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On Hamiltonian Cycles in Two-Triangle Graphs*

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We consider the question of the minimum number of Hamiltonian cycles occurring in Hamiltonian planar triangulations extended to the so called two-triangle graphs.

Uvažujeme otázku minimálního počtu Hamiltonovských kružnic v rovinných Hamiltonovských triangulacích rozšířenou na tzv. 2Δ-grafy.

Изучается вопрос минимального числа гамильтоновых циклов в гамильтоновых плоских триангуляциях обощенный на так называемые 2Δ-графы.

1. Preliminary

All graphs considered are finite, undirected, without loops and multiple edges. The vertex set and edge set of a given graph G is denoted by $V(G)$ and $E(G)$, respectively. The complete graph on n vertices is denoted by K_n, K_5^- stands for the complete graph on five vertices with one edge deleted. By a triangle in a given graph we mean any of its subgraphs isomorphic to K_3. A *planar triangulation* is any maximum planar graph (with respect to the set of edges on a given set of vertices), a graph is called *triangulated* if it does not contain an induced cycle of length greater than three. By a 3-connected graph we always mean a vertex-3-connected one.

Definition 1. A graph is called a *two-triangle graph* (shortly a *2Δ-graph*) if each of its edges lies in at least two of its triangles.

Remark 1. Clearly every planar triangulation on at least four vertices and every 3-connected triangulated graph are 2Δ-graphs.

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Definition 2. Let G be a graph and e one of its edges. The number of Hamiltonian cycles containing e is denoted by $c_G(e)$ and the total number of Hamiltonian cycles occurring in G is denoted by $c(G)$. For a given integer $n \geqq 4$, we denote by $c_{2\Delta}(n)$, $c_{3CT}(n)$, $c_{PT}(n)$ the minimum possible number of Hamiltonian cycles that may occur in a Hamiltonian 2Δ-graph, 3-connected triangulated graph and planar triangulation on n vertices, respectively.

Remark 2. According to Remark 1, $c_{2\Delta}(n) \leqq c_{3CT}(n)$ and $c_{2\Delta}(n) \leqq c_{PT}(n)$ hold true for every $n \geqq 4$.

Hakimi, Schmeichel and Thomassen proved in [1], that $c_{PT}(n) \leqq 4$ for all $n \geqq 12$. The main problem stated in [1] is then to determine the numbers $c_{PT}(n)$ precisely. For $n \geqq 12$, we have solved this problem in [12] and extended the result to two-triangle graphs:

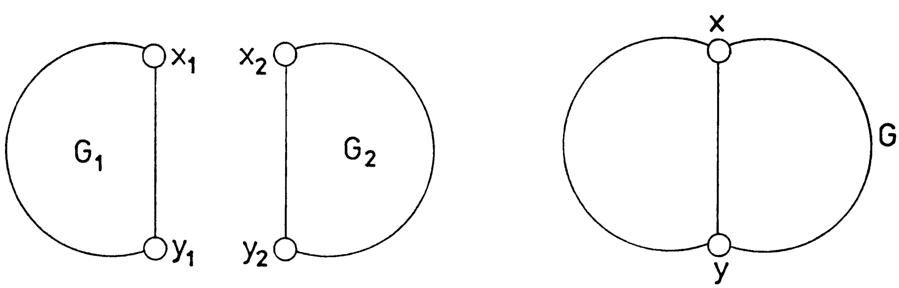
Theorem 1. For $n \geqq 5$, $c_{2\Delta}(n) \geqq 4$ holds true.

Corollary. For $n \geqq 12$, $c_{2\Delta}(n) = c_{3CT}(n) = c_{PT}(n) = 4$ holds true.

The aim of the following chapters is to deal with the case of $4 \leqq n \leqq 11$.

2. Two-triangle graphs

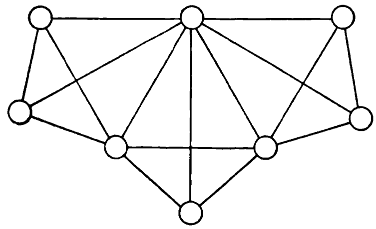
Lemma 1. Let G_1 (G_2, resp.) be a graph and $\{x_1, y_1\}$ ($\{x_2, y_2\}$, resp.) one of its edges, such that $c_{G_1}(\{x_1, y_1\}) = c_{G_2}(\{x_2, y_2\}) = 2$. Suppose the graph G is obtained by amalgamation of the graphs G_1 and G_2 in such a way, that the vertices x_1 and x_2 (y_1 and y_2, resp.) are unified and designated by x (y, resp.) (see Figure 1). Then G contains exactly four Hamiltonian cycles and G also contains vertices u and v, such that $c_G(\{u, v\}) = 2$.



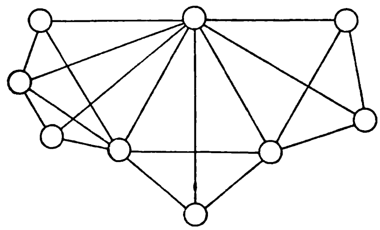
Proof. Since every Hamiltonian cycle in G is a union of Hamiltonian paths from x to y in G_1 and G_2, the first part of the statement is clear. Now G_2 contained exactly two Hamiltonian paths from x_2 to y_2, and these paths differ in at least one edge, say, $\{u, v\}$. Then $\{u, v\}$ lies on just one of them, and hence $c_G(\{u, v\}) = 2$. ∎

Theorem 2. i) $c_{2\Delta}(4) = 3$,
ii) $c_{2\Delta}(5) = 6$,
iii) $c_{2\Delta}(n) = 4$ for $n \geqq 6$.

Proof. The statements i) and ii) are trivial $\left(c(K_4) = 3 \text{ and } c(K_5^-) = 6\right)$. Since $c_{K_4}(e) = 2$ for any $e \in E(K_4)$ and $c_{K_5^-}(e) = 2$ for any edge $e \in E(K_5^-)$ nonadjacent with the nonedge of K_5^-, the statement iii) follows by inductive use of Lemma 1 (see also examples in Figure 2). ■



n even



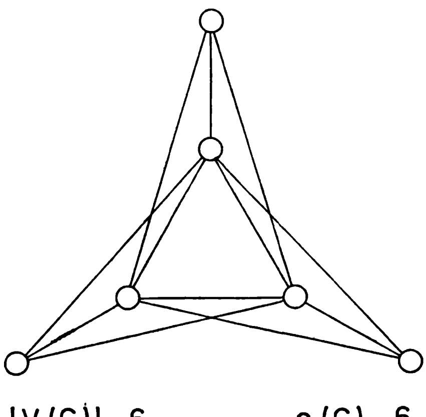
n odd

3. Three-connected triangulated graphs

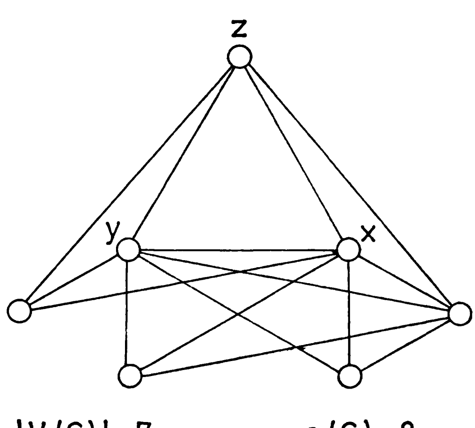
Lemma 2. Let G be a graph with at least four vertices and let x, y and z be vertices forming a triangle in G. Suppose the graph G' is obtained by adding a new vertex w to G adjacent to and only to the vertices x, y and z. Then the following hold true:

i) $c(G') = c_G(\{x, y\}) + c_G(\{y, z\}) + c_G(\{x, z\})$,
ii) $c_{G'}(\{w, x\}) = c_G(\{x, y\}) + c_G(\{x, z\})$,
iii) $c_{G'}(\{x, y\}) \leqq c_G(\{x, y\})$.

Proof is straightforward. ■

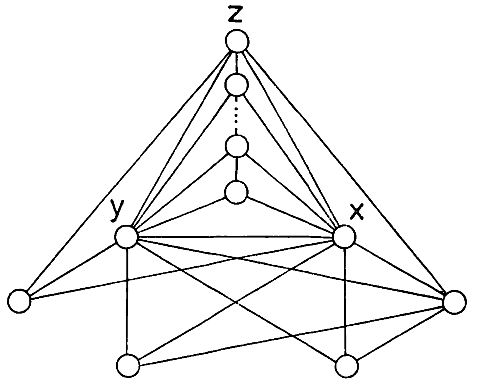


$|V(G)| = 6$ $c(G) = 6$



$|V(G)| = 7$ $c(G) = 8$

Theorem 3. i) $c_{3CT}(4) = 3$,
ii) $c_{3CT}(5) = 6$,
iii) $c_{3CT}(6) = 6$,
iv) $c_{3CT}(7) = 8$,
v) $c_{3CT}(n) = 4$ for $n \geqq 8$.

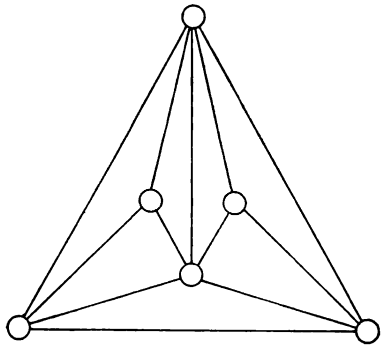


Proof. The statements i)–iv) can be verified by hand, the graphs with 6 and 7 vertices which achieve the minimum numbers of Hamiltonian cycles are depicted in Figures 3 and 4. Now consider the graph G of Figure 4 and the vertices x, y and z shown in the figure. One can easily check that $c_G(\{x, y\}) = 0$ and $c_G(\{y, z\}) = c_G(\{x, z\}) = 2$. The statement v) then follows by inductive use of Lemma 2 as seen from Figure 5 (there are $n - 7$ vertices placed inside the triangle xyz). ■

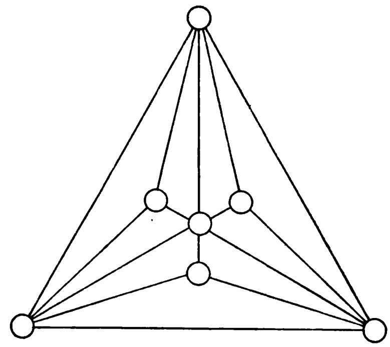
4. Planar triangulations

Theorem 4. i) $c_{PT}(4) = 3$,
ii) $c_{PT}(5) = 6$,
iii) $c_{PT}(6) = 10$,
iv) $c_{PT}(7) = 12$,
v) $c_{PT}(8) = 6$,
vi) $c_{PT}(9) = 8$,
vii) $c_{PT}(10) = 6$,
viii) $4 \leqq c_{PT}(11) \leqq 6$.

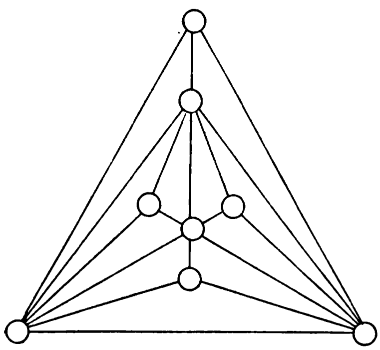
As expected, the case of planar triangulation has turned out to be the most interesting one. However, we have not been able to settle it easier than by counting Hamiltonian cycles in all necessary planar triangulations. The statements i)–v) have been worked out by hand and then checked and extended to vi) and vii) by the second author using computer search based on [3] (Lemma 2 is useful there). The case of $n = 11$ is under computation by now, it seems to us that $c_{PT}(11) = 6$ is more likely to hold true. The graphs achieving the minimum numbers of Hamiltonian cycles are depicted in figure 6.



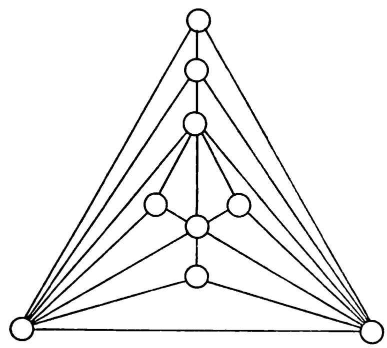
$|V(G)|=6$ $c(G)=10$



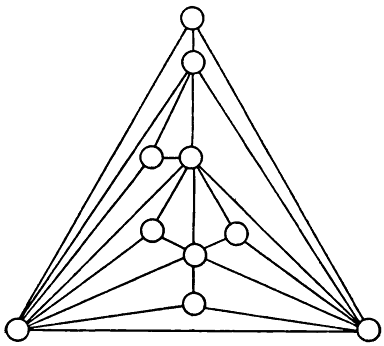
$|V(G)|=7$ $c(G)=12$



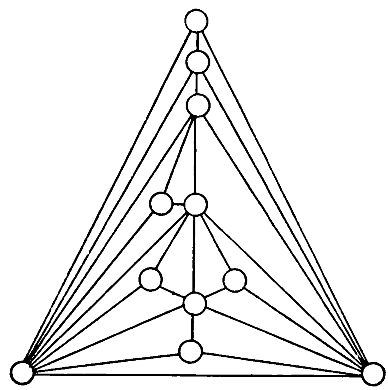
$|V(G)|=8$ $c(G)=6$



$|V(G)|=9$ $c(G)=8$



$|V(G)|=10$ $c(G)=6$



$|V(G)|=11$ $c(G)=6$

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