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A Note on Zero-Dimensional Preimages of Compact Spaces

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The aim of this note is to give a partial answer to the following question posed by Eric van Douwen [1]: is every compact (Hausdorff) space a continuous image of a zero-dimensional compact space of the same power? (of the same character?). This question is motivated by the fact that if we ask for the weight the answer is positive. Indeed, a compact space of weight τ can be embedded into the Tychonoff cube $[0, 1]^\tau$ which is a continuous image of the (zero-dimensional) Cantor cube $\{0, 1\}^\tau$. Of course, in the case of power or character this argument cannot be used. However, we shall show that, by some additional assumptions, van Douwen's question can be answered in positive. In particular, under CH, every compact space of power continuum is a continuous image of a zero-dimensional compact space of the same power and every first-countable space is a continuous image of a zero-dimensional first-countable compact space.

We shall use the method of inverse limits. For terminology and basic theorems we refer the reader to R. Engelking [2].

Theorem. Every compact space of power continuum is a continuous image of a zero-dimensional compact space of power not greater than $\sum\{2^\tau: \tau < 2^\omega\}$. Moreover, if the character at every point of the given space is less than the continuum, then the resulting space has the same property.

Proof. Let X be a compact space of the power continuum. Then $X = \{x_\alpha: \alpha < 2^\omega\}$ and for every $\alpha < 2^\omega$ we can fix a base of neighbourhoods $\{U_\alpha^\xi: \xi < \tau_\alpha\}$ at the point x_α, where τ_α is the character of the space X at x_α. Clearly, $\tau_\alpha \leqq 2^\omega$ for every $\alpha < 2^\omega$. Hence there exists a one-to-one function F from 2^ω into $2^\omega \times 2^\omega$ (not necessary onto) such that

(1) for every $\alpha < 2^\omega$ and every $\xi < \tau_\alpha$ there exists $\nu < 2^\omega$ such that $F(\nu) = (\xi, \alpha)$.

We shall construct, by induction, an inverse sequence $\{X_\alpha, p_\alpha^\beta; \alpha < \beta < 2^\omega\}$ consisting of compact spaces X_α and continuous "onto" bonding mappings p_α^β such that

(2) $X_0 = X$

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(3) if α is a limit ordinal, then $X_\alpha = \varprojlim \{X_\beta, p_\beta^\nu;\ \beta < \nu < \alpha\}$ and for $\beta < \alpha$, p_β^α is the projection from the inverse limit onto X_β.

Now we shall need the following claim:

Claim. If Z is a subset of a Tychonoff space X and $|Z| < 2^\omega$, then for every $x \in X$ and for every open neighbourhood U of x there exists an open set $V \subset X$ such that $x \in V \subset U$ and $\mathrm{Bd}\, V \cap Z = \emptyset$, where $\mathrm{Bd}\, V$ is the boundary of V.

For the proof of the claim fix a continuous function $f: X \to [0, 1]$ such that $f(x) = 0$ and $f(X - U) = \{1\}$. Since $|f(Z)| < 2^\omega$, there exists a point $p \in [0, 1] - f(Z)$. It suffices to set $V = f^{-1}([0, p))$.

Assume X_α is just defined for some $\alpha < 2^\omega$. If $F(\alpha) = (\xi, \beta)$, then by the Claim, there exists an open set $V_\alpha \subset X_0$ such that

(4) $x_\beta \in V_\alpha \subset U_\beta^\xi$ and $\mathrm{Bd}\, V_\alpha \cap \{x_\mu$: there exist $\nu < \alpha$ and $\xi < \tau_\mu$ such that $F(\nu) = (\xi, \mu)\} = \emptyset$.

Then the space $X_{\alpha+1}$ is defined to be a subspace of the product $X_\alpha \times \{0, 1\}$. Namely we set

(5) $$X_{\alpha+1} = (X_\alpha - W_\alpha) \times \{0\} \cup \mathrm{cl}\, W_\alpha \times \{1\},$$

where $W_\alpha = (p_0^\alpha)^{-1}(V_\alpha)$. The bonding mapping $p_\alpha^{\alpha+1}$ is the restriction of the projection from $X_\alpha \times \{0, 1\}$ onto X_α.

We shall show that the space $\bar{X} = \varprojlim \{X_\alpha, p_\beta^\alpha;\ \beta < \alpha < 2^\omega\}$ is a zero-dimensional compact space. Compactness follows from the fact that all X_α's are compact. For every $\alpha < 2^\omega$, the canonical projection p_α is the mapping from the limit space $\bar{X}$, onto X_α; note that all bonding mappings are "onto", so the projections are "onto" as well. In particular, by the condition (2), X is a continuous image of $\bar{X}$. To prove zero-dimensionality fix a point $y \in \bar{X}$. Then there exists an index $\beta(y) < 2^\omega$ such that $p_0(y) = x_{\beta(y)}$. Let $\alpha_0 < 2^\omega$ be the smallest index such that for some $\xi < \tau_{\beta(y)}$, $F(\alpha_0) = (\xi, \beta(y))$. By the condition (5), for every $\alpha < 2^\omega$, both $(X_\alpha - W_\alpha) \times \{0\}$ and $\mathrm{cl}\, W_\alpha \times \{1\}$ are closed-open subsets of $X_{\alpha+1}$. Hence, by the condition (4), the family

$$\begin{aligned} B_y = {} & \{p_{\alpha+1}^{-1}(\mathrm{cl}\, W_\alpha \times \{1\}): F(a) = (\xi, \beta(y)) \text{ for some } \xi < \tau_{\beta(y)})\} \cup \\ & \cup \{p_{\alpha+1}^{-1}(\mathrm{cl}\, W_\alpha \times \{1\}): \alpha < \alpha_0 \text{ and } p_{\alpha+1}(y) \in \mathrm{cl}\, W_\alpha \times \{1\}\} \cup \\ & \cup \{p_{\alpha+1}^{-1}((X_\alpha - W_\alpha) \times \{0\}): \alpha < \alpha_0 \text{ and } p_{\alpha+1}(y) \notin \mathrm{cl}\, W_\alpha \times \{1\}\} \end{aligned}$$

consists of closed-open neighbourhoods of y. Clearly, $|B_y| < 2^\omega$ whenever $\tau_{\beta(y)} < 2^\omega$. We shall show by that B_y is a base at the point y. Since $\bar{X}$ is compact it suffices to prove that for every point $z \in \bar{X} - \{y\}$ there exists $H \in B_y$ such that $z \notin H$. If $p_0(y) \neq p_0(z)$, then there exists $\xi < \tau_{\beta(y)}$ such that $p_0(z) \notin U_{\beta(y)}^\xi$. Hence, by conditions (4) and (5), $p_{\alpha+1}(z) \notin \mathrm{cl}\, W_\alpha \times \{1\}$, where $F(\alpha) = (\xi, \beta(y))$; such an α exists by the condition (1). Hence $z \notin p_{\alpha+1}^{-1}(\mathrm{cl}\, W_\alpha \times \{1\})$, which proves our assertion in case $p_0(y) \neq p_0(z)$. In the opposite case there exists a minimal ordinal α such that $0 < \alpha < 2^\omega$ and $p_\alpha(y) \neq p_\alpha(z)$. By the condition (3), α has to be a successor ordinal, say $\alpha = \mu + 1$. Note that $p_0(z) = p_0(y) = x_{\beta(y)}$ and, by the condition (4), for every $\alpha > \alpha_0$ both $p_0(z)$ and $p_0(y)$ do not belong to Bd V_α. Hence, by the condition (5), $p_\nu(y) \neq p_\nu(z)$ for every $\nu > \alpha_0$. Thus $\mu + 1 \leqq \alpha_0$, $p_{\mu+1}(y) \neq p_{\mu+1}(z)$ and $p_\mu(y) = p_\mu(z)$. It follows that if $p_{\mu+1}(y) \in \mathrm{cl}\, W_\mu \times \{1\}$, then $p_{\mu+1}(z) \notin \mathrm{cl}\, W_\mu \times \{1\}$ and conversely. But it proves that z and y are separated by an element of the family B_y.

It remains to show that $|\bar{X}| \leqq \sum\{2^\tau: \tau < 2^\omega\}$. Since for every limit $\alpha < 2^\omega$, X_α is contained in the product $\Pi\{X_\beta: \beta < \alpha\}$, it is easy to check, going by induction, that

$$|X_\alpha| \leqq 2^{|\alpha| + \omega}$$

for every $\alpha < 2^\omega$. Now by conditions (4) and (5), it follows that for every $x \in X$ there exists an index $\alpha(x) < 2^\omega$ such that for every β, $\alpha(x) < \beta < 2^\omega$, the mapping $p^\beta_{\alpha(x)}$ restricted to $(p^\beta_0)^{-1}(x)$ is one-to-one. Hence we have

$$|p^{-1}_{\alpha(x)}((p_0^{\alpha(x)})^{-1}(x))| = |(p_0^{\alpha(x)})^{-1}(x)| \leqq |X_{\alpha(x)}| \leqq 2^{|\alpha(x)| + \omega}.$$

Since $\bar{X} = \cup\{p^{-1}_{\alpha(x)}((p_0^{\alpha(x)})^{-1}(x)): x \in X\}$, we get

$$|\bar{X}| \leqq \sum\{2^\tau: \tau < 2^\omega\};$$

which completes the proof of the theorem.

Corollary 1. If $2^\tau = 2^\omega$ for every τ such that $\omega \leqq \tau < 2^\omega$ then every compact (Hausdorff) space of power continuum is a continuous image of a zero-dimensional compact space of the same power.

Corollary 2. Under the assumption of the continuum hypothesis every first-countable compact space is a continuous image of a zero-dimensional first-countable compact space.

Added in proof: The question studied here was stated before by Arhangel'skij and Ponomarev and was answered under CH (for 1^{st}-countable) by A. V. Ivanov [Usp. Mat. Nauk 35 (1980), 161–162].

References

[1] van Douwen E. K., Topology Proceedings vol. 6 (1981). Problem Section, page 439.
[2] Engelking R.: General Topology, Polish Scientific Publishers, Warszawa 1977.