

Masami Sekizawa

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NATURAL TRANSFORMATIONS OF AFFINE CONNECTIONS ON  
MANIFOLDS TO METRICS ON COTANGENT BUNDLES

Masami Sekizawa

Dedicated to Professor Shun-ichi Tachibana on the  
occasion of his 60th birthday

Let  $M$  be a smooth manifold and  $T^*M$  its cotangent bundle. There is a well-known "natural" construction which yields, for any affine connection  $\nabla$  on  $M$ , a pseudo-Riemannian metric  $\bar{g}$  on  $T^*M$ , the so called Riemann extension of  $\nabla$  (see [6] - [10]). If a local coordinate system is given in  $M$  then the components of the metric  $\bar{g}$  at each point  $(x,w) \in T^*M$  depend only on the symmetrized components of the connection  $\nabla$  and the components of the given co-vector  $w$ . The more detailed analysis shows that this construction involves the geometry of the second order, and thus we can consider the Riemann extension as an example of "natural transformation of the second order".

The aim of this paper is to describe explicitly all second order natural transformations of a symmetric affine connection on a manifold into a metric (not necessarily regular) on its cotangent bundle. To solve this problem, we shall use the precise definitions as well as the general method established by D.Krupka[2] - [4], which reduces our geometric problem to the classification of corresponding "differential invariants" and then to solving a system of partial differential equations. The main result of this paper is the following

Theorem 1. A pseudo-Riemannian metric  $G$  (not necessarily regular) on  $T^*M$  comes from a second order natural transformation of a symmetric connection  $\nabla$  on  $M$  if and only if  $G = a\bar{g} + b\theta^2$ , where  $\bar{g}$  is the Riemann extension of  $\nabla$ ,  $\theta^2$  is the tensor square of the canonical 1-form of  $T^*M$ , and  $a, b$  are constants.

Let us notice that the metric  $G = a\bar{g} + b\theta^2$  is regular (of the signature  $(n,n)$ ) if and only if  $a \neq 0$ .

O.Kowalski and the present author[1] have recently classified all second order natural transformations of a Riemannian metric  $g$  given on a base manifold  $M$  into a pseudo-Riemannian metric  $G$  given on the tangent bundle  $TM$ . Since the cotangent bundle  $T^*M$  over  $(M,g)$  is dual to  $TM$ , the analogous problem for  $T^*M$  is automatically settled through this duality.

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### 1. Canonical 1-form and Riemann extension

We shall adopt the Einstein summation convention in sections 1 and 2.

Let  $(U; x^1, x^2, \dots, x^n)$  and  $(\bar{U}; \bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$  be two systems of local coordinates in a smooth manifold  $M$  of dimension  $n$  such that the domain  $U \cap \bar{U}$  is not empty. The coordinate vector fields  $E_i = \partial/\partial x^i$  and  $\bar{E}_i = \partial/\partial \bar{x}^i$  ( $1 \leq i \leq n$ ) are related by the transformation formulas

$$(1.1) \quad E_i = A_i^a \bar{E}_a \quad \text{or} \quad \bar{E}_i = B_i^a E_a \quad (1 \leq i \leq n)$$

where  $(A_i^a) = \begin{bmatrix} \partial \bar{x}^a \\ \partial x^i \end{bmatrix}$  and  $(B_i^a) = \begin{bmatrix} \partial x^a \\ \partial \bar{x}^i \end{bmatrix}$  are the (mutually

inverse) Jacobi matrices. If  $w = w_h dx^h = \bar{w}_h d\bar{x}^h$  is a 1-form on  $U \cap \bar{U}$  then we get by (1.1)

$$(1.2) \quad \bar{w}_h = B_h^a w_a \quad (1 \leq h \leq n).$$

Further if  $\nabla$  is an affine connection on  $M$  then its components  $\Gamma_{ij}^h$  and  $\bar{\Gamma}_{ij}^h$  ( $1 \leq h, i, j \leq n$ ) (i.e.,  $\nabla_{X_i} X_j = \Gamma_{ij}^a X_a$ ,  $\nabla_{\bar{X}_i} \bar{X}_j = \bar{\Gamma}_{ij}^a \bar{X}_a$ ) are related by

$$(1.3) \quad \bar{\Gamma}_{ij}^h = A_a^h (B_i^b B_j^c \Gamma_{bc}^a + B_{ij}^a) \quad (1 \leq h, i, j \leq n),$$

where we put  $B_{ij}^h = \partial^2 x^h / \partial \bar{x}^i \partial \bar{x}^j$  ( $1 \leq h, i, j \leq n$ ).

Now let us denote  $p: T^*M \rightarrow M$  the natural projection of the cotangent bundle. Let  $(p^{-1}U; x^1, x^2, \dots, x^n, w_1, w_2, \dots, w_n)$  and  $(p^{-1}\bar{U}; \bar{x}^1, \bar{x}^2, \dots, \bar{x}^n, \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n)$  be two systems of local coordinates in  $T^*M$  induced from  $(U; x^1, x^2, \dots, x^n)$  and  $(\bar{U}; \bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ , respectively. Then, whenever  $U \cap \bar{U} \neq \emptyset$ , the transformation law on  $p^{-1}(U \cap \bar{U})$  is given by

$$\begin{cases} \bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^n), \\ \bar{w}_i = B_1^a w_a, \quad (1 \leq i \leq n). \end{cases}$$

We set  $X_i = \partial / \partial x^i$ ,  $X^i = \partial / \partial w_i$  and  $\bar{X}_i = \partial / \partial \bar{x}^i$ ,  $\bar{X}^i = \partial / \partial \bar{w}_i$  for  $1 \leq i \leq n$ ; then the two bases  $\{X_1, X_2, \dots, X_n, X^1, X^2, \dots, X^n\}$  and  $\{\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n, \bar{X}^1, \bar{X}^2, \dots, \bar{X}^n\}$  are related to each other by

$$(1.4) \quad \begin{cases} \bar{X}_i = B_i^a X_a + B_{i\ ca}^c B_b^d w_d X^a, \\ \bar{X}^i = A_a^i X^a, \quad (1 \leq i \leq n), \end{cases}$$

where we put  $A_{ij}^h = \partial^2 \bar{x}^h / \partial x^i \partial x^j$  ( $1 \leq h, i, j \leq n$ ).

Let now  $G_{ij}^h = G(X_i, X_j)$ ,  $G_i^h = G(X_i, X^h)$ ,  $G^{ij} = G(X^i, X^j)$  be the local components of a symmetric  $(0,2)$ -tensor field  $G$  on  $T^*M$ . We shall always write such components in the block matrix form

$$(1.5) \quad G = \begin{bmatrix} G_{ij} & G_i^j \\ G_j^i & G^{ij} \end{bmatrix}.$$

Using (1.4), we obtain the following transformation formulas:

$$(1.6) \quad \begin{cases} \bar{G}_{ij} = B_i^a B_j^s G_{as} + B_{i\ ca}^c B_b^d w_d B_j^s G_s^a \\ \quad + B_1^a B_j^u B_{us}^t B_{t\ v}^v G_a^s + B_{i\ ca}^c B_b^d w_d B_j^u A_{us}^t B_{t\ v}^v G^{as}, \\ \bar{G}_i^h = A_a^h B_j^s G_s^a + A_a^h B_j^u A_{us}^t B_{t\ v}^v G^{as}, \\ \bar{G}^{ij} = A_a^i A_s^j G^{as}, \quad (1 \leq h, i, j \leq n). \end{cases}$$

(In section 2, we shall show that the formulas (1.2), (1.3) and (1.6) define actions of the second order differential group  $L_n^2$  on some vector spaces).

Let  $p_*:T(T^*M)\rightarrow TM$  be the differential of the natural projection  $p:T^*M\rightarrow M$  and let  $q$  be the natural projection of  $T(T^*M)$  to  $T^*M$ , where "T" stands for the tangent bundle. Then the canonical 1-form  $\theta$  on  $T^*M$  is defined by

$$\theta(\tilde{X}) = q(\tilde{X})(p_*\tilde{X})$$

for all  $\tilde{X} \in T(T^*M)$ . The exterior derivative  $d\theta$  of  $\theta$  is called the canonical 2-form on  $T^*M$ . In terms of the induced system of local coordinates in  $T^*M$ ,  $\theta$  and  $d\theta$  are expressed as

$$\theta = w_i dx^i \quad \text{and} \quad d\theta = dw_i \wedge dx^i.$$

Let  $\nabla$  be an affine connection on the base manifold  $M$ . This induces a unique connection in the vector bundle  $T^*M$ , and thus each tangent space  $(T^*M)_{(x,w)}$  splits into the horizontal and the vertical subspace:

$$(T^*M)_{(x,w)} = H_{(x,w)} \oplus V_{(x,w)}.$$

Let  $\tilde{X} = h\tilde{X} + v\tilde{X}$  be the decomposition of a vector field  $\tilde{X}$  on  $T^*M$  into the horizontal and the vertical part. The Riemann extension  $\bar{g}$  of the affine connection  $\nabla$  on  $M$  to  $T^*M$  is a pseudo-Riemannian metric defined by

$$\bar{g}(\tilde{X}, \tilde{Y}) = (d\theta)(v\tilde{X}, h\tilde{Y}) + (d\theta)(v\tilde{Y}, h\tilde{X})$$

for all vector fields  $\tilde{X}$  and  $\tilde{Y}$  on  $T^*M$ . If  $\tilde{X} = \xi^i X_i + \xi_1 X^1$ , then we obtain easily

$$h\tilde{X} = \xi^i X_i + w_a \Gamma_{ib}^a \xi^b X^1 \quad \text{and} \quad v\tilde{X} = (\xi_1 - w_a \Gamma_{ib}^a \xi^b) X^1.$$

Thus the components of  $\bar{g}$  with respect to the induced system of local coordinates are

$$\bar{g} = \begin{bmatrix} -w_a (\Gamma_{ij}^a + \Gamma_{ji}^a) & \delta_1^j \\ \delta_j^1 & 0 \end{bmatrix},$$

where  $\delta_j^i$  denotes the Kronecker's symbol. This shows that the components of  $\bar{g}$  depend only on the symmetrized components of the affine connection and on the components of the co-vector  $w$ , and they do not depend on the local coordinates  $(x^1, x^2, \dots, x^n)$  in  $M$ . In case of a symmetric connection we get

$$\bar{g} = \begin{bmatrix} -2w_a \Gamma_{ij}^a & \delta_i^j \\ \delta_j^i & 0 \end{bmatrix} .$$

Obviously,  $\bar{g}$  is a pseudo-Riemannian metric with the signature  $(n, n)$ . (See [6] - [10] for more information about Riemann extensions).

2. Differential invariants

Let us now recall the general theory of natural transformations due to D.Krupka. We refer to [2] - [4] for more details, and to [5] for the general philosophy of naturality.

Let  $L_n^r$  be the  $r$ -th order differential group of the  $n$ -dimensional Euclidean space  $R^n$ , that is, the Lie group of all  $r$ -jets of local diffeomorphisms of  $R^n$  with source and target at the origin  $o \in R^n$ , here  $r$  is any non-negative integer. Let  $P, Q$  be smooth manifolds on which the group  $L_n^r$  acts to the left. An  $r$ -th order differential invariant  $f: P \rightarrow Q$  is an  $L_n^r$ -equivariant map of the left  $L_n^r$ -space  $P$  to the left  $L_n^r$ -space  $Q$ , i.e, a map satisfying  $f(j_o^r \alpha \cdot p) = j_o^r \alpha \cdot f(p)$  for all  $j_o^r \alpha \in L_n^r$  and all  $p \in P$ . Here the dot  $\cdot$  denotes the action of  $L_n^r$  on  $P$  (or on  $Q$ , respectively).

Further let  $F^r M$  denote the bundle of all frames of  $r$ -th order over  $M$  which carries a natural structure of a principal  $L_n^r$ -bundle  $F^r M(M, L_n^r, \kappa_n^r)$ . We get a natural functor  $F^r$  from the category  $D_n$  of  $n$ -manifolds and injective immersions into the category of principal  $L_n^r$ -bundles and  $L_n^r$ -bundle morphisms. Here, for a given morphism  $\varphi: M_1 \rightarrow M_2$  of  $D_n$  the corresponding morphism  $F^r \varphi: F^r M_1 \rightarrow F^r M_2$  is given in a familiar way (see [5]).

Finally, for a left  $L_n^r$ -space  $P$ , let  $F_P^r M$  denote the fibre bundle with fibre  $P$ , associated to the principal  $L_n^r$ -bundle  $F^r M$ . We obtain a natural functor  $F_P^r$  from the category  $D_n$  into the category of fibre bundles and their morphisms. Here, for any morphism  $\varphi: M_1 \rightarrow M_2$  of  $D_n$  the corresponding morphism  $F_P^r \varphi: F_P^r M_1 \rightarrow F_P^r M_2$  is given by

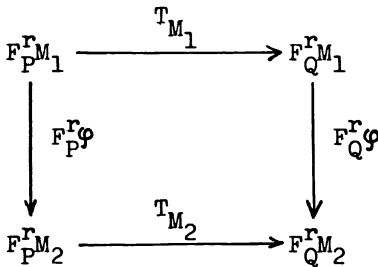
$$F_P^r \varphi([y, p]) = [F^r \varphi(y), p]$$

for any  $[y, p] \in F_{P M_1}^r$  ( $[y, p]$  is the equivalence class of a pair  $(y, p) \in F_{M_1}^r \times P$  with respect to the equivalence relation defined by the right action  $(y, p) \cdot j_0^r \alpha = (y \cdot j_0^r \alpha, j_0^r \alpha^{-1} \cdot p)$  of  $L_n^r$  on  $F_{M_1}^r \times P$ ).

For each manifold  $M$  and each differential invariant  $f: P \rightarrow Q$  we can define a morphism  $f_M: F_P^r M \rightarrow F_Q^r M$  over the identity map  $id: M \rightarrow M$  by

$$f_M([y, p]) = [y, f(p)]$$

for all  $[y, p] \in F_P^r M$ . This morphism  $f_M$  is called the realization of a differential invariant  $f$  on the manifold  $M$ . Further, an  $r$ -th order natural transformation  $T$  of the functor  $F_P^r$  into the functor  $F_Q^r$  is a collection of bundle morphisms  $T_M$  over the identity map, where  $M$  is an object of  $D_n$ , such that the following diagram is commutative



for every morphism  $\varphi: M_1 \rightarrow M_2$  of  $D_n$ .

The following theorem due to Krupka[2] says that a problem to find all  $r$ -th order natural transformations of  $F_P^r$  to  $F_Q^r$  is equivalent to a problem to find all  $r$ -th order differential invariants  $f$  from  $P$  to  $Q$ .

Theorem A. Let  $f: P \rightarrow Q$  be an  $r$ -th order differential invariant. Then the correspondence  $T_f: M \rightarrow f_M$ , where  $M$  is an object of  $D_n$ , is a natural transformation of the functor  $F_P^r$  to the functor  $F_Q^r$ . Moreover, the correspondence  $f \rightarrow T_f$  is a bijection between the set of all  $r$ -th order differential invariants from  $P$  to  $Q$  and the set of all  $r$ -th order natural transformations of  $F_P^r$  to  $F_Q^r$ .

Remark. Often  $P$  and  $Q$  are vector spaces and thus  $F_P^R$  and  $F_Q^R$  are vector bundles. Yet, a morphism  $T_M: F_P^R \rightarrow F_Q^R$  need not be a morphism in the category of vector bundles because it may be non-linear on fibres.

In order to apply the method by Krupka to our problem, we shall restrict ourselves to second order differential invariants. We define functions  $A_i^h, A_{ij}^h$  ( $1 \leq h \leq n, 1 \leq i \leq j \leq n$ ) on  $L_n^2$  by

$$A_i^h(j_0^2\alpha) = D_i\alpha^h(o), \quad A_{ij}^h(j_0^2\alpha) = D_iD_j\alpha^h(o)$$

for any local diffeomorphism  $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$  with  $\alpha(o) = o \in R^n$ , here  $D_i$  denotes the partial derivative with respect to the  $i$ -th variable in  $R^n$ . The system of the canonical (global) coordinates of  $L_n^2$  is a system of coordinates  $\{B_i^h, B_{ij}^h\}$  ( $1 \leq h \leq n, 1 \leq i \leq j \leq n$ ) of  $L_n^2$  which are defined by

$$B_i^h(j_0^2\alpha) = A_i^h(j_0^2\alpha^{-1}), \quad B_{ij}^h(j_0^2\alpha) = -A_{ij}^h(j_0^2\alpha^{-1})$$

$$(1 \leq h \leq n, 1 \leq i \leq j \leq n).$$

The multiplication law  $j_0^2\alpha \cdot j_0^2\beta = j_0^2(\alpha \cdot \beta)$  in  $L_n^2$  is described in terms of the canonical coordinates as

$$\begin{cases} B_i^h(j_0^2\alpha \cdot j_0^2\beta) = B_a^h(j_0^2\alpha)B_1^a(j_0^2\beta), \\ B_{ij}^h(j_0^2\alpha \cdot j_0^2\beta) = B_{ab}^h(j_0^2\alpha)B_1^a(j_0^2\beta)B_j^b(j_0^2\beta) + B_a^h(j_0^2\alpha)B_{ij}^a(j_0^2\beta), \\ (1 \leq h \leq n, 1 \leq i \leq j \leq n). \end{cases}$$

Since  $B_i^h(j_0^2(id)) = \delta_i^h, B_{ij}^h(j_0^2(id)) = 0$  ( $1 \leq h \leq n, 1 \leq i \leq j \leq n$ ) for the identity map  $id$  of  $R^n$ , we obtain that

$$(2.1) \quad B_a^h(j_0^2\alpha)A_1^a(j_0^2\alpha) = \delta_i^h \quad (1 \leq h, i \leq n),$$

$$(2.2) \quad A_{ab}^h(j_0^2\alpha)B_1^a(j_0^2\alpha)B_j^b(j_0^2\alpha) + A_a^h(j_0^2\alpha)B_{ij}^a(j_0^2\alpha) = 0,$$

$$(1 \leq h \leq n, 1 \leq i \leq j \leq n)$$

for all  $j_0^2\alpha \in L_n^2$ . These formulas will be used in section 3.

Let us consider the vector space  $P = R^{n*} \oplus (R^n \otimes (R^{n*} \otimes R^{n*}))$  (of



dimension  $n+n^2(n+1)/2$ ) where  $R^{n*}$  is the dual space to  $R^n$  and  $\odot$  denotes the symmetric product. We denote by  $\{w_h\} (1 \leq h \leq n)$  and  $\{\Gamma_{ij}^h\} (1 \leq h \leq n, 1 \leq i \leq j \leq n)$  the canonical coordinates on  $R^{n*}$  and  $R^n \otimes (R^{n*} \otimes R^{n*})$ , respectively. Then  $\{w_h, \Gamma_{ij}^h\} (1 \leq h \leq n, 1 \leq i \leq j \leq n)$  form a canonical system of coordinates on  $P$ . We define an action of  $L_n^2$  on  $P$  by the formulas

$$(2.3) \quad \begin{cases} \bar{w}_h = B_h^a w_a, \\ \bar{\Gamma}_{ij}^h = A_a^h (B_i^b B_j^c \Gamma_{bc}^a + B_{ij}^a). \quad (1 \leq h \leq n, 1 \leq i \leq j \leq n), \end{cases}$$

which are modelled according to (1.2) and (1.3).

Further, consider the vector space  $Q = R^{n*} \oplus ((R^n \otimes R^{n*}) \odot (R^n \otimes R^{n*}))$  (of dimension  $2n(n+1)$ ). Here we can define a canonical system of coordinates on  $Q$  in the form  $\{z_h, G_{ij}, G_i^h, G^{ij}\} (1 \leq h \leq n, 1 \leq i \leq j \leq n)$ . Then we define an action of  $L_n^2$  on  $Q$  by

$$(2.4) \quad \begin{cases} \bar{z}_h = B_h^a z_a, \\ \bar{G}_{ij} = B_i^a B_j^s G_{as} + B_i^a A_{ca}^b B_b^d B_j^s G_s^a \\ \quad + B_i^a B_j^u A_{us}^t B_t^v G_s^a + B_i^c B_{ca}^b B_b^d B_j^u A_{us}^t B_t^v G_s^a, \\ \bar{G}_i^h = A_a^h B_i^s G_s^a + A_a^h B_i^u A_{us}^t B_t^v G_s^a, \\ \bar{G}^{ij} = A_a^i A_s^j G^{as}, \quad (1 \leq h \leq n, 1 \leq i \leq j \leq n), \end{cases}$$

which is in the agreement with (1.2) and (1.6).

One can see easily that, for the corresponding associated  $L_n^2$ -bundles  $F_P^2 M$  and  $F_Q^2 M$  over a manifold  $M$ , we always have

$$(2.5) \quad F_P^2 M = T^*M \oplus \mathcal{P}'_M, \quad F_Q^2 M = T^*M \oplus \mathcal{Q}'_M,$$

where  $\mathcal{P}'_M$  and  $\mathcal{Q}'_M$  are some vector bundles over  $M$ . (Here  $\mathcal{P}'_M$  is an associated bundle to  $F_P^2 M$  but  $\mathcal{Q}'_M$  is not, as we see from the transformation rules (2.3) and (2.4)).

Now, we define the problem to find all second order natural transformations of a symmetric affine connection on a manifold to a metric on its cotangent bundle as the problem to find all those natural transformations of  $F_P^2$  to  $F_Q^2$  which, with respect to the splittings (2.5), induce the identity map  $id: T^*M \rightarrow T^*M$  for each  $M$ .

According to Theorem A, our main Theorem 1 is reduced to the following

Theorem 2. All differential invariants  $f : (w_h, \Gamma_{ij}^h) \mapsto (z_h, G_{ij}, G_i^h, G^{ij})$  from  $P = R^{n*} \otimes (R^n \otimes (R^{n*} \otimes R^{n*}))$  into  $Q = R^{n*} \otimes ((R^n \otimes R^{n*}) \otimes (R^n \otimes R^{n*}))$  such that  $z_h = w_h$  ( $1 \leq h \leq n$ ) are given, in the canonical coordinates, by

$$G_{ij} = -2aw_s \Gamma_{ij}^s + bw_i w_j \quad (1 \leq i \leq j \leq n),$$

$$G_i^h = a \delta_i^h \quad (1 \leq h, i \leq n),$$

$$G^{ij} = 0 \quad (1 \leq i \leq j \leq n),$$

where  $a$  and  $b$  are constants.

### 3. Proof of Theorem 2

The method of the proof is that we attach to each equivariant map  $P \rightarrow Q$  the corresponding Lie algebra homomorphism for the fundamental vector fields. These Lie algebra homomorphisms are then characterized by a system of differential equations to solve. We find all solutions and decide which of them really represent differential invariants.

First of all, it will be useful to extend the symbols  $\Gamma_{ij}^h$ ,  $A_{ij}^h$ ,  $B_{ij}^h$ ,  $G_{ij}$ ,  $G^{ij}$  also for the case  $i \geq j$  by putting  $\Gamma_{ij}^h = \Gamma_{ji}^h$ ,  $A_{ij}^h = A_{ji}^h$ , and so on. Thus the range of all indices will be  $\{1, 2, \dots, n\}$ , and all indices will be independent. We note that, under this notation, we have to use the following conventions (cf. Krupka [3]):

$$\frac{\partial \Gamma_{ij}^h}{\partial r^p} = \frac{1}{2} \delta_p^h (\delta_i^q \delta_j^r + \delta_j^q \delta_i^r), \quad \frac{\partial B_{ij}^h}{\partial B_{qr}^p} = \frac{1}{2} \delta_p^h (\delta_i^q \delta_j^r + \delta_j^q \delta_i^r),$$

$$\frac{\partial G_{ij}}{\partial G_{qr}} = \frac{1}{2} (\delta_i^q \delta_j^r + \delta_j^q \delta_i^r), \quad \frac{\partial G^{ij}}{\partial G^{qr}} = \frac{1}{2} (\delta_q^i \delta_r^j + \delta_r^i \delta_q^j)$$

for  $1 \leq h, i, j, p, q, r \leq n$ .

The fundamental vector fields on  $P$  relative to the action (2.3) are

$$\begin{aligned} \xi_p^q &= \sum_{a,b,c} \frac{\partial \bar{\Gamma}_{bc}^a}{\partial B_q^p}(e) \frac{\partial}{\partial \Gamma_{bc}^a} + \sum_a \frac{\partial \bar{w}_a}{\partial B_q^p}(e) \frac{\partial}{\partial w_a} \\ &= \sum_{a,b,c} (-\delta_p^a \Gamma_{bc}^q + \delta_b^q \Gamma_{pc}^a + \delta_c^q \Gamma_{bp}^a) \frac{\partial}{\partial \Gamma_{bc}^a} + w_p \frac{\partial}{\partial w_q}, \\ \xi_p^{qr} &= \sum_{a,b,c} \frac{\partial \bar{\Gamma}_{bc}^a}{\partial B_{qr}^p}(e) \frac{\partial}{\partial \Gamma_{bc}^a} = \frac{\partial}{\partial \Gamma_{qr}^p}, \quad (1 \leq p, q, r \leq n), \end{aligned}$$

where  $e$  denotes the identity element  $j_0^2(\text{id})$  of  $L_n^2$ . Here we also used the formula

$$\frac{\partial A_1^h}{\partial B_q^p}(e) = -\frac{\partial B_1^h}{\partial B_q^p}(e) = -\delta_p^h \delta_1^q \quad (1 \leq h, i, p, q \leq n)$$

which is derived from (2.1) by differentiation with respect to  $B_q^p$ .

The corresponding fundamental vector fields on  $Q$  relative to the action (2.4) are given by

$$\begin{aligned} \bar{\xi}_p^q &= \sum_{a,b} \left( \frac{\partial \bar{G}_{ab}}{\partial B_q^p}(e) \frac{\partial}{\partial G_{ab}} + 2 \frac{\partial \bar{G}_a^b}{\partial B_q^p}(e) \frac{\partial}{\partial G_a^b} + \frac{\partial \bar{G}^{ab}}{\partial B_q^p}(e) \frac{\partial}{\partial G^{ab}} \right) + \sum_a \frac{\partial \bar{z}_a}{\partial B_q^p} \frac{\partial}{\partial z_a} \\ &= 2 \sum_a \left( G_{ap} \frac{\partial}{\partial G_{aq}} + G_p^{a\partial} \frac{\partial}{\partial G_q^a} - G_a^{q\partial} \frac{\partial}{\partial G_a^p} - G^{aq\partial} \frac{\partial}{\partial G^{ap}} \right) + z_p \frac{\partial}{\partial z_q}, \\ \bar{\xi}_p^{qr} &= \sum_{a,b} \left( \frac{\partial \bar{G}_{ab}}{\partial B_{qr}^p}(e) \frac{\partial}{\partial G_{ab}} + 2 \frac{\partial \bar{G}_a^b}{\partial B_{qr}^p}(e) \frac{\partial}{\partial G_a^b} + \frac{\partial \bar{G}^{ab}}{\partial B_{qr}^p}(e) \frac{\partial}{\partial G^{ab}} \right) \\ &= -z_p \sum_a \left( G_a^{q\partial} \frac{\partial}{\partial G_{ar}} + G_p^{a\partial} \frac{\partial}{\partial G_q^a} + G_a^{r\partial} \frac{\partial}{\partial G_{aq}} + G^{ar\partial} \frac{\partial}{\partial G_q^a} \right), \\ &(1 \leq p, q, r \leq n). \end{aligned}$$

Here we also used the formula

$$\frac{\partial A_{ij}^h}{\partial B_{qr}^p}(e) = -\frac{\partial B_{ij}^h}{\partial B_{qr}^p}(e) = -\frac{1}{2} \delta_p^h (\delta_i^q \delta_j^r + \delta_j^q \delta_i^r) \quad (1 \leq h, i, j, p, q, r \leq n)$$

which is derived from (2.2) by differentiation with respect to  $B_{qr}^p$ .  
 Since any  $L_n^2$ -equivariant map  $f$  of  $P$  to  $Q$  satisfies

$$f_*(\xi_p^q) = \zeta_p^q \quad \text{and} \quad f_*(\xi_p^{qr}) = \zeta_p^{qr} \quad (1 \leq p, q, r \leq n),$$

we get

$$\xi_p^q(Z^\alpha \circ f) = \zeta_p^q(Z^\alpha) \quad \text{and} \quad \xi_p^{qr}(Z^\alpha \circ f) = \zeta_p^{qr}(Z^\alpha), \quad (1 \leq p, q, r \leq n),$$

where  $Z^\alpha$  denotes any canonical coordinate on  $Q$ . We have to use also the conditions  $z_h \circ f = w_h$  ( $1 \leq h \leq n$ ). Hence we obtain an explicit system of partial differential equations:

$$(3.1) \quad \sum_{a,b,c} (-\delta_p^a \Gamma_{bc}^q + \delta_b^q \Gamma_{pc}^a + \delta_c^q \Gamma_{bp}^a) \frac{\partial G_{ij}^a}{\partial \Gamma_{bc}^a} + w_p \frac{\partial G_{ij}^a}{\partial w_q} = G_{ip} \delta_j^q + G_{jp} \delta_i^q,$$

$$(3.2) \quad \sum_{a,b,c} (-\delta_p^a \Gamma_{bc}^q + \delta_b^q \Gamma_{pc}^a + \delta_c^q \Gamma_{bp}^a) \frac{\partial G_i^j}{\partial \Gamma_{bc}^a} + w_p \frac{\partial G_i^j}{\partial w_q} = -G_i^q \delta_p^j + G_p^j \delta_i^q,$$

$$(3.3) \quad \sum_{a,b,c} (-\delta_p^a \Gamma_{bc}^q + \delta_b^q \Gamma_{pc}^a + \delta_c^q \Gamma_{bp}^a) \frac{\partial G^{ij}}{\partial \Gamma_{bc}^a} + w_p \frac{\partial G^{ij}}{\partial w_q} = -G^{iq} \delta_p^j + G^{jq} \delta_p^i,$$

$$(3.4) \quad \frac{\partial G_{ij}^a}{\partial \Gamma_{qr}^p} = -\frac{1}{2} w_p (G_i^q \delta_j^r + G_j^q \delta_i^r + G_i^r \delta_j^q + G_j^r \delta_i^q),$$

$$(3.5) \quad \frac{\partial G_i^j}{\partial \Gamma_{qr}^p} = -\frac{1}{2} w_p (G^{jq} \delta_i^r + G^{jr} \delta_i^q),$$

$$(3.6) \quad \frac{\partial G^{ij}}{\partial \Gamma_{qr}^p} = 0, \quad (1 \leq i, j, p, q, r \leq n).$$

We shall solve the system (3.1) - (3.6) step by step. To avoid confusion, we do not use the Einstein summation convention up to the end of this section.

(3.6) implies that  $G^{ij}$  ( $1 \leq i, j \leq n$ ) do not depend on  $\Gamma_{qr}^p$  ( $1 \leq p, q, r \leq n$ ). Hence  $G^{ij} = \lambda^{ij}(w_1, w_2, \dots, w_n)$  for  $1 \leq i, j \leq n$ . Then we get from (3.3)

$$(3.7) \quad w_p \partial \lambda^{ij} / \partial w_q = -\lambda^{iq} \delta_p^j - \lambda^{jq} \delta_p^i \quad (1 \leq i, j, p, q \leq n).$$

Now, if  $n \geq 3$ , we can always choose  $p \neq i, j$  and we get, at any generic point (for  $w_1 w_2 \dots w_n \neq 0$ ),  $\partial \lambda^{ij} / \partial w_q = 0$  for all  $i, j, q$ . Then (3.7) reduces to  $\lambda^{iq} \delta_p^j + \lambda^{jq} \delta_p^i = 0$ , and contracting with respect to  $i=p$ , we get  $(n+1) \lambda^{jq} = 0$ , i.e.,

$$(3.8) \quad G^{ij} = \lambda^{ij} = 0$$

for all  $i, j \in \{1, 2, \dots, n\}$ .

Let now  $n = 2$ . Putting  $i=j=p=q$  in (3.7), we get, at a generic point

$$\lambda^{ii} = c^{ii} / (w_i)^2 \quad (i = 1, 2),$$

where  $c^{ii}$  does not depend on  $w_i$ . Further, putting  $i=1, j=2, p=q$  in (3.7), we obtain the equations

$$w_1 \partial \lambda^{12} / \partial w_1 = w_2 \partial \lambda^{12} / \partial w_2 = -\lambda^{12}.$$

By an elementary calculation we get

$$\lambda^{12} = c^{12} / w_1 w_2,$$

where  $c^{12}$  is a constant. Summarizing, we get, at a generic point,

$$G^{ij} = c^{ij} / w_i w_j \quad (i, j = 1, 2)$$

where  $c^{ij}$  ( $i, j = 1, 2$ ) are constants. But  $G^{ij}$  and  $w_i$  must satisfy the transformation laws (2.3) and (2.4). This is possible if and only if  $c^{ij} = 0$  ( $i, j = 1, 2$ ). Thus we get again

$$(3.9) \quad G^{ij} = 0 \quad (i, j = 1, 2)$$

at a generic point, and hence at any point by continuity.

Substitution of (3.8) into (3.5) implies

$$\partial G_1^j / \partial \Gamma_{qr}^p = 0 \quad (1 \leq i, j, p, q, r \leq n),$$

hence  $G_1^j$  ( $1 \leq i, j \leq n$ ) do not depend on  $\Gamma_{qr}^p$  ( $1 \leq p, q, r \leq n$ ). Put  $G_1^j = \mu_1^j(w_1, w_2, \dots, w_n)$ . Then we get from (3.2)

$$(3.10) \quad w_p \partial \mu_1^j / \partial w_q = -\mu_1^q \delta_p^j + \mu_p^j \delta_1^q \quad (1 \leq i, j, p, q \leq n).$$

Using (3.10) for  $i=j=p=q$  and also for  $i=j \neq p=q$ , we see that, at a generic point,  $\mu_i^1$  ( $1 \leq i \leq n$ ) are constants. Putting  $i=j=q \neq p$  in (3.10), we obtain  $\mu_p^j = 0$ . Thus  $\mu_i^j = 0$  for  $1 \leq i, j \leq n, i \neq j$ . Putting  $i=q \neq j=p$  in (3.10), we get immediately  $\mu_j^j - \mu_i^1 = 0$  ( $1 \leq i, j \leq n, i \neq j$ ). This can be written in the form  $\mu_i^j = a\delta_i^j$  ( $1 \leq i, j \leq n$ ) for some constant  $a$ , i.e., at a generic point,

$$(3.11) \quad G_i^j = a\delta_i^j \quad (1 \leq i, j \leq n),$$

where  $a$  is a constant. (3.11) then holds at any point.

Substituting (3.11) into (3.4), we get

$$(3.12) \quad \partial G_{ij} / \partial \Gamma_{qr}^p = -aw_p(\delta_j^q \delta_i^r + \delta_i^q \delta_j^r) \quad (1 \leq i, j, p, q, r \leq n)$$

from which we get by integration with respect to  $\Gamma_{qr}^p$ ,

$$(3.13) \quad G_{ij} = -2a \sum_s w_s \Gamma_{ij}^s + \nu_{ij} \quad (1 \leq i, j \leq n),$$

where  $\nu_{ij}$  ( $1 \leq i, j \leq n$ ) do not depend on  $\Gamma_{qr}^p$  ( $1 \leq p, q, r \leq n$ ). Substituting (3.13) into (3.1) and using (3.12), we obtain, after some calculations,

$$(3.14) \quad w_p \partial \nu_{ij} / \partial w_q = \nu_{ip} \delta_j^q + \nu_{jp} \delta_i^q \quad (1 \leq i, j, p, q \leq n).$$

By calculations similar to those for solving (3.7) in the case  $n = 2$  we get from (3.14), at a generic point,

$$(3.15) \quad \nu_{ij} = b_{ij} w_i w_j \quad (1 \leq i, j \leq n),$$

where  $b_{ij}$  ( $1 \leq i, j \leq n$ ) are constants.

Substitute (3.15) into (3.14) for  $i=q \neq j$ . We get  $b_{ij} w_j w_p = b_{jp} w_j w_p$  ( $1 \leq i, j, p \leq n, i \neq j$ ) and hence, at a generic point,  $b_{ij} = b_{jp}$  whenever  $i \neq j, p$  arbitrary. Hence all  $b_{ij}$  are equal to the same constant  $b$ . Thus we get

$$G_{ij} = -2a \sum_s w_s \Gamma_{ij}^s + b w_i w_j \quad (1 \leq i, j \leq n)$$

at a generic point, and hence at any point.

This completes the proof of Theorem 2.

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AUTHOR'S ADDRESS: Department of Mathematics, Tokyo Gakugei University, Nukuikita-Machi4-1-1, Koganei-Shi, Tokyo, 184, Japan