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THE COHOMOLOGY OF THE DIFFEOMORPHISM GROUP OF A MANIFOLD IS A GELFAND-FUKS COHOMOLOGY[1)]

Peter W. Michor

Abstract: The real singular cohomology of the connected component of the identity of the group of diffeomorphisms with compact support of a smooth manifold M is shown to coincide with Gelfand-Fuks cohomology of the Lie algebra $\mathfrak{X}_c(M)$ of smooth vector fields with compact support on M, with coefficients in the representation space of all smooth functions on the group $\mathrm{Diff}_0(M)$:

$$H^*(\mathrm{Diff}_0(M);\mathbb{R}) = H^*_{GF}(\mathfrak{X}_c(M);\ C^\infty(\mathrm{Diff}_0(M),\mathbb{R})).$$

0. INTRODUCTION

For compact connected Lie groups the real cohomology coincides with the cohomology of the Lie algebra (with real coefficients), which in turn is the exteriour algebra over the graded vector space of primitive elements in the dual of the Lie algebra. This uses invariant integration. For noncompact Lie groups one can reduce to the compact case by Iwasawa decomposition. This is no longer true for diffeomorphism groups. The orientation preserving diffeomorphisms on the 2-sphere contain $SO(3,\mathbb{R})$ as a strong deformation retract (Smale 1959), so $H^*(\mathrm{Diff}^+(S^2)) = H^*(\mathbb{R}P^3)$, but the continuous cohomology (also called Gelfand-Fuks cohomology) of the Lie algebra $\mathfrak{X}_c(S^2)$ of vector fields on S^2 has 10 independent and free generators. A similar statement is true for S^3 (Hatcher 1983).

In this paper I will show that for a large class of connected Lie groups including all diffeomorphism groups the singular cohomology with real coefficients equals the continuous cohomology of the Lie algebra with coefficients in the representation space of all smooth functions on the group. This class consists of all groups G, which are manifolds modelled on certain locally convex spaces, paracompact with smooth partitions of unity, such that multiplication and inversion are smooth. For an explanation of these spaces and the notion of differentiability we use see section 1.

1) This paper is in final form and no version of it will be submitted for publication elsewhere.

The algebraic topological properties of the diffeomorphism group have been attacked by D. Burghelea, W. Thurston, J. Mather, Dusa McDuff. The continuous cohomology of $_c(M)$ has been treated by I. M. Gelfand, D. B. Fuks, G. Segal, V. Guillemin, A. Haefliger, T. Tsujishita and others, mainly using spectral sequences and the theory of minimal models. $H^*_{GF}(\ _c(M); C^\infty(\mathrm{Diff}_0(M))$ has not yet been treated in the literature. There are some obvious spectral sequences one can set up but I was not able to compute the first step of one of them or to decide whether one of them converges or not.

The method presented here also opens a new approach for computing the cohomology for finite dimensional Lie groups.

The plan of the paper is as follows: in the first section we collect background material on calculus on locally convex vector spaces, on manifolds of mappings and the diffeomorphism group and set up the conventions. The second section is devoted to Analysis on Lie groups. The third section studies vector fields and differential forms on Lie groups. These two sections can be read independently of the rest of the paper if one imagines each Lie group appearing to be finite dimensional. In the fourth section the formalism obtained so far is interpreted in such a way as to give the main result.

1. BACKGROUND MATERIAL ON DIFFEOMORPHISM GROUPS

1.1. Let E, F be locally convex vector spaces. A mapping $f: E \to F$ is called C^1_c, if $df(x)v = \lim_{t\to 0} (f(x+tv)-f(x))/t$ exists for all x, v in E and the mapping $df: E\times E \to F$ is continuous. By iteration one gets the notion of differentiability C^r_c for all natural r and C^∞_c. Clearly $df(x)$ is linear and higher derivatives are symmetric, also the chain rule holds. See (Keller 1974) for exhausting information on this concept. Here we just note that it is not cartesian closed: the equation $C^\infty_c(E, C^\infty_c(F, G)) = C^\infty_c(E\times F, G)$ does not hold in general.

1.2. The best remedy for this fact is the calculus of Frölicher-Kriegl on convenient vector spaces. (Frölicher 1982, Kriegl 1982, 1983). Let us say that a mapping $f: E \to F$ is smooth (or C^∞) if $f\ c$ is C^∞ for any smooth curve $c: \mathbb{R} \to E$. Here E and F may be arbitrary locally convex spaces. It follows then that $df: E\times E \to F$ exists, is again smooth and is linear in the second variable. The chain rule is of course true. A linear mapping is C^∞ if and only if it is bounded. The smoothness structure depends only on the set of smooth curves into E (and F), not on the topology; it turns out that it depends only on the bornology. But, alas, in general there are smooth maps which are not continuous, even on bornological locally convex vector spaces. On the space $\mathfrak{D}$ of test functions on the real line there are quadratic smooth functions with real values which are not continuous. This follows directly from the property of cartesian closedness which holds for the notion of differentiability explained here. The equation

$C^\infty(E, C^\infty(F, G)) = C^\infty(E \times F, G)$ holds for all spaces for a suitable topology on $C^\infty(F, G)$. All smooth mappings are continuous with respect to the final topology induced by all smooth curves - call the outcome $c^\infty E$. The finest locally convex topology coarser than $c^\infty E$ is the bornologicalisation of E. If E is a Frèchet space or sequentially determined then $c^\infty E = E$. So on Frèchet spaces the notions C^∞ and C^∞_c coincide; so also on finite dimensional spaces (this was proved first by Boman).

A locally convex vector space is called convenient, if for any smooth curve $c: \mathbb{R} \to E$ there is an antiderivative $f: \mathbb{R} \to E$ with $f' = c$. This is the case if and only if E is bornologically complete (so it bornologicalisation is an inductive limit of Banach spaces). $C^\infty(E,F)$ is convenient if and only F is it. One may regard convenient spaces with their bornological locally convex topology, with the $c^\infty E$ topology explained above, or just with the given topology; in any case the category of convenient spaces and smooth mappings is cartesian closed and complete, but it is badly behaved with respect to quotients. As an offspring we also get that the category of convenient spaces and bounded linear mappings is monoidally closed with a certain completed tensor product $\tilde{\otimes}$: $L(E,L(F,G)) = L(E\tilde{\otimes}F,G)$ holds in full generality. For more information see the papers of Frölicher, Kriegl, the forthcoming book by Kriegl, or also chapter 1 in (Michor 1984). The notion of differentiability described here is the weakest of all notions admitting a chain rule, by its very definition.

1.3. Let X, Y be smooth finite dimensional second countable manifolds. Consider the space $C^\infty(X,Y)$ of smooth mappings from X to Y, equipped with the Whitney C^∞-topology. This space is not locally contractible. In fact any continuous curve $c: [0,1] \to C^\infty(X,Y)$ has image in the set of all f which equal c(0) off some compact set in X. Refining the Whitney C^∞-topology in such a way that all these sets become open one gets the $(F\mathcal{D})$-topology on $C^\infty(X,Y)$. With this topology, $C^\infty(X,Y)$ is locally contractible and even a manifold, modelled on spaces $\Gamma_c(f^*TY)$ of smooth sections with compact support of pullbacks to X of the tangent bundle of Y, equipped with the usual inductive limit topology over all compact subsets of X. The carts are constructed with the help of an exponential mapping on Y. See (Michor 1980) for all this. The chart changes are C^∞_c ore stronger. Composition $C^\infty(X,Y) \times C^\infty_{prop}(Z,X) \to C^\infty(Z,Y)$ is C^∞_c, where C^∞_{prop} denotes the subset of all proper mappings f (so f^{-1}(compact) is compact).

1.4. Diff(X), the group of all diffeomorphisms of X, is open in $C^\infty_{prop}(X,X)$, composition and inversion are C^∞_c. $T_{Id}\mathrm{Diff}(X)$, the tangent space at the identity, turns out to be the space $\Gamma_c(TX) = \mathfrak{X}_c(X)$ of all smooth vector fields with compact support on X, equipped with the usual inductive limit topology. But the usual Lie-bracket on $\mathfrak{X}_c(X)$ corresponds to the Lie bracket of right invariant vector fields on Diff(X). This fact cannot be avoided by changing convention. So for us $\mathfrak{X}_c(X)$ will bear the negative of the usual Lie bracket when regarded as Lie-algebra of

Diff(X). The exponential mapping Exp: $\mathcal{X}_c(X) \to$ Diff(X) is given by integrating vector fields with compact support. Exp is a smooth mapping, but is not analytic in any sense. Exp is not locally surjective, but its image generates $\mathrm{Diff}_0(X)$ as a group, where $\mathrm{Diff}_0(X)$ is the connected component of Diff(X), consisting of all diffeomorphisms with compact support which are diffeotopic (within a compact subset) to the identity. This is due to the fact, that $\mathrm{Diff}_0(X)$ is a perfect group (Epstein 1970). All this is true if X is a smooth finite dimensional manifold with corners. See (Michor 1980, 1983) for it.

1.5. If X is compact and μ is a smooth positive measure of total mass 1 on X (a density), then Diff(X) splits topologically and smoothly as Diff(X) = $= \mathrm{Diff}_\mu(X) \times \mathfrak{M}(X)$ where $\mathrm{Diff}_\mu(X)$ is the subgroup of μ-preserving diffeomorphisms and $\mathfrak{M}(X)$ is the set of all smooth positive measures of total mass 1, an open convex subset in an affine hyperplane in the Frèchet space of all densities. So Diff(X) is homotopy equivalent to $\mathrm{Diff}_\mu(X)$. See (Michor 1985) for this.

1.6. The theorem of De Rham: In (Michor 1980, 1983) it is shown that Diff(X),with the ($F\mathcal{D}$)-topology described above, is paracompact (a slight, easily correctable mistake there). It is also shown, that Diff(X) admits C^∞_c-partitions of unity. This is then used in (Michor 1983) to show that the theorem of De Rham holds for Diff(X), in fact for any C^∞_c-manifold, which is paracompact and modelled on (NLF)-spaces (nuclear LF spaces): The cohomology of C^∞_c-differential forms equals the singular cohomology. The notion of C^∞_c-differential form is a little complicated there due to the lack of cartesian closedness. Here we prefer to work with (Frölicher-Kriegl-) smooth differential forms. An inspection of the proof shows that this does not change the result. The uses fine sheafs. So we have:

1.7. Theorem: Let M be a paracompact topological space and a smooth manifold (in the sense of Frölicher Kriegl) which admits smooth partitions of unity. Then the singular cohomology of M (and many others) with real coefficients equals the De Rham cohomology of smooth differential forms.

Remark: If M is paracompact and modelled on (NLF)-spaces, then it admits automatically smooth partitions of unity. This is shown in (Michor 1983); mistakenly it is not assumed there, that M is paracompact.

1.8. So in the following pages we assume that G is a paracompact topological space, a (Frölicher-Kriegl-) smooth manifold modelled on convenient locally convex vector spaces, which admits smooth partitions of unity. Furthermore G is group, multiplication and inversion are smooth. The Lie algebra of G is denoted by $\mathfrak{g}$, it is a convenient space and the bracket is (bilinear-) bounded. In parts of section 2 we also need that G has a smooth exponential mapping exp: $\mathfrak{g} \to$ G with the usual properties with respect to smooth one parameter groups; local surjectivity is not assumed. Note that the regular Frèchet Lie groups of Omori et. al. (see Kobayashi et al. 1985) satisfy all these requirements. For a smooth finite dimensional paracompact manifold (possibly with corners) the diffeomorphism group Diff(X) will be equipped with the $(F\mathcal{D})$-topology described in 1.3. It is a C_c^∞-manifold then modelled on (NLF)-spaces, paracompact and admits C_c-partitions of unity. This is stronger than the requirements above. But we will use the Frölicher-Kriegl-calculus on Diff(X), so be aware that smooth functions, vector fields and differential forms are not continuous on Diff(X). We could take the c^∞-topology on Diff(X), as described in 1.2, but it is possible that we loose paracompactness then and cannot apply the theorem. If we stick to C_c^∞-calculus on Diff(X) then we are forced to work with rather awkward constructions of spaces of smooth multilinear mapping since we have to circumvent the lack of cartesian closedness.

2 ANALYSIS ON LIE GROUPS

2.1. Let G be a Lie group as described in 1.8 above with Lie algebra .
So the bracket on $\mathfrak{g}$ comes from the left invariant vector fields on G.
We will denote elements of G by x, y, z and elements of $\mathfrak{g}$ by X, Y, Z and so on.
The mappings $\lambda_x, \rho_x : G \to G$ will denote left and right translation by x.
Let $\mu : G \times G \to G$ denote the group multiplication and let $\nu : G \to G$ be the inversion.
Then as usual we have the following formulas for their tangent mappings:

$$T_{(x,y)}\mu(\xi_x, \eta_y) = T_y(\lambda_x)\eta_y + T_x(\rho_y)\xi_x \quad \text{for } \xi_x \in T_xG \text{ and } \eta_y \in T_yG.$$

$$T_{x^{-1}}(\lambda_x)\, T_e(\rho_{x^{-1}}) = Ad(x) : \mathfrak{g} \to \mathfrak{g}\ ,\ Ad \in C^\infty(G, L(\mathfrak{g},\mathfrak{g})).$$

$$(T_eAd.X)(Y) = [X,Y]\ . \qquad T_x\nu = -\,T_e(\lambda_{x^{-1}})\, T_x(\rho_{x^{-1}}).$$

2.2. Now let V be a convenient vector space. For $f \in C^\infty(G, V)$ we have $df \in \Omega^1(G;V)$, a 1-form on G with values in V. We define $\delta f : G \to L(\mathfrak{g},V)$ by $\delta f(x).X \doteq df\, T_e(\lambda_x)\, X$.
Then $f \in C^\infty(G, L(\mathfrak{g},V))$.

2.3. Lemma: For $f \in C^\infty(G,\mathbb{R})$ and $g \in C^\infty(G,V)$ we have $\delta(f.g) = f.\delta g + \delta f \otimes g$,
where $\mathfrak{g}^* \otimes V \hookrightarrow L(\mathfrak{g},V)$. This is true for any bounded bilinear operation.

Proof: $\delta(f.g)(x)\,X = d(f.g)\,(T_e(\lambda_x)\,X) = df((T_e(\lambda_x)\,X).g(x) + f(x).dg\,(T_e(\lambda_x)\,X)$
$= (\delta f \otimes g + f.\delta g)(x)\,X.$ □

2.4. Lemma: For $f \in C^\infty(G,V)$ we have $\delta\delta f(x)(X,Y) - \delta\delta f(x)(Y,X) = \delta f(x)\,[X,Y]$.

Proof: Let L_X be the left invariant vector field associated with $X \in \mathfrak{g}$, so $L_X(x) = T_e(\lambda_x)\,X$. Then $\delta f(x)\,X = df\,(L_X(x)) = (L_X\,f)(x)$. So we have

$\delta\delta f(x)(X)(Y) = (\delta(\delta f)(x)\,X)\,Y = \delta(\delta f(.)\,Y)(x)\,X$, since evaluation is bounded linear $L(\mathfrak{g},V) \to V$. Then we continue:

$$\delta\delta f(x)(X)(Y) = \delta(L_Y\, f)(x)\,X = L_X\,L_Y\,f\,(x).$$

$$\delta\delta f(x)(X)(Y) - \delta\delta f(x)(Y)(X) = (L_X\,L_Y - L_Y\,L_X)\,f\,(x).$$ □

2.5. Fundamental theorem of calculus on Lie groups: If G admits an exponential mapping, then for $f \in C^\infty(G,V)$, $X \in \mathfrak{g}$, $x \in G$ we have

$$f(x.\exp(X)) - f(x) = \left(\int_0^1 \delta f(x.\exp(tX)\,dt)(X)\right).$$

Proof: $\frac{d}{dt} f(x.\exp(tX)) = \frac{d}{dt} (f \circ \lambda_x \circ \exp)(tX) = df\ T(\lambda_x) \frac{d}{dt} \exp(tX) =$

$= df\ T(\lambda_x)\ L_X(\exp tX) = df\ T(\lambda_x)\ T(\lambda_{\exp tX})\ X = \delta f(x.\exp tX)\ X.$

$f(x.\exp X) - f(x) = \int_0^1 \frac{d}{dt} f(x.\exp tX)\ dt = \int_0^1 \delta f(x.\exp tX)\ X\ dt =$

$= (\int_0^1 \delta f(x.\exp tX)\ dt)\ X.$ □

2.6. Theorem on Taylor expansion with remainder on Lie groups: For $f \in C^\infty(G,V)$:

$$f(x.\exp X) = \sum_{j=0}^{k} \frac{1}{j!} \delta^j f(x)(X^j) + \int_0^1 \frac{(1-t)^k}{k!} \delta^{k+1} f(x.\exp tX)\, dt\ (X^{k+1}),$$

where $X^j = (X,X,\dots,X)$ j times.

Proof: Use the fundamental theorem of calculus and repeated partial integration. If you are afraid of doing so in a convenient vector space, apply a continuous linear functional, do it in $\mathbb{R}$, and use the theorem of Hahn Banach. □

Remark: Taylor expansion on a Lie group for analytic functions can be found in (Varadarajan 1974). The remainder seems to be new.

2.7. Power series. Let $f \in C^\infty(G,V)$. Then $\delta^j f(x)\colon \mathfrak{g}\times\dots\times\mathfrak{g} \to V$ is a j-linear bounded mapping, $\delta^j f : G \to L^j(\mathfrak{g};V)$ is smooth. Define $P_x f = \sum_{j=0}^{\infty} \delta^j f(x)$ in $\prod_j L^j(\mathfrak{g};V) = \prod_j L(\tilde{\otimes}^j \mathfrak{g}\,;V) = L(\oplus_j \tilde{\otimes}^j \mathfrak{g}\,;V) = L(\tilde{\otimes}\mathfrak{g}\,;V)$, where $\tilde{\otimes}^j$ is the j-th iterated (convenient) tensor product $\tilde{\otimes}$, and where $\tilde{\otimes}\mathfrak{g}$ is the "convenient tensor algebra".

Definition: The universal envelopping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is $\tilde{\otimes}\mathfrak{g}$ modulo the closed ideal generated by all elements of the form $X\otimes Y - Y\otimes X - [X,Y]$, X,Y in $\mathfrak{g}$.

Attention: The quotient need not be bornologically complete again, so $U\mathfrak{g}$ might be not convenient. This does not matter very much since we only consider $L(U\mathfrak{g},V)$ which is convenient if and only if V is it.

Theorem: For $f \in C^\infty(G,V)$ we have $P_x f \in L(U\mathfrak{g}, V)$, $P f \in C^\infty(G, L(U\mathfrak{g}, V))$.

Proof: $P_x f \in L(\tilde{\otimes}\mathfrak{g},V)$ and we have:

$$P_x f\ (X_1\otimes\dots\otimes X_n\otimes(X\otimes Y - Y\otimes X - [X,Y])\otimes Y_1\otimes\dots\otimes Y_m) =$$
$$= \delta^{n+m+2} f(x)(X_1,\dots,X_n,X,Y,Y_1,\dots,Y_m) - \delta^{n+m+1} f(x)(X_1,\dots,X_n,Y,X,Y_1,\dots,Y_m) -$$
$$- \delta^{n+m+1} f(x)(X_1,\dots,[X,Y],Y_1,\dots,Y_m) =$$
$$= \delta^n(\ \delta^2(\ \delta^m f(.)(Y_1,\dots,Y_m))(.)(X,Y) - \delta^2(\ \delta^m f(.)(Y_1,\dots,Y_m))(.)(Y,X) -$$
$$- \delta(\ \delta^m f(.)(Y_1,\dots,Y_m))(.)[X,Y]\ )(x)(X_1,\dots,X_n) = \delta^n 0(x)(\dots) = 0.$$

The last assertion follows from the categorical properties of the Frölicher Kriegl calculus. Note the heavy use of cartesian closedness in the computation. □

2.8. If G is finite dimensional, then $U\mathfrak{g}$ is isomorphic to the algebra of all left invariant differential operators on G. For $A \in U\mathfrak{g}$ let L_A be the corresponding left invariant differential operator.

Lemma: Then $(P_x f)(A) = L_A(f)(x)$.

Proof: Look again at the proof of 2.4. One may conclude inductively from it that $\delta^k f(x)(X_1,\dots,X_k) = (L_{X_1} L_{X_2} \dots L_{X_k} f)(x)$. □

Remark: The Taylor formula looks like $f(x.\exp X) = P_x f\ (1 + X + \frac{1}{2!}X^2 + \dots + \frac{1}{k!}X^k)$ $+ \int_0^1 (1-t)^k (P_{x.\exp tX} f)\ dt\ (\frac{1}{k!} X^{k+1})$.

If f is analytic then we can write for small X the following formula: $f(x.\exp X) = P_x f\,(1 + X + \frac{1}{2!}X^2 + \dots) \doteq (P_x f)(e^X) = L(e^X)\, f$, where e^X is in a suitable locally convex completion of $U\mathfrak{g}$ (rapidly decreasing with degree, say). $L(e^X)$ is then a left invariant differential operator of infinite order, with kernel a hyperfunction.

3. VECTOR FIELDS AND DIFFERENTIAL FORMS

3.1. Let $f \in C^\infty(G,\mathfrak{g})$. Then f defines a smooth vector field $L_f \in \mathfrak{X}(G)$ on G by $L_f(x) = T_e(\lambda_x)\, f(x)$. Clearly any vector field on G is of this form. If $g \in C^\infty(G,V)$ then $L_f\, g\,(x) = L_f(x)\, g = dg\,(L_f(x)) = dg\,(T_e(\lambda_x)\, f(x)) = \delta g(x)\, f(x)$. We will write $L_f\, g = \delta g.f$ to express this formula.

3.2. Lemma: For $f,g \in C^\infty(G,\mathfrak{g})$ we have $[L_f, L_g] = L(K(f,g))$, where

$$K(f,g)(x) = [f(x), g(x)]_{\mathfrak{g}} + \delta g(x)\, f(x) - \delta f(x)\, g(x), \text{ or shorter}$$

$$K(f,g) = [f,g]_{\mathfrak{g}} + \delta g.f - \delta f.g\,.$$

Proof: Let $h \in C^\infty(G,\mathbb{R})$. Then $L_g\, h\,(x) = \delta h(x)\, g(x)$. So we have (using 2.3)

$$(L_f\, L_g\, h)(x) = \delta(\delta h(.)\, g(.))(x)\, f(x) = \delta(\delta h(.)\, g(x))(x)\, f(x) + \delta h(x)(\delta g(x)\, f(x))$$
$$= \delta^2 h(x)(f(x),g(x)) + \delta h(x)\, \delta g(x)\, f(x).$$

$L_f\, L_g\, h = \delta^2 h.(f,g) + \delta h.\delta g.f.$ Now we will use 2.4.

$$[L_f, L_g]\, h = L_f\, L_g\, h - L_g\, L_f\, h = \delta^2 h.(f,g) + \delta h.\delta g.f - \delta^2 h.(g,f) - \delta h.\delta f.g$$
$$= \delta h.([f,g]_{\mathfrak{g}} + \delta g.f - \delta f.g) = L([f,g]_{\mathfrak{g}} + \delta g.f - \delta f.g).$$

Remark: 1. So $L: C^\infty(G,\mathfrak{g}) \to \mathfrak{X}(G)$ is a linear isomorphism and a Lie algebra homomorphism, where $C^\infty(G,\mathfrak{g})$ is equipped with the bracket K.

2. For (NLF)-manifolds the C^∞_c-vectorfields are the continuous derivations of the algebra $C^\infty_c(M)$, see (Michor 1983). In general one may see first that $[\xi,\eta]$ exists as a derivation on local smooth functions, and then one can check in local coordinates that it is given by a smooth vector field.

3. Note that for f, g in $C^\infty(G,\mathfrak{g})$ and h in $C^\infty(G,\mathbb{R})$ we have $K(h.f,g) = h.K(f,g) - (\delta h.g).f$ and $K(f,h.g) = h.K(f,g) + (\delta h.f).g$.

3.3. Let $L^p_{alt}(\mathfrak{g})$ denote the convenient vector space of all bounded alternating p-linear mappings $\mathfrak{g}\times\dots\times\mathfrak{g} \to \mathbb{R}$. (Attention: $\Lambda^p\mathfrak{g}^*$ is in general smaller).
For g in $C^\infty(G, L^p_{alt}(\mathfrak{g}))$ we define a differential form L_g in $\Omega^p(G)$ for $\xi_i \in T_xG$ by $(L_g)_x(\xi_1,\dots,\xi_p) = g(x)(T_x(\lambda_{x^{-1}})\,\xi_1,\dots, T_x(\lambda_{x^{-1}})\,\xi_p)$, or $(L_g)_x = L^p_{alt}(T_x(\lambda_{x^{-1}}))\, g(x)$. Clearly any p-form φ in $\Omega^p(G)$ can be written in the form $\varphi = L_g$ for a unique g in $C^\infty(G, L^p_{alt}(\mathfrak{g}))$. For f_i in $C^\infty(G,\mathfrak{g})$ we have

$$L_g\,(L_{f_1},\dots, L_{f_p})(x) = g(x)\,(T_x(\lambda_{x^{-1}})\, L(f_1)(x), \dots) = g(x)\,(f_1(x),\dots,f_p(x)) =$$
$$= g.(f_1,\dots,f_p)\,(x).$$

3.4. Exteriour derivative. For g in $C^\infty(G, L^p_{alt}(\mathfrak{g}))$ it suffices to test the exteriour derivative $d\,L_g$ on leftinvariant vector fields $L(X_i)$, X_i in $\mathfrak{g}$:

$$(d\, L_g)\,(L(X_0),\dots,L(X_p)) = \Sigma_{i=0}^{p}\; L(X_i)\,(L_g\,(L(X_0),\dots, \widehat{L(X_i)},\dots, L(X_p))) +$$
$$+ \Sigma_{i<j}\,(-1)^{i+j}\, L_g\,([L(X_i),L(X_j)],L(X_0),\dots,\widehat{L(X_i)},\dots,\widehat{L(X_j)},\dots,L(X_p)) =$$

$= \Sigma_{i=o}^{p} (-1)^i (\delta g.X_i)(X_o,\dots,\hat{X}_i,\dots,X_p) + \Sigma_{i<j} (-1)^{i+j} g([X_i,X_j],X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots)$

Now let $d^{\mathfrak{g}} : L^p_{alt}(\mathfrak{g}) \to L^{p+1}_{alt}(\mathfrak{g})$ be the exteriour derivative of the Lie algebra ,
so $(d^{\mathfrak{g}}\omega)(X_o,\dots,X_p) = \Sigma_{i<j} (-1)^{i+j} ([X_i,X_j],X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$.
Furthermore we define the mapping $\delta^\wedge : C^\infty(G, L^p_{alt}(\mathfrak{g})) \to C^\infty(G, L^{p+1}_{alt}(\mathfrak{g}))$ by
$(\delta^\wedge g)(X_o,\dots,X_p) = \Sigma_{i=o}^{p} (-1)^i (\delta g.X_i)(X_o,\dots,\hat{X}_i,\dots,X_p)$.

Lemma: $d L_g = L((\delta^\wedge + d^{\mathfrak{g}})g)$ and $-(\delta^\wedge)^2 = d^{\mathfrak{g}}\delta^\wedge + \delta^\wedge d^{\mathfrak{g}}$.

Proof: The first formula has been checked above and the second follows from $d^2 = 0$ and $(d^{\mathfrak{g}})^2 = 0$.

3.5. Lie derivative. For f in $C^\infty(G,\mathfrak{g})$, g in $C^\infty(G, L^p_{alt}(\mathfrak{g}))$ and X_i in $\mathfrak{g}$ we have for the Lie derivative $\Theta(L_f)(L_g)$ of the p-form L_g along the vector field L_f :

$\Theta(L_f)(L_g)(L(X_1),\dots,L(X_p)) = L_f (L_g(L(X_1),\dots,L(X_p)) +$
$\quad + \Sigma_{i=0}^{p} (-1)^i L_g ([L_f,L(X_i)],L(X_1),\dots,\hat{L(X_i)},\dots)$
$\quad = (\delta g.f)(X_1,\dots,X_p) + \Sigma(-1)^i g.(K(f,X_i),X_1,\dots,\hat{X}_i,\dots,X_p)$.

Now $K(f,X) = [f,X] + 0 - \delta f.X$, and so we get:

$\Theta(L_f)(L_g)(L(X_1),\dots,L(X_p)) = (\delta g.f)(X_1,\dots,X_p) - \Sigma(-1)^i g.(\delta f.X_i,X_1,\dots,\hat{X}_i,\dots X_p)$
$\quad + \Sigma(-1)^i g.([f,X_i],X_1,\dots\hat{X}_i,\dots X_p)$.

Now consider the Lie derivation of $\mathfrak{g}$:
$({}^{\mathfrak{g}}\Theta_X \omega)(X_1,\dots,X_p) = \Sigma(-1)^i \omega([X,X_i],X_1,\dots,\hat{X}_i,\dots X_p)$. For f in $C^\infty(G,\mathfrak{g})$ we will apply this pointwise.

We also consider ${}^\delta\Theta : C^\infty(G,\mathfrak{g}) \times C^\infty(G, L^p_{alt}(\mathfrak{g})) \to C^\infty(G, L^p_{alt}(\mathfrak{g}))$, defined by
$({}^\delta\Theta_f\, g)(x)(X_1,\dots,X_p) = (\delta g(x)\, f(x))(X_1,\dots,X_p) -$
$\quad - \Sigma (-1)^i g(x)(\delta f(x)X_i,X_1,\dots,\hat{X}_i,\dots,X_p)$.

Lemma: $\Theta(L_f)(L_g) = L(({}^{\mathfrak{g}}\Theta_f + {}^\delta\Theta_f)g)$.

For shortness sake we also write $\Theta_f : C^\infty(G,L^p_{alt}(\mathfrak{g})) \to C^\infty(G,L^p_{alt}(\mathfrak{g}))$ for the mapping defined by $\Theta_f = {}^{\mathfrak{g}}\Theta_f + {}^\delta\Theta_f$ or equivalently $\Theta(L_f)(L_g) = L(\Theta_f\, g)$.

3.6. Collection of definitions: For the convenience of the reader we collect here all definitions given so far and some more. Let f,f_i in $C^\infty(G,\mathfrak{g})$, g,g_i in $C^\infty(G,L^p_{alt}(\mathfrak{g}))$ and X_i in $\mathfrak{g}$.

1. $K(f_1,f_2) = [f_1,f_2]_{\mathfrak{g}} + \delta f_2.f_1 - \delta f_1.f_2$ is a Lie bracket on $C^\infty(G,\mathfrak{g})$.
2. $(d^{\mathfrak{g}} g)(X_o,\dots,X_p) = \Sigma_{i<j} (-1)^{i+j} g([X_i,X_j],X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$
   $(\delta^\wedge g)(X_o,\dots,X_p) = \Sigma (-1)^i (\delta g.X_i)(X_o,\dots,\hat{X}_i,\dots,X_p)$.
   We have $d = d^{\mathfrak{g}} + \delta^\wedge$ if we put $dL_g = L(dg)$.
3. $({}^{\mathfrak{g}}\Theta_f\, g)(X_1,\dots,X_p) = - \Sigma\, g(X_1,\dots,[f,X_i]_{\mathfrak{g}},\dots,X_p)$
   $({}^\delta\Theta_f\, g)(X_1,\dots,X_p) = (\delta g.f)(X_1,\dots,X_p) + \Sigma\, g(X_1,\dots,\delta f.X_i,\dots,X_p)$.
   Then $\Theta_f = {}^{\mathfrak{g}}\Theta_f + {}^\delta\Theta_f$ satisfies $\Theta(L_f)(L_g) = L(\Theta_f g)$.
4. $C^\infty(G, L^*_{alt}(\mathfrak{g}))$, with the pointwise exteriour product, is a graded commutative algebra. We will write $g_1 \wedge g_2$ for this product.
5. We have the insertion operator $i_f : C^\infty(G, L^p_{alt}(\mathfrak{g})) \to C^\infty(G,L^{p-1}_{alt}(\mathfrak{g}))$, given by $(i_f g)(x)(X_1,\dots,X_{p-1}) = g(x)(f(x),X_1,\dots,X_{p-1})$. We have $L(i_f g) = i(L_f)(L_g)$.

3.7. <u>Theorem</u>: Let f, f_i in $C^\infty(G,\mathfrak{g})$, let g, g_i in $C^\infty(G, L^*_{alt}(\mathfrak{g}))$, u,v in $C^\infty(G,\mathbb{R})$.

1. $i_f(g_1 \wedge g_2) = i_f g_1 \wedge g_2 + (-1)^{\deg g} g_1 \wedge i_f g_2$ and $i(f_1)\, i(f_2) = -\, i(f_2)\, i(f_1)$.
2. $i(K(f_1,f_2)) = \Theta(f_1) i(f_2) - i(f_2)\Theta(f_1) = [\,\Theta(f_1), i(f_2)]$.
   $\Theta_f(g_1 \wedge g_2) = (\Theta_f g_1) \wedge g_2 + g_1 \wedge (\Theta_f g_2)$.
   $\Theta(K(f_1,f_2)) = \Theta(f_1)\Theta(f_2) - \Theta(f_2)\Theta(f_1) = [\,\Theta(f_1),\ \Theta(f_2)]$.
   $\Theta_{uf} = u.\Theta_f + \delta u \wedge i_f$
3. $\Theta_f = i_f\, d + d\, i_f$ and $d\,\Theta_f = \Theta_f\, d$.
4. $i([f_1,f_2]_{\mathfrak{g}}) = {}^{\mathfrak{g}}\Theta(f_1)\, i(f_2) - i(f_2)\, {}^{\mathfrak{g}}\Theta(f_1) = [{}^{\mathfrak{g}}\Theta(f_1),\ i(f_2)]$.
   ${}^{\mathfrak{g}}\Theta_f$ is a derivation of degree 0 for the exteriour product.
   ${}^{\mathfrak{g}}\Theta([f_1,f_2]_{\mathfrak{g}}) = {}^{\mathfrak{g}}\Theta(f_1)\, {}^{\mathfrak{g}}\Theta(f_2) - {}^{\mathfrak{g}}\Theta(f_2)\, {}^{\mathfrak{g}}\Theta(f_1) = [{}^{\mathfrak{g}}\Theta(f_1),\ {}^{\mathfrak{g}}\Theta(f_2)]$.
   ${}^{\mathfrak{g}}\Theta(u.f) = u.{}^{\mathfrak{g}}\Theta_f$.
5. ${}^{\mathfrak{g}}\Theta_f = i_f\, {}^{\mathfrak{g}}d + {}^{\mathfrak{g}}d\, i_f$, $({}^{\mathfrak{g}}d)^2 = 0$, ${}^{\mathfrak{g}}d\, {}^{\mathfrak{g}}\Theta_f = {}^{\mathfrak{g}}\Theta_f\, {}^{\mathfrak{g}}d$.
6. $i(\delta f_2.f_1 - \delta f_1.f_2) = {}^{\delta}\Theta(f_1)\, i(f_2) - i(f_2)\, {}^{\delta}\Theta(f_1) = [{}^{\delta}\Theta(f_1),\ i(f_2)]$.
   ${}^{\delta}\Theta_f$ is a derivation of degree 0 for the exteriour product.
   ${}^{\delta}\Theta(u.f) = u.{}^{\delta}\Theta_f + \delta u \wedge i_f$.
7. For F in $C^\infty(G, L(\mathfrak{g},\mathfrak{g}))$ define $\bar\Theta_F$: $C^\infty(G, L^p_{alt}(\mathfrak{g})) \to C^\infty(G, L^p_{alt}(\mathfrak{g}))$ by
   $(\bar\Theta_F g)(x)(X_1,\dots,X_p) = -\,\Sigma\, g(x)(X_1,\dots,F(x)(X_i),\dots,X_p)$.
   Furthermore define A: $C^\infty(G,\mathfrak{g}) \times C\ (G,\mathfrak{g}) \to C^\infty(G, L(\mathfrak{g},\mathfrak{g}))$ by
   $A(f_1,f_2) = [\delta f_2,\ \mathrm{ad}\ f_1] - [\delta f_1,\ \mathrm{ad}\ f_2]$. Then we have
   $[{}^{\delta}\Theta(f_1),\ {}^{\delta}\Theta(f_2)] = {}^{\delta}\Theta(K(f_1,f_2)) - \bar\Theta(A(f_1,f_2))$.
   $\bar\Theta_F$ is a derivation for the exteriour product, $[i_f,\ \bar\Theta_F] = i(F(f))$,
   and $[\bar\Theta(F_1),\ \bar\Theta(F_2)] = \bar\Theta([F_1,F_2]) = \bar\Theta(F_1F_2 - F_2F_1)$.
8. ${}^{\delta}\Theta_f = i_f\, \delta\hat{} + \delta\hat{}\ i_f$. $\delta\hat{}$ is a derivation of degree 1 (like d).
   $-(\delta\hat{})^2 = \delta\hat{}\ d^{\mathfrak{g}} + d^{\mathfrak{g}}\ \delta\hat{}$.
   $(\delta\hat{})^2 g\ (X_o,\dots,X_p) = -\,\Sigma_{i<j}(-1)^{i+j}(\delta g.[X_i,X_j])(X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$.
9. $[d^{\mathfrak{g}},{}^{\delta}\Theta_f]g\ (X_o,\dots,X_p) = \Sigma_{i<j}(-1)^{i+j} g(\delta f.[X_i,X_j] - [\delta f.X_i,X_j] - [X_i,\delta f.X_j], X_o, \dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$
   $[\delta\hat{},{}^{\mathfrak{g}}\Theta_f]g\ (X_o,\dots,X_p) = \Sigma(-1)^i(\delta g.[f,X_i])(X_o,\dots,\hat{X}_j,\dots,X_p) + $
   $+ \Sigma_{i<j}(-1)^{i+j}\ g([\delta f.X_i,X_j] + [X_i,\delta f.X_j],\ X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$
   $[\delta\hat{},{}^{\delta}\Theta_f]g\ (X_o,\dots,X_p) = -\,\Sigma(-1)^i(\delta g.[f,X_i])(X_o,\dots,X_i,\dots,X_p)$
   $-\,\Sigma_{i<j}(-1)^{i+j}\ g(\delta f.[X_i,X_j],X_o,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p)$.

The proof of all these formulas does not offer difficulties. Some of them are valid on the Lie algebra $\mathfrak{g}$, others are known from calculus on manifolds. For the others just plug in the definitions and compute, using the previously checked formulas. Only the proof of 9 is a little longer.

4. COHOMOLOGY OF LIE GROUPS

4.1. Let G be a Lie group in the sense of 1.8. with Lie algebra $\mathfrak{g}$, but we do not need the existence of an exponential mapping here. By theorem 1.7 we know that the singular cohomology of G coincides with the De Rham cohomology of smooth differential forms (all with real coefficients). By the results of section 3 we see that the space of smooth differential forms is isomorphic to $C^\infty(G,L^*_{alt}(\mathfrak{g}))$, with the differential d. Thanks to cartesian closedness of the category of convenient vector spaces and smooth mappings and its relation to the cartesian closed category of the same spaces and the bounded linear mappings we see that $C^\infty(G,L^p_{alt}(\mathfrak{g}))$ is isomorphic to the space $L^p_{alt}(\mathfrak{g}; C^\infty(G,\mathbb{R}))$ of bounded p-linear alternating mappings from ^{p} in $C^\infty(G,\mathbb{R})$. We will use the same symbols as in section 3 for all mappings treated there. In particular we have several actions of the Lie algebra $\mathfrak{g}$ on $L^p_{alt}(\mathfrak{g},C^\infty(G,\mathbb{R}))$, namely Θ_X, ${}^{\mathfrak{g}}\Theta_X$, ${}^{\delta}\Theta_X$, $\bar{\Theta}_{ad\ X}$. We use Θ_X as the main action. If $G = Diff_0(X)$ as in section 1, then the complex $L^p_{alt}(\mathfrak{X}_c(X),C^\infty(Diff_0(X))$ is exactly the differential complex for the Gelfand Funks cohomology of $\mathfrak{X}_c(X)$ with coefficients in the representation space $C^\infty(Diff_0(X),\mathbb{R})$, up to differences which come from the non-continuity of smooth and bounded multilinear mappings. Thus we get the following result:

Theorem: Let X be a finite dimensional smooth paracompact manifold with corners. Then the singular cohomology with real coefficients of $Diff_0(X)$, the group of diffeomorphisms with compact support diffeotopic to the identity through diffeomorphisms with compact support, equals the cohomology of the differrential complex $L^p_{alt}(\mathfrak{X}_c(X), C^\infty(Diff_0(X),\mathbb{R}))$ with differential d.

4.2. For the rest of this paper let $L^p_{alt}(\mathfrak{g},C^\infty(G,\mathbb{R})) =: K^p$, let the differential be $d = d' + d''$, where $d' = d$ and $d'' = \delta\hat{}$. Then $d^2 = 0$, $d'^2 = 0$, but $d''^2 \neq 0$. Define the following filtration of K: $K_n := \{g \in K: d''^n g = 0\}$. This filtration is graded and increasing. Let $i: K^p_q \to K^p_{q+1}$ be the embedding. Then $d = d' + id''$. For the spectral sequence associated with the filtration we have the following results:

Lemma: Let $a_i \in K^{q-1}_{p-i}$ and $b_j \in K^q_{p-j}$. Then $d(\Sigma\, i^n a_n) = \Sigma i^m b_m$ if and only if for suitable $c_j \in K^q_{p-j}$ we have:

$$d'a_0 = ic_1 + b_0,\quad c_j + d''a_{j-1} + d'a_j = ic_{j+1} + b_j\ (1\leq j\leq k),\quad c_{k+1} + d''a_k = 0.$$

Lemma: For the first term $E_1^{p,q} = H^q_{d'}(K_p/K_{p-1})$ of the spectral sequence we have: $E_1^{1,0} = \mathbb{R}$. If $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ then $E_1^{2,0} = 0$ (true if $G = Diff_0(X)$).

4.3. Some other observation. Of course one may consider first the d'-cohomology of K. Unfortunately d" does not induce a mapping in the d'-cohomology, but $(d'')^2$ does and $(d'')^2$ is even chain homotopic to 0 in the d'-chain complex. So $H_{(d'')^2}(H_{d'}(K)) = H_{(\delta\hat{})^2}(C^\infty(G,H^*(\mathfrak{g})))$ makes sense, but the relation to the d-cohomology is not clear.

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