ERGODIC THEOREMS IN σ-LATTICE CONES

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ABSTRACT. We extend the maximal ergodic theorem of Hopf to the case of σ-lattice cones of Cornea and Licea ([1]). As consequences we prove some abstract potential theory results of maximal type and an abstract pointwise ergodic theorem.

The concept of σ-lattice cone of Cornea and Licea can be viewed as an abstract setting of the cone of positive measurable functions over a measurable space. The aim of this paper is to extend the pointwise ergodic theorem to this abstract case. The large class of nontrivial examples of σ-lattice cones can be used to obtain applications of these results.

For the beginning let us recall some facts from [1].

An ordered convex cone $(C,\leq,+)$ is called a σ-lattice cone if the following conditions are fulfiled:

a) For any $x\in C$ we have $x\geq 0$;

b) For any $x,y\in C$ such that $x\leq y$ there exists $z\in C$ such that $x+z=y$;

c) The ordered set C is a σ-complete lattice;

d) Denoting as usual by "∧" (resp. "∨") the "inf" (resp. the "sup") operation, for every $x\in C$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in C, we have:

$$x\vee(\wedge x_n)=\wedge(x\vee x_n);$$
$$x\wedge(\vee x_n)=\vee(x\wedge x_n);$$
$$x+\wedge x_n=\wedge(x+x_n);$$
$$x+\vee x_n=\vee(x+x_n)\ .$$

If C is a σ-lattice cone, an element $x\in C$ is called finite if for every y such that $y\leq x$, the element $z\in C$ such that $x=y+z$ is unique; that is equivalent with $\bigwedge_{n\geq 1}(1/n)x=0$. The cone of finite elements will be denoted by C_s.

The set $|C|$ defined formally by $|C|=C-C_s$ has a natural lattice structure induced from that of C, in such a way that $|C|$ becomes an upper -σ-complete and conditionally lower -σ- complete lattice. The relations d) hold in $|C|$ also.

If C and C' are σ-lattice cones, a map $T:C \to C'$ is called a kernel if $T0=0$ and if for every sequence $(x_n)_{n\in\mathbb{N}}$ from C we have

$T(\sum_{n=0}^{\infty} x_n)=\sum_{n=0}^{\infty} Tx_n$

A kernel $T:C \to C'$ is called proper if for every $x\in C$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in C, increasing to x, such that $Tx_n\in C'_s$ for every $n\in\mathbb{N}$.

We say that a σ-lattice cone is proper if the identity kernel is proper.

For any $x\in X$ we denote by I_x the map $I_x:C \to C$ defined by:

$$I_x y= \bigvee_{n\in\mathbb{N}} [(nx)\wedge y].$$

It is easy to see that for any $x\in C$, I_x is a kernel with the following properties:

(1) $I_x y\leq y$ for every $y\in C$

(2) $I_x^2=I_x$;

(3) $I_x(\vee x_n)=\vee I_x x_n$

$I_x(\wedge x_n)=\vee I_x x_n$;

$I_{\vee x_n}=\vee I_{x_n}$,

for every sequence $(x_n)_{n\in\mathbb{N}}$ in C.

I_x will be called the indicator of x.

Moreover, if for $z\in|C|=C-C_s$ we set $z^+=z\vee 0$ and $z^-=-z\wedge 0$ (in $|C|$), we have $z=z^+-z^-$ and for every $x\in C$, $y\in C_s$:

$$I_{(x-y)^+}(x-y)^-=0;$$

$$I_{(x-y)^+}x\geq I_{(x-y)^+}y.$$

A measure on C is a kernel $\mu:C \to \bar{\mathbb{R}}_+$. The set of measures on C is a σ-lattice cone which is complete.

If T is a kernel on C, an element $x\in C$ (respectively, a measure μ on C) is called T-supermedian if $Tx\leq x$ (respectively, $\mu(Tx)\leq \leq\mu(x)$ for every $x\in C$). An element $x\in C$ (respectively a measure μ) will be called T-invariant if equalities hold,

If $x\in C_s$ is T-supermedian, the Riesz decomposition theorem asserts that there exist unique $u, v \in C_s$ such that:

$$x = G_T u + v \ ,$$

where $G_T = I + T + \ldots + T^n + \ldots$ and $v = \bigwedge_{n \geq 0} T^n x$ satisfies $Tv = v$.

We also need the following natural construction.

If μ is a measure on the σ-lattice cone C denote by C_o^μ the σ-complete subcone of C of those elements $x \in C$ having zero μ-measure (i.e. $\mu(x)=0$).

Defining in C the equivalence relation $\sim$ by $x \sim y$ iff there exists $x_o \in C_o^\mu$ such that $x \leq y + x_o$ and $y \leq x + x_o$, the set of classes C/C_o^μ becomes a σ-lattice cone. If we denote by $\dot{x}$ the class of $x \in C$, the following then hold:

(1) $\dot{x} \in (C/C_o^\mu)_s$ iff $\bigwedge_{n \geq 1} (1/n) x \in C_o^\mu$;

(2) $\dot{\mu}: C/C_o^\mu \to \overline{\mathbb{R}}_+$ defined by $\dot{\mu}(\dot{x}) = \mu(x)$ is a measure on C/C_o^μ and $\dot{\mu}(\dot{x})=0$ implies $\dot{x}=\dot{0}$;

(3) if μ is T-supermedian the map $\dot{T}$ on C/C_o^μ defined by $\dot{T}\dot{x} = \dot{\widehat{Tx}}$ is a kernel on C/C_o^μ .

Two elements $x, y \in C$ are called μ-almost everywhere (a.e.) equal if $\dot{x} = \dot{y}$.

For a sequence $(x_n)_{n \in \mathbb{N}}$ in C we shall define as usual the upper limit and the lower limit by:

$$\limsup x_n = \bigwedge_n \bigvee_{m \geq n} x_m \ ;$$

$$\liminf x_n = \bigvee_n \bigwedge_{m \geq n} x_m \ .$$

We shall say that the limit of the sequence $(x_n)_{n \in \mathbb{N}}$ exists if $\limsup x_n = \liminf x_n$ and that the limit exists μ-a.e. if $\limsup \dot{x}_n = \liminf \dot{x}_n$. In particular, if $\mu(\limsup x_n) < \infty$ and $\mu(\limsup x_n) = \mu(\liminf x_n)$ the limit exists μ-a.e.

The results of the paper can now be formulated.

The first one is the natural extension of Hopf's maximal ergodic lemma. In order to formulate it, let us introduce the following notation: if T is a kernel on the σ-lattice cone C satisfying

$TC_s \subset C_s$ and $x \varepsilon |C|$, let us denote by $r_n(x,T)=r_n(x)$ the elements defined inductively by $r_o(x)=0$, $r_n(x)=x+Tr_{n-1}(x)$, $n \geq 1$.

PROPOSITION. (Maximal ergodic lemma) *Let* C *be a* σ-*lattice cone,* T *a kernel on* C *satisfying* $TC_s \subset C_s$ *and* μ *a proper* T-*supermedian measure. If* $x=x'-x" \varepsilon |C|$ $(x' \varepsilon C,\ x" \varepsilon C_s)$ *and* $X_N = \bigvee_{n=1}^{N} r_n(x,T)$, $N \geq 1$, *we have:*

$$\mu(I_{X_N^+} x') \geq \mu(I_{X_N^+} x") \quad \textit{for every} \quad N \geq 1.$$

<u>Proof</u>. We shall use the same trick as in the proof of Garcia for the classical ergodic lemma ([2]).

First, let us suppose that $\mu(x')$ and $\mu(x")$ are finite. From the fact that $X_N^+ \geq r_n(x,T)$ we infer that $TX_N^+ \geq Tr_n(x,T)$ for every $n=0,\ldots,N-1$ (we put $r_o=0$). Adding x in both sides of the last inequality, we obtain that $TX_N^+ + x \geq r_{n+1}$ for $n=0,\ldots,N-1$, that is:

$$TX_N^+ + x \geq X_N ,$$

or:

$$TX_N^+ + x' + X_N^- \geq x" + X_N^+ .$$

If we apply the kernel $I=I_{X_N^+}$ to the last inequality, we obtain:

$$ITX_N^+ + Ix' \geq Ix" + IX_N^+ = Ix" + X_N^+ ,$$

and

$$\mu(ITX_X^+) + \mu(Ix') \geq \mu(Ix") + \mu(X_N^+) .$$

Using the facts that I≤identity and that μ is T-supermedian together with $\mu(X_N^+) < \infty$, we obtain the anounced inequality.

If $x' \varepsilon C$ or $x" \varepsilon C_s$ have infinite measure, it will suffice to use the fact that μ is proper; standard limit arguments will conclude the proof.

The following consequences of the preceding result can be viewed as abstract potential theory results.

THEOREM 1. *Let* C, T *and* μ *satisfy the assumptions of the proposition and let* $x, y \varepsilon C_s$, y *being* T-*invariant. The following are then true:*

(i) $y \geq \bigwedge_{n=1}^{\infty} (1/n) r_n(x,T)$ *implies* $\mu(y) \geq \mu(I_y x)$;

(ii) $y \leq \bigvee_{n=1}^{\infty} (1/n) r_n(x,T)$ *implies* $\mu(y) \leq \mu(I_y x)$.

Proof. We shall apply the preceding proposition for $z=\varepsilon y-x$, where $\varepsilon>1$ is arbitrary. We have:

$$(I_{Z_N^+} \varepsilon y) \geq \mu(I_{Z_N^+} x) ,$$

where $Z_N = \bigvee_{n=1}^{N} r_n(z,T)$. Let N tend to infinity (the sequence Z_N^+ being increasing). We obtain:

$$(*) \qquad \mu(I_{Z^+} \varepsilon y) \geq \mu(I_{Z^+} x) ,$$

where

$$I_{Z^+} = I_{[\bigvee_{n=1}^{\infty} r_n(z,T)]^+} = I_{[\varepsilon y - \bigwedge_{n=1}^{\infty} 1/n(x+Tx+\ldots+T^{n-1}x)]^+}$$

the last equality being an easy consequence of the T-invariance of y and the distributivity laws in $|C|$.

Moreover, the inequalities $y \geq \bigwedge_{n=1}^{\infty} ((1/n)\cdot r_n(x,T))$ and $\varepsilon>1$ imply, as a direct consequence of the definition of the indicator kernel, that $I_{Z^+}=I_y$. Thus the inequality (*) can be written:

$$\varepsilon\mu(y) = \mu(I_y \varepsilon y) \geq \mu(I_y x) .$$

In order to obtain the inequality (i) it is sufficient to consider ε 1.

The proof of (ii) goes along the same way if we apply the ergodic lemma to $x-\varepsilon y$, where $0<\varepsilon<1$.

The following is an immediate consequence of theorem 1.

COROLLARY 1. *Let* C, T *and* μ *satisfy the preceding assumptions and let* $x \in C_s$ *have finite* μ*-measure. Then every* T*-invariant finite element* $y \in C$ *satisfying* $x \leq y \leq \bigvee_{n=1}^{\infty} (1/n) r_n(x,T)$ *equals* x μ-a.e. *Similarly, every* T*-invariant* *element* $y \in C$ *having the same support as* x μ-a.e. *(that is* $\mu(I_y x) = \mu(x)$*) and satisfying* $\bigwedge_{n=1}^{\infty} (1/n) r_n(x,T) \le y \le x$*, equals* x μ-a.e.

Proof. For the first part we have from Theorem 1 (i) that $\mu(y) \le \mu(I_y x)$. But $\mu(I_y x) \le \mu(x)$ so $\mu(x) = \mu(y)$, which combined with $y \ge x$ and $\mu(x) < \infty$ concludes the proof.

Similarly the proof of the second part makes use of Theorem 1 (ii).

It is interesting to apply this corollary in the case when C is a cone of positive measurable functions on a σ-finite measure space, $(X, \mathcal{X}, \mu)$ and T restricted to $L_1(X, \mathcal{X}, \mu) \cap C$ is a positive contraction. For example if $f \in L_1 \cap C$ and $\sup_{n \ge 1} (1/n)(f + Tf + \ldots + T^{n-1} f) = \infty$ μ-a.e., our results asserts that there exists no T-invariant finite positive measurable function greater than f μ-a.e. Also if $f \ne 0$ is in $L_1 \cap C$ and $\inf_{n \ge 1} (1/n)(f + Tf + \ldots + T^{n-1} f) = 0$ μ-a.e., than there exists no T-invariant measurable positive function less than f μ-a.e. and having μ-a.e. the same support as f.

The second corollary can be viewed as a disjointness result in the Riesz decomposition.

COROLLARY 2. *Let* C, T *and* μ *be as above. Suppose that* $x \in C_s$ *is* T*-supermedian and* $x = G_T u + v$ *is the Riesz decomposition. Then:*

$$\mu(v) = \mu(I_v x).$$

In particular if $\mu(x) < \infty$ *we have* $\mu(I_v G_T u) = 0$*, that is the invariant part and the potential part have* μ-a.e. *disjoint supports.*

Proof. From theorem 1 (i) we have that $\mu(v) \ge \mu(I_v x)$ because v is invariant and $v = \bigwedge_{n \ge 1}^{\infty} T^n x = \bigwedge_{n=1}^{\infty} (1/n) r_n(T,x)$. The opposite inequality is obvious. For the second part apply the kernel I_v and the measure μ to $x = G_T u + v$.

Our generalisation of the pointwise ergodic theorem is also a consequence of theorem 1. However the abstract setting and the absence of units involves some more assumptions.

THEOREM 2. (Ergodic theorem). *Let* C, T *be as above and let* μ

be a T-invariant proper measure. Let $x \in C$ *and suppose that* $\mu(\limsup_{n\to\infty} (1/n)r_n(T,x))<\infty$. *Then the following are equivalent:*

a) $\limsup(1/n)r_n(T,x)$ *and* $\liminf(1/n)r_n(T,x)$ *have* μ-a.e. *the same support;*

b) *the limit of* $(1/n)r_n(T,x)$ *exists* μ-a.e. *Moreover in every case we have:*

$$\mu(\liminf(1/n)r_n(T,x))=\mu(\limsup(1/n)r_n(T,x))=\mu(I_{\liminf(1/n)r_n(x,T)}x)$$

<u>Proof</u>. Let us use the following notations:

$$x^*=\limsup(1/n)r_n(T,x)$$

$$x_*=\liminf(1/n)r_n(T,x)$$

By standard arguments we have $Tx_*\leq x_*$ and $\dot{T}\dot{x}^*\geq\dot{x}^*$, which implies, by the T-invariance of the measure μ and the supposition that x^* has μ-finite measure that $\dot{x}^*$ and $\dot{x}_*$ are $\dot{T}$-invariant in $(C/C_o^{\mu})_s$.

The implication b) $\Rightarrow$ a) being obvious, in order to prove the opposite one, let us remark that $x^*\leq \bigvee_{n=1}^{\infty} (1/n)r_n(T,x)$ and $x_*\geq \bigwedge_{n=1}^{\infty} (1/n)r_n(T,x)$, so by Theorem 1 used in $C/_{C_o^{\mu}}$, we have:

$$\dot{\mu}(\dot{x}_*)\geq\dot{\mu}(I_{\dot{x}_*}\dot{x})$$

and

$$\mu(\dot{x}^*)\leq\dot{\mu}(I_{\dot{x}^*}\dot{x})$$

As, by usual arguments, it is easily seen that $\dot{\mu}(I_{\dot{x}_*}\dot{x})=\mu(I_{x_*}x)$ and $\dot{\mu}(I_{\dot{x}^*}\dot{x})=\mu(I_{x^*}x)$, the last two inequalities conclude the proof.

Finally, let us remark that in the classical L_1-case discussed above, Theorem 2 gives necessary and sufficient conditions that, for $f\in L_1$, $f\geq 0$, the ergodic average converges μ-a.e., in the case that $\limsup_{n\to\infty} 1/n(f+Tf+\ldots+T^{n-1}f)$ is integrable, without knowing the L_∞-behaviour of T.

REFERENCES

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