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ON GENERALIZED BLUMBERG SETS

Tibor Neubrunn

A close connection between quasicontinuity and Blumberg sets of functions was observed by J.C. Neugebauer [10]. Since that time many authors have been interested in the study of Blumberg sets of functions and related problems. The connection between quasicontinuity and Blumberg sets of functions were studied e.g. in [8], [11], [12]. The paper [8] studies the last mentioned connections for multi-valued mappings.

The main aim of the present note is to show that when we study the Blumberg sets in connection with the quasicontinuity, then sometime a weaker form of the Blumberg set is more appropriate. So we introduce quasi-Blumberg (shortly q-Blumberg) sets, showing that some results may be stated in a more general, and in any case, in a more natural way. The results are stated mostly for multi-valued mappings. The corresponding ones for single-valued mappings are obtained as corollaries.

Notations and notions

Given two topological spaces X, Y, we call a mapping $F: X \to P(Y)$, where $P(Y)$ is the power set of Y, a multi-valued mapping or a multifunction. We suppose $F(x) \neq \emptyset$ for any $x \in X$ and write $F: X \to Y$ instead of $F: X \to P(Y)$, for shortness.

A multifunction $F: X \to Y$ is said to be upper (lower) quasicontinuous at $x \in X$ if for any open $V \subset Y$, $V \supset F(x)$ $(V \cap F(x) \neq \emptyset)$ and any open U containing x, there exists a nonempty open set $G \subset U$ such that $F(t) \subset V$ $(F(t) \cap V \neq \emptyset)$ for any $t \in G$. F is said to be upper (lower) quasicontinuous if it is upper (lower) quasicontinuous at any $x \in X$. We use the notation u.q.c (l.q.c) to denote upper (lower) quasicontinuity.

Note that u.q.c (l.q.c) is a generalization of the well-known upper (lower) semi-continuity of a multifunction (see e.g. [5] p. 405).

Further, note that a single-valued mapping $f: X \to Y$ may be considered as a multi-valued mapping. We identify $f(x)$ with the singleton $\{f(x)\}$. In the case of a single-valued mapping both the u.q.c and l.q.c coincide with the usual notion of the quasicontinuity (see e.g [4], [7]). Similarly upper and lower semi-continuity in case of a single-valued mapping coincide with the usual notion of the continuity.

A collection $\mathcal{A}$ of subsets of a set X is said to be upper (lower) dense in a collection $\mathcal{B}$ of subsets of X if for any $B \in \mathcal{B}$ and any open V such that $V \supset B$ ($V \cap B \neq \emptyset$), there exists $A \in \mathcal{A}$ such that $A \subset V$ ($A \cap V \neq \emptyset$). We say that $\mathcal{A}$ is upper (lower) dense if it is upper (lower) dense in the collection of all nonempty open sets. Obviously any of the last two notions gives the notion of the usual density if $\mathcal{A}$, $\mathcal{B}$ are taken as subsets of X, while the elements x are identiffied with the singletons $\{x\}$.

Recall that the usual Blumberg set for a single-valued mapping $f: X \to Y$ is introduced as a dense set $D \subset X$, such that the restriction f/D is continuous.

Now we define various kinds of Blumberg sets for multifunction $F: X \to Y$.

A dense set $A \subset X$ is called an upper (lower) Blumberg set for a multifunction $F: X \to Y$ (see [8]) if the restriction F/D is upper (lower) semi-continuous. It is called an upper (lower) q-Blumberg set for F if F/D is u.q.c (l.q.c).

If f is single-valued the notions of upper (lower) Blumberg sets of f, both coincide with the above defined Blumberg set for f. The notions of upper and lower q-Blumberg set give in that case the quasi-Blumberg set as defined in [9].

A Blumberg set D for the multifunction $F: X \to Y$ is called full upper Blumberg (full lower Blumberg set) if $F(G \cap D)$ is upper (lower) dense in $F(G)$ for any open set $G \subset X$. Here $F(A) = \{F(x): x \in A\}$. In case of a single-valued mapping f theese two notions coincide and give the usual full Blumberg set for f [11].

In an obvious way the full upper q-Blumberg set and full lower q-Blumberg set is defined.

Results related to quasicontinuity

One of the reasons why the q-Blumberg sets are suitable when we study the relations to quasicontinuity gives the following

Proposition 1. A multifunction $F: X \to Y$ is u.q.c. (l.q.c) if and only if any dense set D is its upper (lower) q-Blumberg set.

Proof. To prove the sufficiency it suffices to take $D = X$. To prove the necessity we only have to prove that the upper (lower) quasicontinuity is a hereditary property with respect to dense sets. (The reader may observe that it is not hereditary with respect to arbitrary set). We prove the herediterness of l.q.c. with respect to dense set. So let $D \subset X$ be dense and F be l.q.c. Take $x \in D$ and $V \subset Y$ open such that $F(x) \cap V \neq \emptyset$ and W open in D such that $x \in W$. Then $W = U \cap D$ where U is open. By l.q.c of F at x there is G open, $G \neq \emptyset$, $G \subset U$ such that $F(t) \cap V \neq \emptyset$ for any $t \in G$. Then $G \cap D \neq \emptyset$, $G \cap D \subset U$, $G \cap D$ is open in D and $F(t) \subset V$ for any $t \in G \cap D$. The lower quasicontinuity of F/D at x is proved.

To characterize the quasicontinuity of a single-valued function the full Blumberg theorem is used. ([11]). Certain characterizations of u.q.c and l.q.c of multifunctions is also possible by means of full upper and full lower Blumberg sets respectively ([8]). To give such characterization it is sufficient to consider full upper and full lower q-Blumberg sets. We obtain the following theorems. We omit the proofs which are modifications of the proofs of more special theorems proved in [8].

Theorem 1. Any upper (lower) q-Blumberg set D of an u.q.c (l.q.c) multifunction $F: X \to Y$ is its full upper (lower) q-Blumberg set, i.e. $F(G \cap D)$ is upper (lower) dense in $F(G)$ for any open set G.

Theorem 2. If Y is a regular space and $F: X \to Y$ a multifunction having a full lower q-Blumberg set D with such a property that $F(D \cap G)$ is upper dense in $F(G)$ for any G open, then F is l.q.c.

A result analogical to Theorem 2 may be obtained if we suppose normality of the space Y and the existence of such a full upper q-Blumberg set D for which $F(D \cap G)$ is lower dense in $F(G)$ for any G open. As a consequence we obtain that F is u.q.c.

As a corollary of Theorem 1 and Theorem 2 we obtain for single-valued mappings the following result which generalizes a result proved for Blumberg sets in [11].

Corollary. Let Y be a regular space. Let D be a q-Blumberg set for a single-valued mapping $f: X \to Y$. Then f is quasicontinuous if and only if D is full q-Blumberg set for f.

In what follows we shall define certain kind of Blumberg sets which are closely related to a generalized type of quasicontinuity.

Somewhat continuity

One of related types of generalized continuity is the somewhat continuity. It was introduced by Gentry and Hoyle [3] for single-valued mappings. The notion of the upper and lower somewhat continuity is defined in the following way.

A multifunction $F: X \to Y$ is said to be upper (lower) somewhat continuous if for every open $V \subset Y$ such that $F^+(V) \neq \emptyset$ ($F^-(V) \neq \emptyset$), we have Int $F^+(V) \neq \emptyset$ (Int $F^-(V) \neq \emptyset$) (where $F^+(V) = \{x: F(x) \subset V\}$, $F^-(V) = \{x: F(x) \cap V \neq \emptyset\}$).

Both theese notions in case of a single-valued mapping $f: X \to Y$ coincide and gives the somewhat continuity as defined in [3].

The well known characterization ([3]) of somewhat continuous single-valued mappings claims: f is somewhat continuous iff for any $S \subset X$, S dense in X the set f(S) is dense in f(X). The corresponding characterization for upper (lower) somewhat continuity of a multi-valued mapping $F: X \to Y$ is quite analogical.

Proposition 2. A multi-valued mapping $F: X \to Y$ is upper (lower) somewhat continuous iff for any dense set $D \subset X$ the set F(D) is upper (lower) dense in F(X).

One can introduce the somewhat Blumberg set (s-Blumberg set) as follows.

A dense set $D \subset X$ is an upper (lower) s-Blumberg set for the multifunction $F: X \to Y$ if F/D is upper (lower) somewhat continuous.

An analogy of Proposition 1 for s-Blumberg sets is the following

Proposition 3. A multifunction $F: X \to Y$ is upper (lower) s-continuous iff any dense set $D \subset X$ is its upper (lower) s-Blumberg set.

One can obtain for somewhat continuous mapping the following weaker analogy of Theorem 1

Proposition 4. If $D \subset X$ is an upper (lower) s-Blumberg set for a multifunction $F: X \to Y$, then F(D) is upper (lower) dense in F(X).

Proof. It is an immediate corollary of Proposition 2.

So, by Proposition 4, an upper (lower) s-Blumberg set is "full" in some weaker sense.

We say that an s-Blumberg set $D \subset X$ is almost full upper (lower) s-Blumberg set for F if F(D) is upper (lower) dense in F(X).

In connection with considerations concerning quasi-continuity (see Theorem 1, Theorem 2 and its Corollary) and with Proposition 4, one may ask if the existence of an almost full s-Blumberg set does not imply the corresponding somewhat continuity. To be more clear if e.g the existence of an s-Blumberg set for a single-valued mapping f does not imply somewhat continuity of f. The negative answer is known (11 Example 2) shows that there may exist such a dense set D X and a single-valued mapping f: X Y that f/D is continuous (hence somewhat continuous) but f is not somewhat continuous.

Nevertheless for certain type of multifunctions a positive answer may be given

Theorem 3. Let Y be a regular space and F: X Y a multifunction. Let D be such an almost full lower s-Blumberg set for F that F(D G) is upper dense in F(G) for any G open. Let the set of lower quasicontinuity points of F be dense in X. Then F is lower s-continuous.

Proof. Let V Y be such that $F^{-}(V) \neq \emptyset$. We have to prove that

$$\text{Int } F^{-}(V) \neq \emptyset \qquad (1)$$

Take x_0 $F^{-}(V)$. Let y_0 V $F(x_0)$. By the regularity of Y there exists an open set V_1 such that

$$y_0 \quad V_1 \quad \overline{V_1} \quad V \qquad (2)$$

Since D is almost full lower s-Blumberg set for F, we have that F(D) is lower dense in F(X). Thus there exists d D such that F(d) $V_1 \neq \emptyset$. By lower s-continuity of F/D we have

$$\text{Int}_D\, F^{-}(V_1) \quad D \neq \emptyset \qquad (3)$$

So by (3) there exists a nonempty open set G such that

$$G \quad D \quad F^{-}(V_1) \quad D \qquad (4)$$

By the assumption, there is a lower quasicontinuity point z G. We prove that F(z) $V \neq \emptyset$. In the opposite case we would have by (2) F(z) X - $\overline{V_1}$ and by the assumption of the upper density of F(D G) in F(G) there exists a point x D G such that F(x) X - $\overline{V_1}$, in contradiction with (4). So F(z) $V \neq \emptyset$. Now, using lower quasicontinuity of F at z we obtain that a nonempty open U G exists such that U $F^{-}(V)$. Thus (1) is proved.

A special form of the following Corollary was proved in 9 .

Corollary. If Y is a regular space and f: X Y a single-valued mapping with a dense set of quasicontinuity points and if f has an almost full s-Blumberg set, then f is somewhat continuous.

The assumption concerning the density of the set of quasicontinuity points may not be omitted as the following example shows.

Example 2. Let $X = Y = \langle 0,1 \rangle$ with the usual topology. Put $f(x) = x$ if x is rational, $f(x) = 1 - x$, if x is irrational. The mapping f is not somewhat continuous but it has an almost full s-Blumberg set.

The regularity in Corollary 2 may not be omitted as well as the assumption of the upper density of $F(D \cap G)$ in $F(G)$ for any $G \subset X$, G open. The first can be easily deduced from a simple example ([9] Example 2), which shows that the regularity can not be omitted even in a more special case. The second follows from the following.

Example 3. Let $X = Y = \langle 0,1 \rangle$ with the usual topology. Define the multifunction $F: X \to Y$ as $F(x) = \{0\}$ if x is rational and $F(x) = \langle 0,x \rangle$ if x is irrational.

Taking D the set of all irrational numbers in $\langle 0,1 \rangle$, we have that D is almost full lower s-Blumberg set for F. The set of all rational points is a dense set of lower quasicontinuity points of F. But F is not lower somewhat continuous.

Remark. A more detailed examination of the proof of Theorem 3 shows that the assumption $F(D \cap G)$ is upper dense in $F(G)$ was used only partly. In fact we used only the fact that $F(D \cap G)$ is dense in $F(S \cap G)$ where S is the set of all lower quasicontinuity points of F.

Using the last Remark we may prove the following sufficient condition for the lower somewhat continuity of F.

Theorem 4. Let Y be a regular space. Let D be an almost full lower s-Blumberg set for F and let the set S of all such points which are simultaneously lower and upper quasicontinuity points of F be dense in X. Then F is lower somewhat continuous.

Proof. It is sufficient to prove that under our assumptions the assumptions of Theorem 4 are satisfied. The only thing to be veriffied is the upper density of $F(D \cap G)$ in $F(G)$ for every open $G \subset X$. But, in view of Remark, it is sufficient to verify that $F(D \cap G)$ is upper dense in $F(S \cap G)$. So take $F(z)$ where $z \in S \cap G$ and let V be open such that $V \supset F(z)$. By the upper quasicontinuity of F at z there exists a nonempty open set $W \subset G$ such that $F(x) \subset V$ for any $x \in W$. It is sufficient to take $x \in W$ such that $x \in D$ and we have $x \in D \cap G$, $F(x) \subset V$, which ends the proof.

Analogical theorems to Theorem 3 and Theorem 4 may be proved for the upper somewhat continuity if we suppose Y to be normal and F to be closed valued. We omit the obvious formulation and analogical proofs.

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