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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 28 (1987), No. 2, 41--47

Persistent URL: http://dml.cz/dmlcz/701922

Leaves, Excesses, and Neighbourhoods

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Received 14 April, 1987

Recent results on three graph-theoretic problems in combinatorial design theory are surveyed. The leave problem asks for a characterization of leaves ("missing edge graphs") in packings by triples. The excess problem is dual: the characterization of excesses ("extra edge graphs") in coverings by triples. Finally, the neighbourhood problem asks for a characterization of possible neighbourhoods (pairs appearing in triples with a fixed element) in a triple system. Numerous open problems are mentioned.

1. Triple systems, packings, and coverings

In combinatorial design theory, a substantial amount of effort has been invested in the study of triple systems, and the related packing and covering problems for triples. A *triple system* $B[3, \lambda; v]$ of order v and index λ is a pair (V, B); V is a v-set of *elements*, and B is a collection of 3-subsets of V called *triples* or *blocks*, so that each 2-subset of V appears in precisely λ blocks. It is well known that a $B[3, \lambda; v]$ exists if and only if the necessary conditions $\lambda v(v - 1) \equiv 0 \pmod 6$ and $\lambda(v - 1) \equiv 0 \pmod 2$ are satisfied.

When the necessary conditions are not met, one can ask how "close" one can come to producing a triple system. Two natural generalizations are apparent; we could require the maximum number of triples containing each pair at most λ times (a packing problem), or we could require the fewest triples containing each pair at least λ times (a covering problem). Of course, when the necessary conditions for triple systems are met, both the packing and the covering problem are settled by the existence of triple systems. To formalize these ideas, we need some further definitions.

A *partial triple system* PB$[3, \lambda; v]$ is a v-set V and collection B of triples which contain each pair of V at most λ times. It is *maximal* when no 3-subset of V can be added to B without violating this property. It is *maximum* when no $PB[3, \lambda; v]$ exists having more triples.

A *covering by triples* $CT(v, \lambda)$ is a v-set V and a collection B of triples which contain each pair of V at least λ times. It is *minimal* when no triple in B can be

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omitted without destroying this property. It is *minimum* when no $CT(v, \lambda)$ exists having fewer triples.

We are especially concerned here with two problems that have been studied on partial triple systems. The *embedding* problem asks for the smallest triple system with the same index (V', B') containing a specified partial triple system (V, B); that is, $V \subseteq V'$ and $B \subseteq B'$. Lindner [15] conjectures that $|V'| \geqq 2|V| + 1$ is always sufficient for index one, while the best result in this direction to date shows that $|V'| \geqq 4|V| + 1$ is sufficient [1]. One of the main directions taken in attacking the embedding problem is to develop a better understanding of the structure of partial triple systems, and hence it is one of the main motivations for our interest in them here.

The second main problem of interest is the *immersion* problem; given a partial triple system (V, B), one is to produce a containing triple system (V, B') having the smallest possible index, and having $B \subseteq B'$. While finite bounds on the index of the containing system are known here [6], little progress has been made on determining the minimum possible indices for immersion. The key here seems to also be a good understanding of the structure of partial triple systems; but more comes into play. In the immersion problem, to immerse a $PB[3, \lambda; v]$ into a $B[3; \lambda + \mu; v]$ requires the production of a certain $CT(v, \mu)$, and hence the problem is also one of understanding the structure of coverings by triples.

2. Leaves

Both embedding and immersion require that triples be added so as to use all of the pairs which are contained in fewer than λ triples of the partial system; hence, both depend entirely of the structure of the "missing" pairs. With this in mind, we define the *leave* of a $PB[3; \lambda; v]$ (V, B) to be a multigraph (V, E), where E contains edge $\{x, y\}$ s times precisely when the pair $\{x, y\}$ appears in $\lambda - s$ triples in B. A multigraph is a λ-leave when it is the leave of some $PB[3, \lambda; v]$. The key point is that two partial triple systems with the same leave can be embedded identically.

This leads to an important problem, which we call the *leave problem*: which multigraphs are λ-leaves? A complete characterization seems beyond reach, because determining whether a graph is a 1-leave is NP-complete [2]. Nevertheless, strong necessary and/or sufficient conditions seem very promising as a vehicle for addressing the embedding and immersion problems.

Two more conservative problems are to characterize λ-leaves of maximal systems, and of maximum systems. The 1-leaves of maximum partial triple systems are uniquely determined by the congruence class of v modulo 6; see, for example, [16]. Maximum partial triple systems for higher λ have also been studied [17]. However, the characterization of leaves of maximal partial triple systems is very far from settled.

First we consider basic necessary conditions. Let us suppose that G is a v-vertex e-edge multigraph which is a λ-leave. Naturally, no edge in G appears more than λ

times. Consider a $PB[3, \lambda; v]$ whose leave is G; let C be the multigraph whose edges are the unordered pairs (with multiplicities) which appear in triples of the partial triple system. Evidently, $G \cup C$ is precisely λK_v, the complete v-vertex graph with each edge repeated λ times. Now C necessarily has a number of edges which is a multiple of three, and hence we have

$$e \equiv \lambda \binom{v}{2} \pmod 3 \tag{1}$$

Considering a particular vertex x of G, the construction of C ensures that $deg_C(x) \equiv 0 \pmod 2$, and hence

$$deg_G(x) \equiv \lambda(v - 1) \pmod 2 \quad \text{for all} \quad x \tag{2}$$

Conditions (1) and (2) are simple numerical conditions, and are not sufficient for characterizing λ-leaves. One candidate graph which meets conditions (1) and (2) but is not a 1-leave is $C_4 \cup C_5$, with $v = e = 9$. This candidate and many others are ruled out by a simple density condition. Suppose that G is a v-vertex e-edge graph having an edge-cutset of size c whose removal breaks G into two components, of sizes s and $v - s$. Considering the means in which triples can use edges from inside these components, and between these components, leads to the necessary condition:

$$2\left(\lambda\binom{s}{2} + \lambda\binom{v-s}{2} - e + c\right) \geqq \lambda s(v - s) - c \tag{3}$$

For $C_4 \cup C_5$ we have $v = e = 9$, $\lambda = 1$, $c = 0$, and $s = 4$; the condition $14 \geqq 20$ fails, and hence $C_4 \cup C_5$ is not a 1-leave (notice that the condition does not rule out $C_4 \cup C_5$ as a 2-leave; in fact, it is a 2-leave!).

The three necessary conditions developed thus far are still not sufficient; Stinson and Wallis [18] give infinite families of graphs which meet the necessary conditions for 1-leaves given here, but are not 1-leaves. Hence the main open question here is the following.

Problem. Produce stronger necessary conditions for a multigraph to be a λ-leave.

If we restrict our attention to leaves of maximal partial triple systems, we obtain a fourth necessary condition: G is triangle-free. Since the Stinson-Wallis examples are all triangle-free, better necessary conditions are needed in this restricted case as well. However, while determining in general whether a graph is a 1-leave is NP-complete, the following remains open:

Conjecture: Determining whether a graph is a 1-leave of a maximal partial triple system is NP-complete.

If true, this conjecture explains in part the difficulty of obtaining good general necessary conditions.

A second approach is to develop strong sufficient conditions for a graph to be a λ-leave. The only general result in this area is independently proved in [8, 13]:

Theorem. Let G be a graph in which each vertex has degree zero or two. If G meets the necessary conditions (1), (2) and (3) above, G is a 1-leave.

While the proofs of this theorem do not seem amenable to generalization to higher vertex degrees, the theorem suggests the following:

Conjecture: Let G be a graph in which each vertex has maximum degree k. Then there is a constant n_k so that if G satisfies conditions (1) and (2) above, and has more than n_k vertices, then G is a 1-leave.

The theorem above actually establishes that $n_2 = 9$, since $C_4 \cup C_5$ is the largest such graph which fails to meet condition (3). We should remark that this conjecture is a relaxation of an old conjecture of Nash-Williams; he makes the stronger conjecture that $n_k \leqq 4k + 1$. The immense gap between the current state of knowledge and the conjectured state of affairs leaves much room for new ideas here!

Finally, we should remark in support of the last conjecture that every simple graph meeting conditions (1) and (2) is a component of a 1-leave which is polynomially larger than the given graph [3]; hence, any structural necessary condition must take the number of vertices into account.

3. Excesses

There is a natural duality between packing and covering problems; hence we may ask what the covering analogue to the leave problem is. To develop this, we define the *excess* of a $CT(v, \lambda)$ to be a multigraph whose vertices are the elements of the covering, and in which edge $\{x, y\}$ appears s times in the covering precisely when the corresponding pair appears in $\lambda + s$ triples of the covering. A multigraph is a λ-excess when it is the excess of some $CT(v, \lambda)$.

A multigraph is a 0-excess if and only of it has an edge-partition into triangles; hence determining whether a multigraph is a 0-excess is NP-complete [14]. One might therefore consider asking for a characterization of excesses of minimal, or even minimum, coverings. Excesses of minimum coverings for index one are completely determined in [12]; for higher indices, see [11, 17]. As with maximal packing, the characterization of excess of minimal coverings remains open; we suspect that it is computationally difficult:

Conjecture: Determining whether a multigraph is a 1-excess of a minimal covering by triples is NP-complete.

Once again, the lack of a complete characterization leads us to consider strong necessary conditions, and strong sufficient conditions. First we examine necessary conditions. Let G be a v-vertex e-edge excess of a $CT(v, \lambda)$. We note that the graph C

of edges covered in the covering has the property that $\lambda K_v \cup G = C$. Hence we have the conditions:

(1′) $$e \equiv -\lambda \binom{v}{2} \pmod 3$$

(2)′ $$deg_G(x) \equiv \lambda(v-1) \pmod 2$$

Notice that when $v \equiv 0,1 \pmod 3$, any multigraph which meets the necessary conditions to be a leave also meets these conditions to be an excess. At first glance, one might hope that every 1-leave with $v \equiv 0,1 \pmod 3$ is also a 1-excess; this would imply that every $PB[3, 1; v]$ with $v \equiv 0, 1 \pmod 3$ has an immersion into a $B[3; 2; v]$. However, this does not hold [6]. Consider the complete bipartite graph $K_{n,n-2}$ with $n \equiv 3 \pmod 6$. Such graphs are all 1-leaves, but are not 1-excesses. We develop a general necessary condition which excludes $K_{n,n-2}$, among others.

Let G be a λ-excess with v vertices and e edges. Suppose that the vertex set of G is partitioned into two classes, of size s and $v - s$, so that c of the e edges cross between the classes. Examining how triples can use those edges which cross between classes, we find

(3′) $$2\left(\lambda\binom{s}{2} + \lambda\binom{v-s}{2} + e - c\right) \geqq \lambda s(v-s) + c$$

For $K_{n,n-2}$, we have $v = 2n - 2$, $e = n(n - 2)$, $\lambda = 1$, $s = n$, and $c = n(n - 2)$; hence we would require $n(n - 1) + (n - 2)(n - 3) \geqq (n - 1)(n - 2) + n(n - 2)$, which does not hold for $n > 3$. Observe, however, than $K_{n,n-2}$ is a 3-excess. In fact, we can restate a conjecture of [6] in terms of leaves and excesses as follows:

Conjecture: Let G be a 1-leave on v vertices. Then G is a 2-excess provided that $v \equiv 1 \pmod 2$; it is a 3-excess provided that $v \equiv 0, 1 \pmod 3$; and it is always a 5-excess.

It appears to be important to refine the necessary conditions prior to attacking this conjecture. Ultimately, however, what is needed is strong sufficient conditions. As with leaves, the only positive results here concern the case of very small vertex degrees. Colbourn and Rosa [10] prove the following:

Theorem. Let G be a multigraph satisfying conditions (1′) and (2′), and having maximum degree two. Then G is a 1-excess of a (not necessarily minimal) covering by triples.

This suggests the following natural conjecture:

Conjecture. For each degree k, there is an absolute constant m_k so that all graphs meeting conditions (1′) and (2′), having maximum degree k, and at least m_k vertices, are λ-excesses.

One of the most appealing directions here is to try to develop a closer relation between excesses and leaves. In particular, the immersion problem suggests that λ-leaves are all μ-excesses for some μ closely related to λ; this relationship is poorly understood at the present time.

4. Neighbourhoods

In this section, we introduce a third graph-theoretic problem on triple systems, which is also closely related to the leave problem. In a triple system $B[3, \lambda; v]$, the *neighbourhood* $N(x)$ of an element x is a multigraph whose vertices are all elements of the triple system, excluding x, and whose edges are those unordered pairs appearing in triples with x. The neighbourhood problem is to characterize multigraphs which are neighbourhoods. With this in mind, we observe that a multigraph is a neighbourhood if and only if it is λ-regular and is a λ-leave. To see the latter, observe that if $N(x) = G$ in a $B[3, \lambda; v]$, omitting all triples containing x gives a $PB[3, \lambda; v - 1]$ whose leave is G. We suspect that, unlike characterization of leaves or excesses, a simple characterization of neighbourhoods is possible; in fact, we believe the following:

Conjecture: Let G be a λ-regular multigraph which meets conditions (1), (2), and (3) to be a λ-leave. Then G is a λ-leave and hence a neighbourhood.

If true, this implies that all possible neighbourhoods actually arise, and the structure of designs (in this respect) is as rich as one could expect.

Much more progress has been made on the neighbourhood problem than on the (more general) leave problem. For $\lambda = 1$, the conjecture is trivially true. For $\lambda = 2$, Colbourn and Rosa [9] employed the sufficiency condition for 1-leaves to establish the conjecture. For $\lambda = 3$, Colbourn and McKay [7] developed useful path-factorizations of cubic multigraphs to exploit the results on 1-leaves and neighbourhoods with $\lambda = 2$. Hence the conjecture has been verified for $\lambda \leqq 3$. For higher λ, Colbourn [4] established that every *simple* graph meeting the necessary conditions is a neighbourhood. In fact, many of the techniques used apply equally well to multigraphs; for example, the methods used in [4] establish the conjecture for all $\lambda \equiv 2, 4 \pmod 6$ for *multi*graphs. It appears that relatively little work remains to establish the conjecture in full; however, present techniques fall somewhat short of this goal.

A nice generalization of the neighbourhood problem is the *double neighbourhood problem*: for prescribed (labelled) multigraphs G and H, can one construct a triple system with $N(x) = G$ and $N(y) = H$? The basic necessary conditions are that each is a neighbourhood, and that the graphs are consistent: if G contains $\{y, z\}$, H must contain $\{x, z\}$, Moreover, their union must not have an edge covered more than λ times. The double neighbourhood problem is settled only for $\lambda = 1$ [5], where these necessary conditions are in fact sufficient. Settling $\lambda = 2$ seems to require some new ideas. Nevertheless, the double neighbourhood problem may prove to be more tractable than the leave problem, and hence merits further study.

5. Concluding Remarks

Structural properties of triple systems have long been of interest, in part due to the extensive investigation of embedding problems. The recent emphasis on graph-theoretical problems in design theory has spawned a number of interesting problems on the borderline between graph theory and design theory. In each of the three problems mentioned here, the successes to date have resulted from finding the appropriate mixture of graph-theoretic and design-theoretic tools. The development and refinement of such tools seems to be a very important goal in the understanding of embedding, immersion, and the structure of triple systems.

Acknowledgements

This paper is a summary of an invited address at the Fifteenth Winter School on Abstract Analysis and Topology, held in Srní, Czechoslovakia, in January 1987. I thank the organizing committee, and Vojta Rödl in particular, for the opportunity to present this work. Thanks also to Rudi Mathon and Alex Rosa for many useful discussions on this research.

References

[1] ANDERSEN L. D., HILTON A.J.W. and MENDELSOHN, E., "Embedding partial Steiner triple systems", *Proc. London Math. Soc.* 41 (1980) 557—576.

[2] COLBOURN C. J., "Embedding partial Steiner triple systems in NP-complete", *J. Comb. Thy.* A 35 (1983) 100—105.

[3] COLBOURN C. J., "Realizing small leaves of partial systems", *Ars Combinatoria* 23A (1987).

[4] COLBOURN C. J., "Simple neighbourhoods in triple systems", submitted for publication.

[5] COLBOURN C. J., COLBOURN M. J., and ROSA A., "Completing small partial triple systems", *Discrete Math.* 45 (1983) 165—179.

[6] COLBOURN C. J., HAMM, R. C., and ROSA A., "Embedding, immersing, and enclosing", *Proc. Sixteenth Southeastern Conf. Combinatorics, Graph Theory, and Computing* 1985, pp. 229—236.

[7] COLBOURN C. J. and MCKAY B. D., "Cubic neighbourhoods in triple systems", *Ann. Discrete Math.*, to appear.

[8] COLBOURN C. J. and ROSA A., "Quadratic leaves of maximal partial triple systems", *Graphs and Combinatorics* 2 (1986) 317—337.

[9] COLBOURN C. J. and ROSA A., "Element neighbourhoods in twofold triple systems", *J. Geometry*, to appear.

[10] COLBOURN C. J., and ROSA A., "Quadratic excesses of coverings by triples", submitted for publication.

[11] ENGEL K., "Optimalitätsaussagen über Tripelsystems", *Rostock. Math. Kolloq.* 17 (1981) 17—26.

[12] FORT M. K., and HEDLUND G. A., "Minimal coverings of pairs by triples", *Pacifiic J. Math.* 8 (1958) 709—719.

[13] HILTON A. J. W. and RODGER C. A., "Triangulating nearly complete graphs of odd order", in preparation.

[14] HOLYER I., "The NP-completeness of some edge-partition problems", *SIAM J. Computing* 10 (1981) 713—717.

[15] LINDNER C. C., "A survey of embedding theorems for Steiner systems", *Annals of Discrete Math.* 7 (1980) 175—202.

[16] SPENCER J., "Maximal consistent families of triples", *J. Comb. Thy.* 5 (1968) 1—8.

[17] STANTON R. G. and ROGERS M. J., "Packings and coverings by triples", *Ars Combinatoria* 13 (1982) 61—69.

[18] STINSON D. R. and WALLIS W. D., "Graphs which are not leaves of maximal partial triple systems", *Annals of Discrete Math.*, to appear.