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Packing Measures on Euclidean Spaces

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0. Summary. Packing measures were introduced in [8] to calculate the exact measure of sample paths of a Brownian motion in $\mathbb{R}^d$ for a suitable packing measure. Recently, these measures became interest for the actual geometric measure theory [9], since a density theorem holds, which we have further improved. Our interest for packing measure was stimulated by [2]. Indeed, packing measures seem to be an interesting new class of measures, which is in the general case completely different from that of Hausdorff measures. With respect to questions concerning sets of non-σ-finite measure they have some advantages. Proofs are only sketched.

1. Some notations. Let $\mathbb{R}^d$ $(d \in \mathbb{N})$ be the Euclidean space with the usual norm $\|\cdot\|$. Let $\mathbb{H}$ be the family of all Hausdorff functions, i.e. $h \in \mathbb{H}$ iff $h: [0, +\infty] \to \to [0, +\infty]$ and

(1) $q > 0 \Rightarrow h(q) > 0, h(0) = 0.$

(2) $q_1 < q_2 \Rightarrow h(q_1) \leqq h(q_2).$

(3) $\lim\limits_{q \downarrow 0} h(q) = 0.$

In [8] a very general notion of packing for a set $E \subseteqq \mathbb{R}^d$ was introduced, packings by open balls centred at E seem to be a good kind, but we will use closed balls, i.e. a set $\{B(x_n. r_n)\}$ of closed balls is called a packing for $E \subseteqq \mathbb{R}^d$ $(E \neq \emptyset)$ iff

(4) $x_n \in E$ for all n.

(5) $n \neq m \Rightarrow B(x_n, r_n) \cap B(x_m, r_m) = \emptyset.$

For general metric spaces (X, ϱ) it will be usefull to replace (5) by

(6) $n \neq m \Rightarrow \varrho(x_n, x_m) > r_n + r_m.$

This holds obviously in Euclidean spaces. For $h \in \mathbb{H}$ and $E \subseteqq \mathbb{R}^d$ a pre-measure τ^h [8] is defined by

(7) $\tau^h(E) = \inf\limits_{\delta > 0} \sup\limits_{n} \{\sum h(2r_n);\ B(x_{n'}, r_n)$ is a packing for E with $r_n \leqq \delta$ for all $n\}.$

We remark that it is enough to consider finite packings for E and that $\tau^h(E) = +\infty$

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for all unbounded sets E. For a nonempty set $E \subseteq \mathbb{R}^d$

(8) $\operatorname{diam}(E) = \sup\{\|x - y\| \mid x, y \in E\}$

denotes the diameter of E. For $A, B \subseteq \mathbb{R}^d$ $(A \neq \emptyset, B \neq \emptyset)$

(9) $\operatorname{dist}(A, B) = \inf\{\|a - b\|; a \in A, b \in B\}$

is called the distance of A and B. It is well-known that for $E \subseteq \mathbb{R}^d$ and $h \in \mathbb{H}$

(10) $\mu^h(E) = \sup_{\delta>0} \inf\{\sum_n h(\operatorname{diam}(E_n)); E \subseteq \bigcup_n E_n, \operatorname{diam}(E_n) \leq \delta\}$

is the h-Hausdorff measure of E [7]. We consider here the h-packing measure of $E \subseteq \mathbb{R}^d$ [8]

(11) $p^h(E) = \inf\{\sum_n \tau^h(E_n); E \subseteq \bigcup_n E_n\}$.

A map $f: E \to R^d$ of some $E \subseteq \mathbb{R}^d(E \neq \emptyset)$ is called non-expanding iff

(12) for all $x, y \in E$, $\|x - y\| \geqq \|f(x) - f(y)\|$.

A Borel measure μ on $\mathbb{R}^d$ is strongly metrically invariant iff

(13) $\mu(E) \geqq \mu(f(E))$ for every non-expanding map f of E.

For a bounded positive Borel measure μ on $\mathbb{R}^d$ and $h \in \mathbb{H}$ we define the lower h-density of some $x \in \mathbb{R}^d$ w.r.t. $E \subseteq \mathbb{R}^d$ and μ by

(14) $D^h_\mu(E, x) = \sup_{\delta>0} \inf\left\{\frac{\mu(E \cap B(x, r))}{h(2r)}; r \leqq \delta\right\}$.

A Borel measure μ on $\mathbb{R}^d$ is called representable as a Hausdorff measure iff there is some $h \in \mathbb{H}$ and a metric ϱ generating the Euclidean topology such that

(15) $\mu = \mu^h_\varrho$ (μ^h_ϱ means the Hausdorff measure w.r.t. ϱ and h).

A Borel measure μ is called tight iff

(16) $\mu(B) = \sup\{\mu(K); K \subseteq B, K \textit{ compact}\}$ for all Borel sets B.

$E \subseteq \mathbb{R}^d$ has σ-finite measure for an arbitrary positive outer measure μ iff

(17) $E = \bigcup_n E_n$, $\mu(E_n) < +\infty$ for all n.

A measure μ on $\mathbb{R}^d$ is σ-finite iff $\mathbb{R}^d$ has σ-finite measure.

2. Properties of τ^h and p^h.

Theorem 1

(i) $A \subseteq B \Rightarrow \tau^h(A) \leqq \tau^h(B)$.
(ii) $\tau^h(A \cup B) \leqq \tau^h(A) + \tau^h(B)$.
(iii) *dist* $(A, B) > 0 \Rightarrow \tau^h(A \cup B) = \tau^h(A) + \tau^h(B)$.
(iv) $h, g \in \mathbb{H}, \lim_{q\downarrow 0} \frac{g(q)}{h(q)} = 0$ *and* $\tau^h(E) < +\infty \Rightarrow \tau^g(E) = 0$.

(v) For all $x \in \mathbb{R}^d$, $\tau^h(\{x\}) = 0$.
(vi) $\tau^h(E) = \tau^h(\bar{E})$ *($\bar{E}$ closure of E).*
(vii) $h \in \mathbb{H}$, *U open, $U \neq \emptyset$ and*

$$\lim_{q \downarrow 0} \frac{q^d}{h(q)} = 0 \Rightarrow \delta^h(U) = +\infty.$$

(viii) *If E is bounded Lebesgue measurable subset of positive measure, $h(q) = q^d \Rightarrow 0 < \tau^h(E) < +\infty$.*

Proof: see [8], for $\tau^h(\bar{E}) \leqq \tau^h(E)$ to prove (vi) use the following argument: If $\{B(x_n, r_n); n = 1, \dots, m\}$ is a finite packing for $\bar{E}$ then $\|x_n - x_k\| > r_n + r_k$ for $n \neq k$. Choose ε such that $0 < \varepsilon < min\{\|x_n - x_k\| - r_n - r_k; n \neq k\}/2$. Since the balls $B(x_n, \varepsilon)$ cut E, take $y_n \in B(x_n, \varepsilon) \cap E$ to obtain $\{B(y_n, r_n); n = 1, \dots, m\}$ as a packing for E.

(viii) and (vi) $\Rightarrow$ (vii).//

Theorem 2

(i) *p^h is a metric outer measure.*
(ii) *All Borel sets are p^h-measurable.*
(iii) *p^h is Borel regular, i.e. for $E \subseteqq \mathbb{R}^d$ there exists a Borel set $B \supseteqq E$ such that $p^h(B) = p^h(E)$ holds.*
(iv) $E \subseteqq \mathbb{R}^d \Rightarrow p^h(E) \leqq \tau^h(E)$, *$p^h$ is atomless, i.e. $p^h(\{x\}) = 0$.*
(v) $E_n \uparrow E \Rightarrow p^h(E_n) \uparrow p^h(E)$.
(vi) *If $E \subseteqq \mathbb{R}^d$ is p^h-measurable with finite measure* $\Rightarrow p^h(E) = \sup\{p^h(K); K \subseteqq E, K \text{ compact}\}$
(vii) $p^h(E) = inf\{\lim_{n\to\infty} \tau^h(E_n); E_n \uparrow E\}$.
(viii) $p^h(E) = \sup_{\delta>0} \inf\{\sum_n \tau^h(E_n); E \subseteqq \bigcup_n E_n, \operatorname{diam}(E_n) \leqq \delta\}$.
(ix) *If for $h \in \mathbb{H}$ there is some $c \in \mathbb{R}$, $c > 1$ such that $h(2q) \leqq ch(q)$ for all $q \geqq 0 \Rightarrow \mu^h \leqq p^h$.*
(x) *p^h is strongly metrically invariant.*

Proof: (i)–(vii) (ix) [8], (viii) [6]. (x) Use: If $\{B(y_n, r_n)\}$ is a packing for $f(E)$ then $\{B(x_n, r_n)\}$ is one for E where $y_n = f(x_n)$.//

3. A Density Theorem.

We improve the version of [8] making some constant λ_1 and a restriction to h superfluous. Our proof is based on two lemmas.

Lemma 1 ([4, *Besicovitch Covering Lemma*])

Let $E \subseteqq \mathbb{R}^d$ be a bounded set, $(B(x, r(x)))_{x \in E}$ a family of closed balls centred in E

and with radius $r(x) \in (0,1)$ *Then we can find a sequence* (B_k) *of selected balls from the given ones that*

(i) $E \subseteqq \bigcup_k B_k$

(ii) *There is some* $c_d \in \mathbb{N}$ *such that* (B_k) *can be distributed to* c_d *subsequences* $(B_k^1), \ldots$ $\ldots, (B_k^{c_d})$ *of disjoint balls.*

Lemma 2 ([4, *special case of Theorem* 6.2.3])

Let μ *be a* σ*-finite positive Borel measure on* $\mathbb{R}^d$. *If* $\mathbb{V}$ *is a Vitali class of closed balls for E, i.e. for each* $x \in E$ *and* $\delta > 0$ *there is a* $B(x, r) \in \mathbb{V}$ *with* $r \leqq \delta$, *then there is a sequence* (V_n) *from* $\mathbb{V}$ *of disjoint elements such that* $\mu(E - \bigcup_n V_n) = 0$.

Theorem 3

If μ *is a Borel measure on* $\mathbb{R}^d$, $E \subseteqq \mathbb{R}^d$ *a Borel set of finite measure then for all* a, b *with* $a < \inf_{x \in E} D_\mu^h(E, x) \leqq \sup_{x \in E} D_\mu^h(E, x) < b$ *it holds* $ap^h(E) \leqq \mu(E) \leqq bp^h(E)$.

Proof. For the second inequality it is enough to prove

$$b > \sup_{x \in F} D_\mu^h(F, x) \Rightarrow \mu(F) \leqq b\tau^h(F)$$

for Borel subsets $F \subseteqq E$. Take for $\delta > 0$

$$\mathbb{V}_\delta = \left\{B(x, r); x \in F, \frac{\mu(B(x, r) \cap F)}{h(2r)} < b, r \leqq \delta\right\}$$

as a Vitali class for F. Apply Lemma 1 and 2 to obtain a packing $\{B(x_n, r_n)\}$ from $\mathbb{V}_\delta$ for F satisfying $\mu(F - \bigcup_n B(x_n, r_n)) = 0$. This gives us by $\mu(F) \leqq b \,.\, \sum_n h(2r_n)$ the demanded inequality. For $ap^h(E)$ the arguments of [8] can be applied.//

4. Sets of non-σ-finite measure

Theorem 4

Let $K \subseteqq \mathbb{R}^d$ *be compact. Then there are equivalent*

(i) *K has non-*σ*-finite measure for* p^h.

(ii) *There is some* $K_0 \subseteqq K$ *such that for each open set* U *with* $U \cap K_0 \neq \emptyset$ $\tau^h(U \cap K_0) = +\infty$.

Proof: [5, 8].//

Theorem 5

If $A \subseteqq \mathbb{R}^d$ *is analytic and of non-*σ*-finite measure for* p^h *then there is a compact subset* $K \subseteqq A$ *of non-*σ*-finite measure for* p^h.

Proof: [5].//

Remark. In [5] this result was proved for general separable complete metric spaces and continuous h, since open packings were used. The use of closed ones makes the continuity of h superfluous because of Theorem 1 (vi). In the general setting we must take the packing notion with (6) instead of (5) to yield the same.

Theorem 6

p^h is a tight Borel measure.

Proof: Theorem 5 and Theorem 2(vi) give us the result.//

Theorem 7

Every analytic set $A \subseteqq \mathbb{R}^d$ of non-σ-finite measure for p^h contains $2^{\aleph_0}$ pairwise disjoint compact sets each of non-σ-finite measure.

Proof: [5].//

Theorem 8

If $\mathbb{R}^d$ has non-σ-finite measure for p^h then p^h is not representable as a Hausdorff measure.

Proof: Take a closed set $X \subseteqq \mathbb{R}^d$ satisfying that for open U with $U \cap X \neq \emptyset$ $\tau^h(U \cap X) = +\infty$. For each Hausdorff measure μ_ϱ^g on $\mathbb{R}^d$ we can find a dense G_δ – set of X, say Y, such that $\mu_\varrho^g(Y) = 0$ [7], but $p^h(X) = +\infty$ by the Baire Category Theorem.//

Remarks.

1. Hausdorff and packing measure differ by the property to be G_δ-regular [7].
2. If $\mathbb{R}^d$ has σ-finite measure for p^h then we can find an open dense subset $Z \subseteqq \mathbb{R}^d$ such that p^h restricted to Z is representable as a Hausdorff measure [1].

Theorem 9

If $h \in \mathbb{H}$ fulfils $\lim\limits_{q \downarrow 0} (q^d/h(q)) = 0$ then R^d has non-σ-finite measure for p^h.

Proof: Theorem 1 (vii) and the Baire Category Theorem.//

5. Subsets of finite positive packing measure but Hausdorff measure zero — a discussion of ideas

For $x \in \mathbb{R}^d$ let $u_n(x)$ denite the unique cube of side length $1/2^n$ containing x, which projections in the i-th axis $(i = 1, \ldots, d)$ is a half-open interval of the form $[\![k_i(1/2^n), (k_i + 1)(1/2^n)[\![$ with $k_i \in \mathbb{Z}$, $u_n(x)$ is called a dyadic cube. We call a cube semidyadic iff its projection in the i-th axis have the form $[\![(1/2)\,k_i(1/2^n),\ ((1/2)\,k_i + 1)\,(1/2^n)[\![$. Each $x \in \mathbb{R}^d$ belongs to 2^d such cubes. A unique semidyadic $v_n(x)$ for x is defined by

(18) $dist\,(\mathbb{R}^d) - v_n(x),\ u_{n+2}(x)) = 2^{-n-2}$.

For sufficiently large $k_0 \in \mathbb{N}$ let $I = \{ik_0;\ i \in \mathbb{N}\}$ and define then

(19) $G(I) = \{v_n(x);\ x \in \mathbb{R}^d,\ n \in I\}$.

The notion of semidyadic packing for $I = \mathbb{N}$ is due to [8] as well as the related notion of packing measure $\bar{p}_I^h$ pre-measure is $\bar{\tau}_I^h$. Note that for a packing element $v_n(x)$ of a subset $E \subseteqq \mathbb{R}^d$, $x \in E$ is demanded.

Lemma 3

If h satisfies $h(2q) \leqq ch(q)$ $(c > 1)$ for all $q \geqq 0$ then there are $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that $\lambda_1\,\bar{\tau}_I^h(E) \leqq \tau^h(E) \leqq \lambda_2\,\bar{\tau}_I^h(E)$ and $\lambda_1\,\bar{p}_I^h(E) \leqq h^h(E) \leqq \lambda_2\,\bar{p}_I^h(E)$.

Proof: Use $B(x, 2^{-n-2}) \subseteqq v_n(x)$.//

For a Borel set $E \subseteqq \mathbb{R}^d$, a Borel measure μ, $h \in \mathbb{H}$ we define

(20) $$D_{\mu,I}^h(E, x) = \sup_{\delta > 0} \inf \left\{ \frac{\mu(E \cap v_n(x))}{h(\dim(v_n(x))}\,;\ \operatorname{diam}(v_n(x)) \leqq \delta,\ n \in I \right\}$$

for $x \in \mathbb{R}^d$ and for $0 < \inf_{x \in E} D_{\mu,I}^h(E, x)$ it is not hard to see that

Lemma 4

$$\inf_{x \in E} D_{\mu,I}^h(E, x)\,\bar{p}_I^h(E) \leqq \mu(E) \leqq \sup_{x \in E} D_{\mu,I}^h(E, x)\,\bar{p}_I^h(E)\,.$$

Further it seems easily to see that certain self similar sets [3] $K \subseteqq \mathbb{R}^d$ satisfy the condition (A)

(A) *For $x, y \in K$, $n, m \in I$ either $v_n(x) \cap v_m(y) = \emptyset$ or one is contained in the other.* Since such K has then positive Hausdorff-Besicovitch number [3] we can select a compact subset (which satisfies (A) too) having Hausdorff measure zero for all $h(q) = q^\alpha$ $(\alpha > 0)$, but non-σ-finite packing measure for some $h_0(q) = q^{\alpha_0}$ ($\alpha_0 > 0$ is smaller than the Hausdorff-Besicovitch number mentioned above) by Theorem 5. Hence, the premises of the next Theorem are not an empty set.

Theorem 10

Let $X \subseteqq \mathbb{R}^d$ be a compact subset of non-σ-finite measure for p^h and satisfying (A) for X. If h satisfies $h(2q) \leqq ch(q)$ $(c > 1)$ for all $q \geqq 0$ then there is a compact subset $C \subseteqq X$ such that $0 < p^h(C) < +\infty$.

Proof: X has non-σ-finite measure for $\bar{p}_I^h$. Further we can suppose $\bar{\tau}_I^h(u_n(x) \cap X) = +\infty$ for all $x \in X$, $n \in \mathbb{N}$. This makes it possible to find some finite packing for

K taken from $G(I)$, $\{v_{n_k}(x_k)\}_{k\in J}$ ($J \subseteqq \mathbb{N}$, $n_k \in I$) satisfying

(i) $\sum_{k\in J} h(\operatorname{diam}(v_{n_k}(x_k)) > 1$

(ii) $\sum_{k\in J}^{(j)} h(\operatorname{diam}(v_{n_k}(x_k))) \leqq 1$ *for all* $j \in J$ *if the term* $h(\operatorname{diam}(v_{n_j}(x_j))$ *is left*

(iii) *If* $\{v_{m_l}(y_1)\}$ *is any different finite packing for* X *from* $G(I)$ *such that for every* l *there is some* k *such that* $v_{n_k}(x_k) \subseteqq v_{m_l}(y_l)$ *then* $\sum_l h(\operatorname{diam}(v_{m_l}(y_1))) \leqq 1$.

Let S_1 be the family of $v_{n_k}(x_k)$. If now $S_1, \ldots, S_n$ ($n \in \mathbb{N}$) is given, take $v_m(x) \in S_n$. $\bar{\tau}_I^h(u_{m+2}(x) \cap X) = +\infty$ implies that we can find a finite packing $\{v_{n_k}(x_k);\, k \in J_n\}$, $J_n = J_n(v_m(x)) \subseteqq \mathbb{N}$ for the set $u_{m+2}(x) \cap X$ such that

(i)′ $\sum_{k\in J_n} h(\operatorname{diam}(v_{n_k}(x_k))) > h(\operatorname{diam}(v_m(x))$

(ii)′ $\sum_{k\in J_n}^{(j)} h(\operatorname{diam}(v_{n_k}(x_k))) \leqq h(\operatorname{diam}(v_m(x)))$ *for all* $j \in J_n$

(iii)′ *If* $\{v_{m_l}(y_l);\, l \in J'\}$ *is any different packing for* $X \cap u_{m+2}(x)$ *taken from* $(G(I)$ *such that for every* $l \in J'$ *there is some* $k \in J_n$ *such that* $v_{n_k}(x_k) \subseteqq v_{m_l}(y_l)$ *then* $\sum_{l\in J'} h(\operatorname{diam}(v_{m_l}(y_l))) \leqq h(\operatorname{diam}(v_m(x)))$.

S_{n+1} denotes the family of all packing elements obtained in this way for all possible $v_m(x) \in S_n$. Suppose $u_{m+2}(x) \cap X$ for $v_m(x) \in S_n$ is always closed. Further $\bigcup S_{n+1} \subseteqq \subseteqq S_n$ and *sup* $\{\operatorname{diam}(v_m(x));\, v_m(x) \in S_n\} \to 0$ ($n \to \infty$).
This implies that

$$C = \bigcap_{n\in\mathbb{N}} (\bigcup\{X \cap u_{m+2}(x);\, v_m(x) \in S_n\})$$

is a compact subset of X. We define a Borel measure μ ob $\mathbb{R}^d$ by $\mu(C) = 1$ and $\mu(R^d - C) = 0$, further put $P' = u_{n+2}(x)$ for $P = v_n(x)$. Then define

$$\mu(P' \cap C) = \frac{\mu(C)\, h(\operatorname{diam}(P))}{\sum\{h(\operatorname{diam}(Q));\, Q \in S_1\}} \quad \text{for all} \quad P \in S_1\,,$$

for $P \in S_{n+1}$ there exists a unique $T \in S_n$ such that $P \subseteqq T$, put then

$$\mu(P' \cap C) = \frac{\mu(T' \cap C)\, h(\operatorname{diam}(P))}{\sum\{h(\operatorname{diam}(Q));\, Q \in S_{n+1},\, Q \subseteqq T\}}\,.$$

μ extends in a natural way to a Borel measure on $\mathbb{R}^d$. We remark that $\mu(P \cap C) = \mu(P' \cap C)$ for all $P \in S_n$. We evaluate upper and lower boubds for $D^h_{\mu,I}(C, x)$, $x \in C$. Fot $x \in C$ there is a unique sequence $(v_{n_k}(x))_{k\in\mathbb{N}}$, $v_{n_k}(x) \in S_k$, $n_k \in I$, $(n_k)_{k\in\mathbb{N}}$ depends of x. We estimate by (i) (i)′

$$\frac{\mu(C \cap v_{n_k}(x))}{h(\operatorname{diam}(v_{n_k}(x))} \leqq 1\,.$$

For $l \in I$ there is a unique $k \in \mathbb{N}$ such that $n_k < l \leqq n_{k+1}$. If $l = n_{k+1}$ we obtain by (ii)′

$$\frac{\mu(C \cap v_{n_{k+1}}(x)}{h(\operatorname{diam}(v_{n_{k+1}}(x))} \geqq \frac{1}{\left(1 + \dfrac{h(\operatorname{diam}(v_{n_{k+1}}(x))}{h(\operatorname{diam}(v_{\epsilon_k}(x))}\right)} \frac{\mu(C \cap v_{nk-1}(x))}{(h(\operatorname{diam}(v_{n_{k-1}}(x))) + h(\operatorname{diam}(v_{n_k}(x))))}$$

and thus

$$\frac{\mu(C \cap v_{n_{k+1}}(x))}{h(\operatorname{diam}(v_{n_{k+1}}(x))} \geqq \frac{1}{\prod_{i=1}^{k}\left(1 + \dfrac{h(\operatorname{diam}(v_{n_{i+1}}(x)))}{h(\operatorname{diam}(v_{n_i}(x)))}\right)} \cdot \frac{1}{(1 + h(\operatorname{diam}(v_{n_1}(x))))} .$$

We may assume (a suitable thinning of X does it, I think)

$$b = \inf_{\substack{x \in C \\ k \in \mathbb{N}}} \frac{1}{\prod_{i=1}^{k}\left(1 + \dfrac{h(\operatorname{diam}(v_{n_{i+1}}(x)))}{h(\operatorname{diam}(v_{n_i}(x)))}\right)} \cdot \frac{1}{(1 + h(\operatorname{diam}(v_{n_1}(x))))} > 0 .$$

If $n_k < l < n_{k+1}$ it follows by (iii)′

$$h(\operatorname{diam}(v_l(x))) \leqq \sum\{h(\operatorname{diam}(P)); P \in S_{k+1}, P \subseteq v_l(x)\}$$

and one estimates also using (A)

$$\frac{\mu(C \cap v_l(x))}{h(\operatorname{diam}(v_l(x)))} \geqq b .$$

Lemma 3 and 4 gives us then $0 < p^h(C) < +\infty$.//

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