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THE SUPERPOSITION OPERATOR IN MUSIELAK-ORLICZ SPACES OF VECTOR-VALUED FUNCTIONS

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This note is a selection of results obtained by the author lately. It concerns properties of the superposition operator acting from the Musielak-Orlicz space into another one. Moreover, there is formulated a theorem for compactness of the integral Hammerstein operator acting from Musielak-Orlicz space of vector-valued functions into Musielak-Orlicz space of real functions. Finally, there is given, as an application above results, the theorem on the existence of solutions of Hammerstein integral equations in Musielak-Orlicz space of real functions.

These results have been presented on the 14th Winter School on Abstract Analysis in January 1986 which was organized in Srni by the Union of Czechoslovak Mathematicians and Physicists, the Faculty of Mathematics and Physics of Charles University and Institute of Mathematics of Czechoslovak Academy of Sciences.

1. Introduction. Let (T,Σ,μ) be a space of non-atomic, complete, positive, σ-finite and separable measure. $(X,\|\cdot\|_X)$ denotes a reflexive and separable real Banach space.

Definition 1. A function $M:X\times T\to[0,\infty]$ is said to be an $\mathcal{N}$-function, iff

a) M is $\mathcal{B}\times\Sigma$-measurable, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of X,

b) $M(\cdot,t)$ is even, convex, lower semicontinuous, continuous at zero and $M(0,t)=0$ for a.a. $t\in T$,

c) $M(u,t)\to\infty$ as $\|u\|_X\to\infty$ for a.a. $t\in T$.

Let us assume that $\mathcal{N}$-function satisfies the so-called Condition (B), which can be also formulated in the following simple form (see [9] Remark 1.5)

(B): For every natural numbers n and i

$$\int\limits_{T_n} \sup_{\|u\|_X<1} M(u,t)\,d\mu < \infty,$$

where $\{T_n\}$ is an increasing sequence of measurable sets such that $\mu(T_n) < \infty$ and $\bigcup_{n=1}^{\infty} T_n = T$.

In the following by $\mathfrak{X}_X$ we will denote the set of all strongly measurable functions from T into X. The famous Pettis theorem states that the strong measurability and the weak one are equivalent for separable Banach spaces. Therefore, we will say shortly "measurable function".

Remark 1. Elements of the set $\mathfrak{X}_X$ will be denoted $x(\cdot)$, $y(\cdot)$, $z(\cdot)$ or, in order to simplify the notation, we will omit sometimes the brackets when it does not lead to a misunderstanding. Symbols u, v will be used for vectors from the space X.

By Musielak-Orlicz space L_M we mean the set of all functions $x \in \mathfrak{X}_X$ for which there exists a constant $k > 0$ such that

$$I_M(kx) = \int\limits_T M(kx(t),t)\,d\mu < \infty.$$

The functional

$$\|x\|_M = \inf\{a > 0:\ I_M(a^{-1}x) \leq 1\}$$

is a norm in L_M. It is called the Luxemburg norm. If $M(u,t)=M(u,s)$ for every $t,s \in T$, then we say that M is generated the space L_M which is called an Orlicz space.

By E_M we denote a subspace of finite elements i.e.

$$E_M = \{x \in \mathfrak{X}_X:\ I_M(kx) < \infty \text{ for every } k > 0\}.$$

Obviously, $E_M \subset L_M$. The space E_M equals the space of all $x \in L_M$ possessing absolutely continuous norms (see [7] Theorem 1.2). What concerns properties of the space E_M we refer to [1], [2] and [5] (Theorem 1.15).

2. The superposition operator and its fundamental properties.

Definition 2. Suppose the function $f: T \times X \to X$ satisfies the Carathéodory conditions, i.e. it is continuous in $u \in X$ for almost all $t \in T$ and measurable for every $u \in X$. The operator F, defined by the formula

$$[Fx](t) = f(t,x(t)),$$

where $x \in \mathfrak{X}_X$, is called a superposition operator.

Now, we present the fundamental properties of this operator.

Property 1. The operator F transforms measurable functions into measurable functions.

Proof of this fact is in [5] (Corollary 2.2).

Property 2. The superposition operator has a partial additivity property, i.e. for functions $x_1, x_2, \ldots, x_n$ such that for $i \neq j$

$$\operatorname{supp} x_i \cap \operatorname{supp} x_j = \emptyset$$

there holds the equality

$$F(x_1 + x_2 + \ldots + x_n) = Fx_1 + Fx_2 + \ldots + Fx_n - (n-1)F(0),$$

where 0 denotes a function equal to zero.

Proof of this property is obvious.

Property 3. If $\mu(T) < \infty$, then the superposition operator transforms sequences of functions which are convergent in measure into sequences of functions which are convergent in measure also.

This statement is shown in [5] Theorem 2.6.

Let $d(x, E_M)$ be a distance between $x \in L_M$ and the subspace E_M defined by the equality

$$d(x, E_M) = \inf\{\|x - y\|_M : y \in E_M\}.$$

By $\Pi(E_M, r)$ we denote a set of all functions $x \in L_M$ for which $d(x, E_M) < r$. If M satisfies Δ_2-condition, then the set $\Pi(E_M, r)$ is equal to the whole Musielak-Orlicz space L_M. Denote by $S_M(r)$ a ball with radius r and center at the null-function $0 \in L_M$, i.e.

$$S_M(r) = \{x \in L_M : \|x\|_M < r\}.$$

Let M_1 and M_2 be two $\mathcal{N}$-functions.

Property 4.

a) If the operator F acts from a ball $S_{M_1}(r)$ into the space L_{M_2} or E_{M_2}, then the operator F can be extended to the operator acting from $\Pi(E_{M_1}, r)$ into the space L_{M_2} or E_{M_2}, respectively.

If the operator F acts from $S_{M_1}(r)$ into Orlicz class

$$\operatorname{dom} I_{M_2} = \{x \in L_{M_2} : I_{M_2}(x) < \infty\}$$

and if additionaly $F0 = 0$, then F acts from $\Pi(E_{M_1}, r)$ into $\operatorname{dom} I_{M_2}$.

b) If the operator F acts from a ball

$$S^E_{M_1}(r) = \{x \in E_{M_1} : \|x\|_{M_1} < r\}$$

into L_{M_2} or E_{M_2}, then it can be extended to the operator acting from all of E_{M_1} into L_{M_2} or E_{M_2}, respectively.

If $F0 = 0$ and $F[S^E_{M_1}(r)] \subset \mathrm{dom}\, I_{M_2}$, then F acts from all of E_{M_1} into $\mathrm{dom}\, I_{M_2}$.

Proof of this fact can be found in [5] Theorem 2.4.

Definition 3. We will say that the family $\mathcal{R}$ of functions $x \in L_M$ has equi-absolutely continuous norms if for every $\varepsilon > 0$ and for every decreasing sequence of sets $C_n \downarrow \emptyset$ as $n \to \infty$, an n_0 can be found such that

$$\|x\chi_{C_n}\|_M < \varepsilon$$

for all functions of the family $\mathcal{R}$, provided $n > n_0$.

Combining methods from paper [7] with the proof of Theorem 2.5 from paper [5] we obtain

Property 5. If the operator F acts from $\Pi(E_{M_1}, r)$ into E_{M_2}, then the operator F transforms a family of functions with equi-absolutely continuous norms into family with equi-absolutely continuous norms.

Now, we formulate two very important theorems.

Theorem 1. (On continuity of superposition operator).

If the operator F acts from $\Pi(E_{M_1}, r)$ into E_{M_2}, then F is continuous at every point of $\Pi(E_{M_1}, r)$.

For the proof of this theorem we refer to [5] Theorem 3.1.

The boundedness of the superposition operator was proved in [6] by dint of introducing and applying the notion of an absolutely continuous modular.

Theorem 2. (On boundedness of superposition operator)

Suppose the operator F acts from the ball $S_{M_1}(r)$ into $\mathrm{dom}\, I_{M_2}$. Then F is bounded on any ball $S_{M_1}(r_0)$ for $r_0 < r$, i.e.

$$\sup_{\|x\|_{M_1} < r_0} \|Fx\|_{M_2} < \infty .$$

3. Compactness of Hammerstein integral operator.

Let $(Y, \|\cdot\|_Y)$ be the dual space to the space X. Similarly as above, we denote by $\mathcal{X}_Y$ the set of all strongly measurable functions from T into Y. Let $\langle v, u\rangle$ stand for the value of the functional $v \in Y$ at the point $u \in X$. Obviously, for $x \in \mathcal{X}_X$ and $y \in \mathcal{X}_Y$ the function $\langle y(\cdot), x(\cdot)\rangle$ is μ-measurable. For every $\mathcal{N}$-function M we define the complementary function $N: Y \times T \to [0, \infty]$ by the formula

$$N(v,t) = \sup\{\langle v, u\rangle - M(u,t)\}$$

for every $t \in T$, $v \in Y$. The function N is $\mathcal{N}$-function too. The Musielak-Orlicz space generated by the function N is denoted by L_N and it is called conjugate to the space L_M. The Luxemburg norm for L_N, is denoted $\|\cdot\|_N$.

One can consider another norm in the space L_M which is defined by formula

$$\|x\|_M^o = \sup_{I_N(y)<1} \left| \int_T \langle y(t), x(t) \rangle d\mu \right|,$$

where

$$I_N(y) = \int_T N(y(t), t) d\mu .$$

The norm $\|\cdot\|_M$ is called the Orlicz norm. The Orlicz norm for $\mathcal{N}$-function N, we define

$$\|y\|_N^o = \sup_{I_M(x)<1} \left| \int_T \langle y(t), x(t) \rangle d\mu \right|.$$

The Orlicz and Luxemburg norms are equivalent.

Let (T_1, Σ_1, μ_1) and (T_2, Σ_2, μ_2) be non-atomic, complete, separable and σ-finite measure spaces. Assume that $M_1: X \times T_1 \longrightarrow [0, \infty]$ and $M_2: R \times T_2 \longrightarrow [0, \infty]$ are $\mathcal{N}$-functions. Let function $K: T_2 \times T_1 \longrightarrow Y$ be strongly $\mu_2 \times \mu_1$-measurable. Then for every strongly μ_1-measurable vector-valued function $x: T_1 \longrightarrow X$, it is evident that $\langle K(\cdot\cdot, \cdot), x(\cdot) \rangle$ is strongly $\mu_2 \times \mu_1$-measurable on $T_2 \times T_1$.

Theorem 3. (On compactness of the linear integral operator).

If the kernel $K(t,s)$, as a function of the variable s belongs to E_{N_1} for a.a. $t \in T$ and $\|K(\cdot\cdot, \cdot)\|_{N_1} \in E_{M_1}$, then the linear integral operator

$$Ax(t) = \int_{T_1} \langle K(t,s), x(s) \rangle d\mu_1$$

is a compact operator from the space L_{M_1} into the space E_{M_2}.

Definition 4. The non-linear integral operator

$$Hx(t) = \int_{T_1} \langle K(t,s), f(s, x(s)) \rangle d\mu_1,$$

where the function $f: T_1 \times X \longrightarrow X$ satisfies the Carathéodory conditions, is called the Hammerstein operator.

This operator can be represented as the composition of the superposition operator F and the linear operator A i.e. $H = A \circ F$. Let $M_3: X \times T_1 \longrightarrow [0, \infty]$ be $\mathcal{N}$-function. Combining the conditions under which the operator F acting from the space L_{M_1} into the space L_{M_3}

is continuous and bounded with the conditions under which the operator A acts from L_{M_3} into L_{M_2} and it is compact, we arrive at the conditions for compactness of the operator H.

Theorem 4. (On complete continuity of the Hammerstein operator).

Let M_1 satisfies the Δ_2-condition. If the kernel $K(t,s)$ as a function of s belongs to E_{N_3} for almost all $t \in T_2$ and $\|K(\cdot\cdot,\cdot)\|_{N_3} \in E_{M_2}$ and F acts from L_{M_1} into E_{M_3}, then the Hammerstein operator H is a continuous and a compact operator from the space L_{M_1} into the space E_{M_2}.

Remark 2. The symbols $\|K(\cdot\cdot,\cdot)\|_{N_3}$ denotes that the norm is calculated for K as a function of this variable which is denoted by one full-stop. $\|K(\cdot\cdot,\cdot)\|_{N_3}$ is an element of E_{M_2} as a function of this variable which is symbolized by two full-stops.

Finishing, we apply this result to the theorem on existence of solutions of Hammerstein integral equations in Musielak-Orlicz space. To this end, we suppose: $T_1 = T_2 = T$, $\mu_1 = \mu_2 = \mu$, $X = Y = R$, $M_1 = M_2 = M_3 = M$ and $N_1 = N_2 = N_3 = N$. We consider the integral equation

$$\text{(H)} \qquad x(t) = \varkappa\int_T K(t,s)f\big(s,x(s)\big)d\mu + z(t) ,$$

where $\varkappa$ is a real number, $z \in E_M$ and the kernel $K: T\times T \to R$ is $\mu\times\mu$-measurable function.

Theorem 5. (On existence of solutions of Hammerstein integral equations).

Let $\mathcal{N}$-function M satisfies Δ_2-condition and the superposition acts from L_M into itself. If the function $K(t,\cdot) \in L_N$ for almost all $t \in T$ and $\|K(\cdot\cdot,\cdot)\|_N \in L_M$, then for every $r > \|z\|_M^0$ the integral equation (H), where $z \in L_M$, has at least one solution in the ball $S_M^E(r)$ for

$$|\varkappa| < \frac{r - \|z\|_M^0}{a\big\|\|K(\cdot\cdot,\cdot)\|_N\big\|_M} ,$$

where $a = \sup_{x\in S_M^E(r)} \|Fx\|_M^0$.

For the proof we refer to [8].

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