorical products have many advantages but also some disadvantages – e.g. they do not save connectivity of graphs, they are not good for constructions of cubes etc. This is the reason why also subdirect irreducibility with respect to cartesian products became been studied. In [3] some examples of subdirectly irreducible and subdirectly reducible graphs with respect to cartesian products were given but the full characterization was open. In the present note there is given the asked characterization.

1. Conventions and notations

Given a graph G, we denote $V(G)$ its set of vertices and $E(G)$ its set of edges. In the case of an indexed family of graphs $\{G_i;\ i \in I\}$ we shall put $V(G_i) = V_i$, $E(G_i) = E_i$.

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A *cartesian product* $C = \square_I G_i$ is a graph defined by: $V(C) = \prod_I V_i$, $E(C) = \{\{(x_i)_I, (y_i)_I\};\ (\exists j \in I)\ (\{x_j, y_j\} \in V_j)\ \&\ (\forall i \in I)\ (i \neq j \Rightarrow x_i = y_i)\}$. The i-th projection p_i: $V(C) \to V_i$ is a mapping defined by $p_i(x_1, \ldots, x_n) = x_i$.

2. Cartesian subdirect irreducibility

Let us recall the following:

2.1. Definition. A graph is cartesian subdirectly irreducible (abbreviated CSI) if, whenever G is embedded as an induced subgraph (with an embedding m) into a cartesian product $\square G_i$ of graphs G_i $(i \in I)$ such that all $p_i m$ are mappings onto, then at least one $p_i m$ is an isomorphism of graphs.

2.2. Remarks. We shall usually omit a notation of the embedding and write v instead of $m(v)$ etc.

In [3] there was proved that any complete graph is CSI. But there are also incomplete CSI graphs (see [3]).

3. Basic equivalences

We are going to characterize cartesian subdirect irreducibility via equivalences on $E(G) = E$.

3.1. For any edge $e \in E$ define a relation R_e on $V(G) \times V(G)$ as the smallest equivalence satisfying the following conditions:

(i) if $\{b, c\}, \{a, c\} \in E$ and $(e, \{a, b\}) \in R_e$ then $(\{a, b\}, \{a, c\}) \in R_e$, $(\{a, b\}, \{b, c\}) \in R_e$;

(ii) if a, b, c, a', b', c' are 6 distinct vertices of G such that $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a', b'\}$, $\{b', c'\}$, $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\} \in E$, $\{a', c'\} \notin E$, $(e, \{a, b\}) \in R_e$ then $(e, \{x, y\}) \in R_e$ for any $x, y \in \{a, b, c, a', b', c'\}$, $x \neq y$.

3.2. Definition. For any edge $e \in E$ define a relation $B_e \supseteq R_e$ on $V(G) \times V(G)$ as the smallest equivalence satisfying the following conditions:

(a) if a, b, c are 3 distinct vertices of G, $e, f, \{b, c\} \in E$, $(e, \{a, b\}) \in R_e$, $(f, \{a, c\}) \in R_f$, then $B_e \supseteq R_e \cup R_f$;

(b) if a, b, c, d are 4 distinct vertices of G, $e = \{a, b\}$, $f, g, h \in E$, $\{c, d\} \notin E$ and $(f, \{b, c\}) \in R_f$, $(g, \{c, d\}) \in R_g$, $(h, \{a, d\}) \in R_h$, then $B_e \supseteq R_e \cup R_f \cup R_g \cup R_h$;

(c) if a, b, c, d, d' are 5 distinct vertices of G, $e, f, g, h, \{c, d\}, \{c, d'\} \in E$, $(e, \{a, b\}) \in R_e$, $(f, \{b, c\}) \in R_f$, $(g, \{a, d\}) \in R_g$, $(h, \{a, d'\}) \in R_h$ then $B_e \supseteq R_e \cup R_f \cup R_g \cup R_h \cup R_{\{c,d\}} \cup R_{\{c,d'\}}$.

The equivalence B_e defined above will be called a basic equivalence (generated by e).

Before proving the characterization theorem we are going to prove some lemmas:

3.3 Lemma. Let $G=(V, E)$ be an induced subgraph of $\square G_i, \{x, y\}, \{y, u\}, \{x, u\} \in E$ Then there exists $j \in I$ such that $p_j/\{x, y, u\}$ is one-to-one and $p_i(x) = p_i(y) = = p_i(u)$ for any $i \in I, i \neq j$.

Proof. There exist $j, k \in I$ such that $p_j(x) \neq p_j(y)$ and $p_i(x) = p_i(y)$ for any $i \in I, i \neq j, p_k(x) \neq p_k(u)$ and $p_i(x) = p_i(u)$ for any $i \in I, i \neq k$. If $j \neq k$ then $p_j(y) \neq p_j(u), p_k(y) \neq p_k(u)$ which contradicts the assumption $\{y, u\} \in E$. Hence, $j = k$, q.e.d.

3.4. Lemma. Let $G = (V, E)$ be an induced subgraph of $\square G_i$, $\{x, y\}$, $\{x, v\}$, $\{y, v\}$, $\{x, z\}$, $\{y, w\}$, $\{v, u\}$, $\{z, w\}$, $\{w, u\} \in E$, $\{z, u\} \notin E$. Then there exists $j \in I$ such that $p_j/\{x, y, z, u, v, w\}$ is one-to-one and $p_i(x) = p_i(y) = p_i(z) = p_i(u) = = p_i(v) = p_i(w)$ for any $i \in I, i \neq j$.

Proof. By 3.3, one has for any $i \in I, i \neq j$: $p_i(x) = p_i(y) = p_i(v)$. Since $\{x, z\} \in E$ there exists $k \in I$ such that $p_k(x) \neq p_k(z)$. Then $p_i(x) = p_i(z)$ for any $i \in I, i \neq k$. Similarly there exists $n \in I$ such that $p_n(y) \neq p_n(w)$, $p_i(y) = p_i(w)$ for any $i \in I$, $i \neq n$.

If $k \neq n \neq j \neq k$ then $p_k(z) \neq p_k(x) = p_k(y) = p_k(w)$, $p_n(z) = p_n(x) = p_n(y) \neq p_n(w)$ which contradicts $\{z, w\} \in E$.

If $k \neq n = j$ then again $p_k(z) \neq p_k(x) = p_k(y) = p_k(w)$; since $p_i(x) = p_i(y) = = p_i(w)$ for any $i \neq j$ there is $p_j(x) \neq p_j(w)$ and $p_j(z) = p_j(x) \neq p_j(w)$ which contradicts $\{z, w\} \in E$.

If $k = j \neq n$ then $p_n(y) = p_n(x) = p_n(z)$; since $p_i(y) = p_i(z)$ for any $i \neq j$, there is $p_j(y) \neq p_j(z)$ and $p_j(w) = p_j(y) \neq p_j(z)$ which contradicts $\{z, w\} \in E$.

If $k = n \neq j$ then one can prove by a similar technique that $p_k(u) \neq p_k(v)$. Since $(x, z\}$, $\{u, v\} \in E$ we have $p_i(x) = p_i(z)$, $p_i(u) = p_i(v)$ for any $i \neq k$. Hence, $\{p_j(z), p_j(u)\} \in E_j$, $p_i(z) = p_i(u)$ for any $i \neq j, k$ and $p_k(z) = p_k(w) = p_k(u)$ as well. Therefore, $\{z, u\} \in E$ which is a contradiction.

The results above imply that $k = n = j$ and $p_j/\{x, y, z, u, v, w\}$ is one-to-one, q.e.d.

3.5. Lemma. Let $G = (V, E)$ be an induced subgraph of $\square G_i, e, f, g \in E, (e, \{x, y\}) \in \in R_e$, $(f, \{y, z\}) \in R_f$, $(g, \{x, z\}) \in R_g$. Then there exists $j \in I$ such that $p_j/\{x, y, z\}$ is one-to-one and $p_i(x) = p_i(y) = p_i(z)$ for any $i \in I, i \neq j$.

Proof. Lemmas 3.3 and 3.4 imply that there exist j, k, m such that $p_j(x) \neq p_j(y)$, $p_k(x) \neq p_k(z), p_m(y) \neq p_m(z), p_i(x) = p_i(y)$ for any $i \neq j$, $p_i(x) = p_i(z)$ for any $i \neq k$, $p_i(y) = p_i(z)$ for any $i \neq m$. If $j \neq m$ then $p_j(x) \neq p_j(y) = p_j(z)$ and $j = k$. Hence, $p_i(y) = p_i(x) = p_i(z)$ for any $i \neq j$, and $j = m$. It is a contradiction. Similarly, the assumptions $j \neq k$ and $k \neq m$ imply a contradiction, too. Thus, $j = k = m$ q.e.d.

3.6. Lemma. Let $G = (V, E)$ be an induced subgraph of $\square G_i$, $e, f, g, h \in E$, $(e, \{a, b\}) \in R_e$, $(f, \{a, c\}) \in R_f$, $(g, \{b, d\}) \in R_g$, $(h, \{c, d\}) \in R_h$. Then there exists

$j \in I$ such that $p_j(a) \neq p_j(b)$, $p_j(c) \neq p_j(d)$ and $p_i(a) = p_i(b)$, $p_i(c) = p_i(d)$ for any $i \in I$, $i \neq j$.

Proof. Lemmas 3.3. and 3.4 imply that there exist $j, k, m, n \in I$ such that $p_j(a) \neq p_j(b)$, $p_k(a) \neq p_k(c)$, $p_m(b) \neq p_m(d)$, $p_n(c) \neq p_n(d)$, $p_i(a) = p_i(b)$ for any $i \neq j$, $p_i(a) = p_i(c)$ for any $i \neq k$, $p_i(b) = p_i(d)$ for any $i \neq m$, $p_i(c) = p_i(d)$ for any $i \neq n$.

If $j = k \neq m$ then $p_m(c) = p_m(a) = p_m(b) \neq p_m(d)$. Hence, $m = n$ and $p_j(c) = p_j(d) = p_j(b)$ which contradicts $j = k$.

Similarly, $j = m \neq k$ implies a contradiction, too.

If $k \neq j \neq m$ then $p_j(c) = p_j(a) \neq p_j(b) = p_j(d)$; hence, $j = n$, q.e.d.

3.7. Lemma. Let $G = (V, E)$ be an induced subgraph of $\square\, G_i$, a, b, c, d be 4 distinct vertices of G, $e = \{a, b\}, f, g, h \in E$, $\{c, d\} \notin E$, $(f, \{a, c\}) \in R_f$, $(g, \{c, d\}) \in R_g$, $(g, \{c, d\}) \in R_g$, $(h, \{b, d\}) \in R_h$. Then there exists $j \in I$ such that $p_j/\{a, b, c, d\}$ is one-to one and $p_i(a) = p_i(b) = p_i(c) = p_i(d)$ for any $i \in I$, $i \neq j$.

Proof. Lemma 3.6 implies that there are $j, k \in I$ such that $p_j(a) \neq p_j(b)$, $p_j(c) \neq p_j(d)$, $p_i(a) = p_i(b)$, $p_i(c) = p_i(d)$ for any $i \neq j$, $p_k(a) \neq p_k(c)$, $p_i(a) = p_i(c)$, $p_i(b) = p_i(d)$ for any $i \neq k$, and $p_k(b) \neq p_k(d)$, If $j \neq k$ then $\{p_j(c), p_j(d)\} = \{p_j(a), p_j(b)\} \in E_j$ and $\{c, d\} \in E$ which is a contradiction. Hence $j = k$, q.e.d.

3.8. Lemma. Let $G = (V, E)$ be an induced subgraph of $\square\, G_i$, a, b, c, d, d' be 5 distinct vertices of G, e, f, g, h, $\{c, d\}$, $\{c, d'\} \in E$, $(e, \{a, b\}) \in R_e$, $(f, \{b, c\}) \in R_f$, $(g, \{a, d\}) \in R_g$, $(h, \{a, d'\}) \in R_h$. Then there exists $j \in I$ such that $p_j/\{a, b, c, d, d'\}$ is one-to-one and $p_i(a) = p_i(b) = p_i(c) = p_i(d) = p_i(d')$ for any $i \neq j$.

Proof. Lemma 3.6 implies that there are $j, k \in I$ such that $p_j(a) \neq p_j(b)$, $p_j(c) \neq p_j(d)$, $p_j(c) \neq p_j(d')$, $p_i(a) = p_i(b)$, $p_i(c) = p_i(d) = p_i(d')$ for any $i \neq j$, $p_k(b) \neq p_k(c)$, $p_k(a) \neq p_k(d)$, $p_k(a) \neq p_k(d')$, $p_i(b) = p_i(c)$, $p_i(a) = p_i(d) = p_i(d')$ for any $i \neq k$. If $j \neq k$ then $p_i(d) = p_i(d')$ for any $i \in I$ which contradicts $d \neq d'$. Hence, $j = k$, q.e.d.

3.9. Proposition. Let $G = (V, E)$ be an induced subgraph of $\square\, G_i$, $e = \{x, y\} \in E$, $f = \{z, u\} \in V \times V$ and $(e, f) \in B_e$. Then there exists $j \in I$ such that $p_j/\{x, y, z, u\}$ is one-to-one and $p_i(x) = p_i(y) = p_i(z) = p_i(u)$ for any $i \in I$, $i \neq j$.

Proof. Follows from 3.2, 3.5, 3.7 and 3.8.

3.10. Proposition. If a connected graph $G = (V, E)$ is CSI then it has just one basic equivalence.

Proof. Suppose that there are at least two basic equivalences on $V \times V$. Put $(B_e, B_f) \in Q$ if there exist $\{a, b\}, \{u, v\} \in E$ such that $(e, \{a, b\}) \in B_e$, $(f, \{u, v\}) \in B_f$,

$\{a, u\}$, $\{b, v\} \in E$, and define $\sim$ as the smallest equivalence generated by Q on the set of all basic equivalences. According to 3.2(a) whenever there is an $x \in V$ such that $(e, \{x, y\}) \in B_e$, $(f, \{x, z\}) \in B_f$ for some $y, z \in V$ and $B_e \sim B_f$ then $B_e = B_f$. Thus, connectivity of G implies that there are at least two different classes of basic equivalences.

For any equivalence class $C = [e]$ of basic equivalences define an equivalence U_C on V putting $(x, y) \in U_C$ whenever there is $f \in E$ such that $(\{x, y\}, f) \in B_f$ and $[f] \neq [e]$. Put $V_C = V/U_C$, $E_C = \{\{\alpha, \beta\} \in V_C \times V_C;\ \alpha \neq \beta,\ \exists a \in \alpha,\ b \in \beta,\ \{a, b\} \in E\}$, $G_C = (V_C, E_C)$.

We are going to prove that G is an induced subgraph of $\square\, G_C$. For any class C of basic equivalences and $x \in V$ put $x_C = \{x' \in V;\ (x, x') \in U_C\}$. If $\{x, y\} = e \in E$ then $\{x_{[e]}, y_{[e]}\} \in E_{[e]}$ and $x_C = y_C$ for any $C \neq [e]$. Hence, $\{x, y\} \in \square\, E_C$. If $\{x, y\} \notin E$ but $(\{x, y\}, e) \in B_e$ then $x_C = y_C$ for any $C \neq [e]$ and $x_{[e]} \neq y_{[e]}$. By 3.2(b), there is $\{x_{[e]}, y_{[e]}\} \notin E_{[e]}$. Hence, $\{x, y\} \notin \square\, E_C$. In the case $(\{x, y\}, e) \notin B_e$ for any $e \in E$ the connectivity of G implies that there is $f = \{x, z\} \in E$; then, by 3.2(c) $x_{[f]} \neq y_{[f]}$ and hence $\{x, y\} \notin \square\, E_C$.

Thus, G is not CSI, q.e.d.

3.11. Proposition. Any CSI graph is connected.

Proof. Suppose $G = (V, E)$, $V = V_1 \cup V_2$, $E \cap (V_1 \times V_2) = \emptyset$, $a \in V_1$, $b \in V_2$. Put $G' = (V', E')$ with $V' = V - \{b\}$, $E' = E \cap (V' x V')) \cup \{\{a, x\}; \{b, x\} \in E\}$, $G'' = \{0, 1\}$, $E'' = \emptyset$. For any $v \in V_1$ put $m(v) = (v, 0)$ and for any $v \in V_2 - \{b\}$ put $m(v) = (v, 1)$; further, put $m(b) = (a, 1)$. Clearly, m is an embedding of G into $G' \,\square\, G''$. Hence, G is not CSI, q.e.d.

3.12. Proposition. If a connected graph $G = (V, E)$ is not CSI, then it has at least two different basic equivalences.

Proof. Suppose that G is an induced subgraph of $G_1 \,\square\, G_2$ $(G_1 \not\geq G \not\leq G_2)$ and that there is only basic equivalence on G. For any edge $e = \{x, y\}$ and an edge $f = \{z, u\}$ there exists – by 3.9 – $j \in \{1, 2\}$ such that $p_j/\{x, y, z, u\}$ is one-to-one. Connectivity of G implies that p_j os one-to-one which contradicts the assumption $G \not\leq G_j$, q.e.d.

3.13. Theorem. A graph is CSI iff it is connected and it has just one basic equivalence.

Proof. follows from 3.10 – 3.12.

3.14. Corollary. Any complete graph is CSI. A graph with just one edge missing (to completeness) is CSI iff it has at least 4 vertices.

3.15. Remark. Cartesian subdirect irreducibility is not closed to induced subgraphs – e.g. any graph with $n \geqq 4$ vertices and with just one edge missing to completeness is CSI, but its induced subgroup $(2, \varnothing\ )$is not CSI.

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