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In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [345]--351.

Persistent URL: http://dml.cz/dmlcz/701907

SET-VALUED MAPPINGS AND STRUCTURE OF BANACH SPACES

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The theory of monotone (maximal monotone), accretive (maximal accretive) single-valued and multi-valued mappings, intensively studied in the last period, has fruitful applications in the theory of nonlinear partial and ordinary differential and integral equations ([2] , [4] , [8] , [21]).

The aim of this note is to present some known recent results concerning single-valuedness and continuity properties of maximal monotone and the new ones of maximal accretive multivalued mappings and the structure of Banach spaces.

1. Notions and notations.

Let X be a real Banach space, X^* its dual, $\langle\ ,\ \rangle$ the pairing between X and X^* , $S_1(0)$ the unit sphere of X . We shall say that a Banach space X is : (i) smooth if its norm is Gâteaux differentiable on $S_1(0)$; (ii) Fréchet smooth if its norm is Fréchet differentiable on $S_1(0)$; (iii) an Asplund space (a week Asplund space) if each convex continuous functional f on X is Fréchet (Gâteaux) differentiable on a dense G_δ subset of X, (iv) an (H)-space, if X is rotund and the following condition is satisfied: if $(x_n), x \in X$, $x_n \to x$ weakly in X, $\|x_n\| \to \|x\|$, then $x_n \to x$ in the norm of X .

The notions of rotundity (R), local uniform rotundity (LUR), uniform rotundity (UR) of X are used in the usual sense ([15]). Let X,Y be topological spaces, $T: X \to 2^Y$ a multivalued mapping, $D(T) = \{u \in X: T(u) \neq \emptyset\}$ its domain, $G(T) = \{(u,v) \in X \times Y: u \in D(T), v \in T(u)\}$ its graph in the space X x Y . We shall say that T is : (i) upper semi-continuous at $u_0 \in D(T)$ if for each open subset W of Y such that $T(u_0) \subset W$, there exists an open neighborhood U of u_0 such that $T(U) \subset W$; (ii) lower semicontinuous at $u_0 \in D(T)$ if for each open subset W of Y such that $T(u_0) \cap W \neq \emptyset$ there exists an open neighborhood U of u_0 such that $T(u) \cap W \neq \emptyset$ for all $u \in U$. Let X be a real nor-

This paper is in final form and no version of it will be submitted for publication elsewhere.

med linear space. A mapping $T:X \to 2^{X^*}$ is said to be: (i) monotone on $D(T)$ if $\langle u^* - v^*, u - v \rangle \geq 0$ for each $u,v \in D(T)$, $u^* \in T(u)$, $v^* \in T(v)$; (ii) maximal monotone on $D(T)$ if T is monotone on $D(T)$ and its graph $G(T)$ is not properly contained in the graph of any other monotone map.

Now we give some well-known examples of maximal monotone operators.

1°. Let X be a Banach space, f a continuous convex function on X. Then the subdifferential map
$X \ni u \to \partial f(u) = \{u^* \in X^*: \langle u^*, v - u \rangle \leq f(v) - f(u)$ for each $v \in X\}$ is maximal monotone on X. In particular, a duality mapping $J:X \to 2^{X^*}$ defined by $J(u) = \{u^* \in X^* : \langle u^*, u \rangle = \|u\|^2, \|u^*\| = \|u\|\}$, $u \in X$, is maximal monotone on X. In fact, $J(u) = \partial(\frac{1}{2}\|u\|^2)$ for each $u \in X$. Recall that $J(u)$ is convex weakly* compact subset of X^* for each $u \in X$. Moreover, J is single-valued on X if and only if X is smooth.

2°. If $T:X \to X^*$ is linear with $D(T) = X$ and $\langle T(u),u \rangle \geq 0$ for each $u \in X$, then T is maximal monotone.

Let X be a reflexive Banach space, $T: X \supset D(T) \to X^*$ a closed linear and monotone mapping such that $\overline{D(T)} = X$. Then T is maximal monotone if and only if T^* is monotone. If X is reflexive, $T:X \to 2^{X^*}$ is monotone with $D(T) \subset X$, then T is maximal monotone if and only if $(T + J)X = X^*$ ([21]). The following result ([21]) is useful in applications:

Let X be a reflexive real Banach space, $T:X \to 2^{X^*}$ a coercive maximal monotone operator on $D(T) \subset X$. Then $T(X) = X^*$. For the further results and examples concerning the maximal monotone operators see [2], [4] and [21].

2. Single-valuedness and continuity properties of maximal monotone multivalued mappings.

Single-valuedness and continuity properties of monotone operators have been studied by Zarantonello [24], Kenderov [19], [20], Fabián [10], [11], Fitzpatrick [12], [13], Zajíček [23], Christensen and Kenderov [6], [7], Jayne and Rogers [18] and others. We recall here only some results which are related to those stated later concerning the accretive multivalued mappings.

Theorem 1 ([19]). Let X be a Banach space which admits an equivalent norm such that its dual norm is (R) in X^*. If $T:X \to 2^{X^*}$ is maximal monotone with $D(T) = X$, then T is single-valued on a dense G_δ subset of X.

If X satisfies the renorming condition of the above theorem, then X is a weak Asplund space ([19]). In particular, each WCG (and hence each separable Banach space) is a weak Asplund space.

Theorem 2 ([20]). X is an Asplund space if and only if each maximal monotone mapping $T:X \to 2^{X^*}$ with $\operatorname{int} D(T) \neq \emptyset$ is single-valued and upper semicontinuous (with respect to the norm topologies of X and X^*) on a dense G_δ subset of X .

It was proved in [6] that the similar result of Theorem 2 holds even in the case when maximal monotonicity of T is replaced by the condition that T is weak* compact valued and upper semicontinuous on int D(T) from the norm topology of X into the weak* topology of X^* .

Theorem 3 ([9]). Let X be a Banach space such that X^* is (R) and (H)-space, $T:X \to 2^{X^*}$ a maximal monotone mapping such that $\operatorname{int} D(T) \neq \emptyset$. Then: (i) there exists a unique lower selection T_0 of T ; (ii) for each $x \in \operatorname{int} D(T)$ at which T_0 is continuous, T(x) is a singleton and T is upper semicontinuous (with respect to the norm topologies of X and X^*) at x ; (iii) the set $C(T_0)$ of all those points at which T_0 is continuous is a dense G_δ subset of int D(T).

According to [22] a subset $A \subset X$ is said to be an α-angle porous ($\alpha > 0$) if for each $x \in A$ and each $\varepsilon > 0$ there exist $z \in B_\varepsilon(x) = \{u \in X : \|u - x\| < \varepsilon\}$ and $x^* \in X^*$ such that

$$A \cap \{y \in X : \langle y - z, x^* \rangle > \alpha \|x^*\| \cdot \|y - z\|\} = \emptyset \quad .$$

We shall say that A is an angle small ([22]) if $A = \bigcup_{n=1}^{\infty} A_n$, where A_n are α-angle porous.

Theorem 4 ([22]). Let X be a real Banach space such that X^* is separable, $T : X \to 2^{X^*}$ a monotone mapping with $D(T) \subset X$. Then there exists an angle small subset $A \subset D(T)$ such that T is single-valued and upper semicontinuous (with respect to the norm topologies of X and X^*) on A .

Theorem 5 ([7]). Let X be a Banach space, $f : X^* \to R$ a convex functional which is continuous with respect to the Mackey topology $\tau(X^*,X)$. Then f is Fréchet differentiable on a norm-dense G_δ subset of X^* .

According to [18] a map $f : X \to Y$ is said to be a Borel measurable function of the 1st Borel class if for each closed subset H of Y the set $f^{-1}(H)$ is a G_δ set in X .

Theorem 6 ([18]). Let X be a Banach space , $T : X \to 2^{X^*}$ a maximal monotone operator with $\operatorname{int} D(T) \neq \emptyset$. (i) If X admits an equivalent norm whose dual norm on X^* is (R) , then T has a norm-to-weak* Borel measurable selection T_0 of the 1st Borel class on D(T). The set C of points of int D(T) where T_0 is norm-to-weak*

continuous coincides with the set of all points of int D(T) where T_o is point-valued. Further C contains a dense G_δ subset of int D(T).
(ii) If X^* has the Radon-Nikodým property, then T has a norm-to-norm measurable selection T_o of the 1^{st} Borel class on D(T). The set U of all points of int D(T), at which T_o is norm-to-norm continuous, coincides with the set of all points of int D(T), at which T is point-valued and norm-to-norm upper semicontinuous. Furthermore U is dense G_δ subset of int D(T).

3. Accretive and maximal accretive multivalued mappings.

First of all we recall some basic and well-known notions concerning accretive operators. A multivalued mapping $A : X \to 2^X$ is said to be : (i) accretive on D(A) if for each $u,v \in D(A)$ and each $x \in A(u)$, $y \in A(v)$ there exists an element $x^* \in J(u - v)$ such that $\langle x-y, x^* \rangle \geq 0$;
(ii) maximal accretive on D(A) if A is accretive on D(A) and if $(u,x) \in X \times X$ is a given element such that for each $v \in D(A)$ and $y \in A(v)$ there exists $x^* \in J(u - v)$ such that $\langle x - y, x^* \rangle \geq 0$, then $u \in D(A)$ and $x \in A(u)$;
(iii) hemicontinuous at $u_o \in \mathrm{int}_a D(A)$ (an algebraic interior of D(A)) if for each $u \in X$, every null sequence of positive numbers t_n and every $v_n \in A(u_n)$, where $u_n = u_o + t_n u$, (v_n) converges weakly in X to some point $z_o \in A(u_o)$.

Theorem 7. Let X be a reflexive smooth and rotund Banach space, $A : X^* \to 2^{X^*}$ an accretive mapping (with respect to the duality mapping $J : X^* \to X$) such that $D(A) = X^*$ and for each $u^* \in X^*$ $A(u^*)$ is convex and closed in X^*. If A is hemicontinuous on X^*, then A is maximal accretive on X^*.

Let us recall that Fabián [11] stated the following result : If X is a reflexive Banach space such that X, X^* are both (LUR) and $A : X \to 2^X$ is maximal accretive such that int D(A) $\neq \emptyset$ and $(A^{-1} + \lambda I)(X) = X$ for each $\lambda > 0$, then A is single-valued and upper semicontinuous (with respect to the norm topology of X) on a dense G_δ subset of int D(A).

Theorem 8 . Let X be a Banach space, $A : X \to 2^X$ a maximal accretive mapping such that int D(A) $\neq \emptyset$.
(a) If X is reflexive and (F)-smooth, then there is a dense G_δ set $D_o \subset$ int D(A) such that $A|_{D_o}$ is single-valued and continuous from the norm topology of X into the weak topology of X ;
(b) If X is (F)-smooth and the duality mapping $J : X \to X^*$ is open, then A is single-valued and upper semicontinuous (with respect to the norm topology of X) on a dense G_δ subset of int D(A).

Remark 1 . If X is reflexive smooth and (H)-Banach space, then J is open. In particular, if X is smooth and (LUR)-Banach space, then J is open. Note that if X^* is (LUR), then X is (F)-smooth and if X is reflexive and (LUR), then X^* is (F)-smooth. Since X and X^* are both (F)-smooth, J is a homeomorphism of X onto X^* ([16]) .

Proposition 1 . If X is reflexive (F)-smooth Banach space, $A : X \to 2^X$ is accretive on D(A) and lower semicontinuous at $u_0 \in D(A)$ from the norm topology of X into the weak topology of X, then $A(u_0)$ is a singleton.

Theorem 8 and Proposition 1 show that the properties of maximal accretive multivalued mappings deeply rely on the structure of Banach spaces (compare [19]) .

Theorem 9. Let X be a real normed linear space, f a convex continuous functional on X , v_0, w_0^* given points of X and X^*, respectively. Assume that there exists a closed linear subspace E of X such that $\{ u \in E : \varphi_{v_0, w_0^*} (u) \leq c \}$ is non-empty and relatively weakly compact in E for some $c > 0$, where φ is defined by $\varphi(u) = f (u + v_0) - \langle w_0^* , u \rangle$ for each $u \in E$. Then :

(i) There exists a point $u_0 \in E$ such that

(*) $$\partial f (u_0 + v_0) \cap (w_0^* + E^{\perp}) \neq \emptyset .$$

(ii) If f is Gâteaux differentiable at the point $u_0 + v_0$, then the intersection (*) consists of exactly one point.

Corollary 1 . Let X be a real normed linear space, f a convex continuous functional on X . Assume that there exists a reflexive subspace E of X such that $f(u) \cdot \|u\|^{-1} \to +\infty$ as $\|u\| \to +\infty$ Then: (i) If v_0, w_0^* are arbitrary points of X, X^* respectively, then

$$\partial f (u_0 + v_0) \cap (w_0^* + E^{\perp}) \neq \emptyset .$$

(ii) If f is Gâteaux differentiable on X , then the above intersection consists of exactly one point.

Corollary 1 extends the results of Beurling and Livingston [3], Browder [5] , Asplund [1] . Another generalization of the Beurling--Livingston theorem was given by Gobbo [17].

Further results concerning these topics will be published later.

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