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EXTENSIONS OF OPERATORS AND THE DIRICHLET PROBLEM IN POTENTIAL THEORY

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1. Introduction

Suppose that $U \subset \mathbb{R}^m$ is a bounded open set and $\mathcal{H}(U)$ is the space of harmonic functions on U. If f is a continuous function defined on the boundary ∂U of U, the classical Dirichlet problem on U is that of finding a continuous extension F of f to $\overline{U}$, the closure of U, such that $F_{|U}$, the restriction of F to U, is harmonic on U. The set U is said to be regular, provided the classical Dirichlet problem has a solution for an arbitrary continuous function defined on ∂U. Since there exist non-regular sets, one is naturally interested in a generalized Dirichlet problem. Roughly speaking, we want to assign to every, say, continuous function on ∂U a harmonic function on U in such a way that the resulting mapping has some reasonable properties (such as linearity and positivity, for instance) and gives the solution of the classical Dirichlet problem provided one exists. It is expected that the harmonic function assigned will tend to the given boundary condition in most boundary points.

To be more specific, define

$$H(U) = \{h \in C(\overline{U});\ h_{|U} \in \mathcal{H}(U)\}, \quad H(\partial U) = H(U)_{|\partial U}.$$

Thus $f \in H(\partial U)$ if and only if there is a solution of the classical Dirichlet problem for f. On $H(\partial U)$, the operator T of the classical Dirichlet problem is defined, of course, by

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This paper is in final form and no version of it will be submitted for publication elsewhere.

$$T(h_{|\partial U}) = h_{|U}, \qquad h \in H(U),$$

and one is in fact interested in a study of positive linear extensions of T from H(∂U) to C(∂U) and, possibly, to a larger space than C(∂U).

Recall that the so called Perron-Wiener-Brelot solution of the Dirichlet problem provides such an extension to the set of all resolutive functions which is considerably larger than C(∂U). Thus no existence problem arises. The question of uniqueness, however, is far from being evident. Notice that, for a non regular set U, H(∂U) is a proper closed subspace of C(∂U), thus topologically very small (it is, in fact, nowhere dense). Nevertheless, M.V. Keldyš proved in [19] the following remarkable result: All positive linear extensions of the operator T from H(∂U) to C(∂U) coincide.

Intuitively speaking, the space H(∂U) has to be in a sense large in C(∂U), otherwise one could hardly expect a unique extension of T. A.F. Monna in [26] proposes as a problem the investigation of relevant functional analytic properties of the space H(∂U) responsible for uniqueness. He suggests studying extensions of T to discontinuous boundary conditions. He also proposes clarification of the uniqueness question in the case of the Dirichlet problem for partial differential equations other than the Laplace equation or, more generally, in the context of axiomatic potential theory.

A series of papers was published on the subject and led to the solution of the above mentioned (and other) questions; see [8], [22],[23],[34],[27],[29],[14].

The main objective of the present paper is to find a suitable abstract setting appropriate for a better understanding of the nature of the Keldyš theorem.

To this end, in Sec. 2, a question of uniqueness of extensions of operators on Riesz spaces is analyzed. The "domain of uniqueness" is characterized in terms that admit applications to potential theory. It turns out that linearity plays no important role and only the monotonicity of the operators in question appears to be essentially involved.

In Sec. 3, a more special situation, namely that of function spaces, is investigated. The Choquet boundary enters quite naturally into the picture. Validity of an abstract Keldyš theorem is shown to be equivalent to various other conditions. Also relations to Korovkin type theorems are studied.

Finally, in Sec. 4 it is shown how conclusions obtained in an abstract setting can be used to prove results already known as well as new results concerning Keldyš type theorem in abstract potential theory.

Most of results of the present paper was announced in [29], [31] and is included in the unpublished text [30].

2. Extensions of operators in Riesz spaces

2.1 Let B be an ordered vector space, D be a Dedekind complete Riesz space and H be a majorizing vector subspace of B. Thus we suppose that for every $b \in B$ there is an $h \in H$ such that $b \leqq h$.

Let $T: H \longrightarrow D$ be a positive linear mapping. For $b \in B$ denote

$$\check{T}b = \bigvee \{Th;\ h \leqq b,\ h \in H\}, \qquad \hat{T}b = \bigwedge \{Th;\ b \leqq h,\ h \in H\}.$$

Here $\bigvee$ and $\bigwedge$ means the supremum and the infimum in D, respectively. Of course, $\hat{T}b = -\check{T}(-b)$ whenever $b \in B$ and $\check{T}b = \hat{T}b = Tb$ for every $b \in H$. The restriction of a mapping $S: B \longrightarrow D$ to H will be denoted by $S_{|H}$.

The following Hahn-Banach-Kantorovič type theorem will be useful in the sequel. The proof can be found in [35], p.277; see also [18],[21],[10].

2.2 Theorem. Let $S: B \longrightarrow D$ be an increasing mapping such that $S_{|H} = T$. Then $\check{T}b \leqq Sb \leqq \hat{T}b$ whenever $b \in B$. If $b_o \in B$, $d_o \in D$ and $\check{T}b_o \leqq d_o \leqq \hat{T}b_o$, then there exists a positive linear mapping $T': B \longrightarrow D$ such that $T'_{|H} = T$ and $T'b_o = d_o$.

2.3 Let us introduce the following notation:

$$P_T = \{S;\ S: B \longrightarrow D,\ S \text{ increasing},\ S_{|H} = T\},$$

$$P_T^o = \{S \in P_T;\ S \text{ linear}\},$$

$$U_T = \{b \in B;\ S_1 b = S_2 b,\ S_1, S_2 \in P_T\},$$

$$U_T^o = \{b \in B;\ S_1 b = S_2 b,\ S_1, S_2 \in P_T^o\}.$$

By Theorem 2.2, $P_T^o \neq \emptyset$ and clearly $U_T \subset U_T^o$.

2.4 Theorem. The following equalities hold:

$$(1) \qquad U_T = U_T^o = \{b \in B;\ \check{T}b = \hat{T}b\}.$$

Proof. If $S \in P_T$, then $\check{T}b \leqq Sb \leqq \hat{T}b$ for every $b \in B$ by Theorem 2.2. Consequently,

$$(2) \qquad \{b \in B;\ \check{T}b = \hat{T}b\} \subset U_T.$$

Choose $b_o \in B$. By Theorem 2.2 there exist positive linear mappings T', T'' of B into D such that $T'_{|H} = T''_{|H} = T$ and $T'b_o = \hat{T}b_o$, $T''b_o = \check{T}b_o$. Thus $b_o \notin U_T^o$, provided $\check{T}b_o \neq \hat{T}b_o$. It follows that

$$(3) \qquad U_T^o \subset \{b \in B;\ \check{T}b = \hat{T}b\}.$$

Since $U_T \subset U_T^o$, (2) and (3) yield (1).

2.5 It turns out that in applications it is not easy to describe elements $b \in B$ satisfying $\check{T}b = \hat{T}b$. Under suitably chosen additional hypotheses, we are going to establish a more appropriate characterization of the sets U_T, U_T^o. To this end, assume that B is a Dedekind complete Riesz space and put

$$\hat{H} = \{\bigwedge F;\ \emptyset \neq F \subset H \text{ finite}\}.$$

Suppose that there exists a Riesz subspace L of the space B such that $\bigwedge H_1 \subset L$ for every nonempty lower directed lower bounded set $H_1 \subset \hat{H}$. Of course, $H \subset \hat{H} \subset L \subset B$.

Assume finally that there is a mapping $T_o : L \longrightarrow D$ having the following properties:

(a) $T_{o|H} = T$;

(b) T_o is a Riesz homomorphism;

(c) $T_o(\bigwedge H_1) = \bigwedge T_o(H_1)$ for every nonempty lower directed lower bounded set $H_1 \subset \hat{H}$.

For $b \in B$ define

$$\check{b} = \bigvee\{h;\ h \leqq b,\ h \in H\}, \qquad \hat{b} = \bigwedge\{h;\ b \leqq h,\ h \in H\}.$$

2.6 Theorem. The following equalities hold:

$$(4) \qquad U_T = U_T^0 = \{b \in B;\ T_0(\hat{b} - \check{b}) = 0\}.$$

Proof. Since T_0 is a Riesz homomorphism, we have $T_0(\bigwedge F) = \bigwedge T_0(F)$ for every finite $F \subset H$, $F \neq \emptyset$. Consequently, for every $b \in B$,

$$\bigwedge\{Th;\ b \leqq h,\ h \in H\} = \bigwedge\{T_0h;\ b \leqq h,\ h \in H\} = \bigwedge\{T_0k;\ b \leqq k,\ k \in \hat{H}\}.$$

The set $H_1 = \{k \in \hat{H};\ b \leqq k\}$ is lower directed and lower bounded. Thus $\bigwedge H_1 \in L$ and $T_0(\bigwedge H_1) = \bigwedge T_0(H_1)$. Of course, $\bigwedge H_1 = \hat{b}$, thus $\hat{T}b = T_0(\hat{b})$; since $\check{b} = -(\widehat{-b})$, $\check{T}b = T_0(\check{b})$. Now (4) follows from Theorem 2.4.

2.7 Notes and comments. Terminology concerning ordered vector spaces is taken from [25]. The question of uniqueness of extensions of positive linear operators has recently been studied in [21]. Theorem 2.6 which has been announced in [29] represents an abstract version of a theorem of Keldyš type. Operators analogous to $\check{T}$ and $\hat{T}$ have been studied in the context of classical potential theory in [7] and in the abstract potential theory in [34]; see also [27]. The technique of envelopes (like $\check{b}$ and $\hat{b}$) is quite typical in Choquet theory and Korovkin type theorems theory. In connection with the Dirichlet problem, this method was used by M. Brelot [7] and systematically developed by H. Bauer [2]. For applications to the Keldyš theorem, see [34].

3. Theorems of Keldyš and Korovkin type in function spaces

3.1 Let Y be a metrizable compact topological space and $B(Y)$ be the vector space of bounded functions on Y. Endowed with the natural ordering, $B(Y)$ is a Dedekind complete Riesz space. As usual, $C(Y)$ denotes the space of continuous functions on Y.

Suppose that $H(Y) \subset C(Y)$ is a vector space containing a strictly positive function and linearly separating points of Y. The last requirement means that given $y_1, y_2 \in Y$, $y_1 \neq y_2$ and $\alpha \in \mathbb{R}$, there is $h \in H(Y)$ such that $h(y_1) \neq \alpha\, h(y_2)$.

Define

$$\hat{H}(Y) = \{\inf F;\ \emptyset \neq F \subset H(Y) \text{ finite}\}.$$

Then $\hat{H}(Y) - \hat{H}(Y) = \{h_1 - h_2;\ h_1, h_2 \in \hat{H}(Y)\}$ is a Riesz subspace of $C(Y)$ containing a strictly positive function and linearly separating points of Y. Consequently, by the Stone-Weierstrass theorem, $\hat{H}(Y) - \hat{H}(Y)$ is a uniformly dense subspace of the Banach space $C(Y)$.

Let $L(Y)$ stand for the set of all functions $f: Y \longrightarrow \mathbb{R}$ for which there exist lower semicontinuous functions $f_1, f_2 \in B(Y)$ such that $f = f_1 - f_2$. Clearly, $L(Y)$ is a Riesz subspace of $B(Y)$. If $\emptyset \neq H_1 \subset \hat{H}(Y)$ is a lower bounded set, then $\inf H_1$ is an upper semicontinuous bounded function so that $\inf H_1 \in L(Y)$. Note that

$$H(Y) \subset \hat{H}(Y) \subset \hat{H}(Y) - \hat{H}(Y) \subset C(Y) \subset L(Y) \subset B(Y).$$

Let V be a Hausdorff topological space and $D(V)$ be a vector space consisting of continuous functions on V. We shall suppose that $D(V)$ is a Dedekind complete Riesz space (with respect to the natural ordering). The lattice operations in $D(V)$ are denoted by $\bigwedge$ and $\bigvee$. It should be mentioned that, in general, $f \wedge g$ does not coincide with the pointwise infimum of $f, g \in D(V)$.

Suppose finally that $T: H(Y) \longrightarrow D(V)$ is a positive linear mapping and that there is a strictly positive function $h_0 \in H(Y)$ such that $\inf\ (Th_0)(V) > 0$.

3.2 Lemma. There exists at most one mapping $T_0: L(Y) \longrightarrow D(V)$ having the following properties:

(a) $T_{0|H(Y)} = T$;

(b) T_0 is a Riesz homomorphism;

(c′) $T_0(\inf f_n) = \bigwedge \{T_0 f_n;\ n \in \mathbb{N}\}$ for every decreasing lower bounded sequence $\{f_n\}$ of continuous functions on Y.

If such a mapping T_0 exists, then there is a system $M_T = \{\mu_x;\ x \in V\}$ of positive Radon measures on Y uniquely determined by T such that

$$(5) \qquad T_0 f(x) = \int f\, d\mu_x, \qquad x \in V,\ f \in L(Y),$$

and the following condition holds:

(c) $T_0(\inf H_1) = \bigwedge T_0(H_1)$ for every nonempty lower directed lower bounded set $H_1 \subset \hat{H}(Y)$.

Proof. Let T_0, T_0' be mappings from $L(Y)$ into $D(Y)$ satisfying conditions (a),(b),(c′). It follows from (b) that

$$T_0|\hat{H}(Y) = T_0'|\hat{H}(Y),$$

thus T_0 and T_0' coincide on $\hat{H}(Y) - \hat{H}(Y)$. Fix $f \in C(Y)$ and $x \in V$. We shall prove that $T_0 f(x) = T_0' f(x)$. Let $h_0 \in H(Y)$ be a strictly positive function on Y such that $Th_0 > 0$ on V. Put $\beta = Th_0(x)$ and fix $\varepsilon > 0$. We know that there is $g \in \hat{H}(Y) - \hat{H}(Y)$ such that

$$g - (\varepsilon/2\beta)h_0 \leqq f \leqq g + (\varepsilon/2\beta)h_0$$

everywhere on Y. By (b), the mappings T_0 and T_0' are positive and linear and $T_0 g = T_0' g$ and $T_0 h_0 = T_0' h_0 = Th_0$. One easily verifies that

$$|T_0 f(x) - T_0' f(x)| \leqq \varepsilon .$$

The condition (c′) implies that the equality $T_0 f = T_0' f$ holds for every bounded upper semicontinuous function . Consequently, $T_0 = T_0'$ on $L(Y)$.

Suppose now that T_0 is a mapping having properties (a),(b), (c′). If $x \in V$, then

$$f \longmapsto T_0 f(x), \qquad f \in C(Y),$$

is a positive Radon measure on Y which will be denoted by μ_x. Then (5) holds and determines μ_x uniquely.

Let $H_1 \subset \hat{H}(Y)$ be a nonempty lower directed lower bounded set. Denote $d = T_0(\inf H_1)$ and notice that the set $T_0(H_1)$ is lower bounded in $D(V)$. Consequently, $k = \bigwedge T_0(H_1)$ exists in $D(V)$. Obviously, $T_0(\inf H_1)$ is a lower bound of the set $T_0(H_1)$. Therefore, $d \leqq k$ on V. Fix $x \in V$ and prove that $d(x) \geqq k(x)$. Since $k \leqq T_0 h'$ for every $h' \in H_1$, we have

$$k(x) \leqq \inf \{T_0 h'(x);\ h' \in H_1\} = \inf \{\int h' d\mu_x;\ h' \in H_1\}.$$

By [15], p. 35,

$$\inf \{\int h' d\mu_x;\ h' \in H_1\} = \int (\inf H_1) d\mu_x$$

and the inequality $d(x) \geqq k(x)$ is verified. This proves (c).

3.3 In what follows we shall suppose that there is a mapping $T_0 : L(Y) \longrightarrow D(V)$ possessing the properties (a),(b),(c′). Denote by $M_T = \{\mu_x;\ x \in V\}$ the system of measures uniquely determined by T in the sense of the preceding lemma.

A Borel set $Q \subset Y$ is said to be T-negligible, if $\mu_x(Q) = 0$ for every $x \in V$.

Similarly as in Sec. 2, define for $f \in B(Y)$ the functions $\hat{f}$, $\check{f}$ as follows:

$$\hat{f}(y) = \inf \{h(y);\ h \geqq f,\ h \in H(Y)\},$$

$$\check{f}(y) = \sup \{h(y);\ h \leqq f,\ h \in H(Y)\}, \qquad y \in Y.$$

Notice that the functions $\hat{f}$ and $(-\check{f})$ are upper semicontinuous and, of course, $\check{f} \leqq f \leqq \hat{f}$. It follows that

$$\{ y \in Y;\ \hat{f}(y) = \check{f}(y)\}$$

is a G_δ set.

Using a countable dense subset of C(Y), one easily deduces that

$$\{y \in Y;\ \hat{f}(y) = \check{f}(y) \text{ for every } f \in C(Y)\}$$

is also a G_δ set.

Recall that, for $f \in L(Y)$, the function $x \longmapsto \int f \, d\mu_x$ is continuous on V, since $T_0 f \in D(V)$. In particular, the function $x \longmapsto \mu_x(K)$ is continuous on V whenever $K \subset Y$ is a compact set.

3.4 Lemma. Let $Q \subset Y$ be a Borel set, let the set $Y \setminus Q$ be T-negligible and $C \subset V$ be a compact set. Then for every $\varepsilon > 0$ there is a compact set $K \subset Q$ such that $\mu_y(Y \setminus K) < \varepsilon$ whenever $y \in C$.

Proof. Fix $\varepsilon > 0$ and choose $x \in C$. Since $\mu_x(Y) = \mu_x(Q)$, there exists a compact set $K(x) \subset Q$ such that $\mu_x(Y) < \mu_x(K(x)) + \varepsilon$. The function $y \longmapsto \mu_y(Y) - \mu_y(K(x))$ is continuous on V, hence there is a neighbourhood V(x) of the point x such that

$$\mu_y(Y) < \mu_y(K(x)) + \varepsilon$$

for every $y \in V(x)$. By compactness of C, there exist $x_1,\dots,x_m \in C$ such that

$$C \subset V(x_1) \cup \dots \cup V(x_m).$$

Put $K = K(x_1) \cup \dots \cup K(x_m)$. Then K is a compact subset of Q. If $y \in C$, then $y \in V(x_j)$ for a suitable $j \in \{1,\dots,m\}$. It follows that

$$\mu_y(Y) < \mu_y(K(x_j)) + \varepsilon \leqq \mu_y(K) + \varepsilon ,$$

thus $\mu_y(Y \setminus K) < \varepsilon$.

3.5 Let us recall that, by definitions from Sec. 2, U_T (resp. U_T^o) is the set of all $f \in B(Y)$ on which all increasing (resp. positive linear) extensions of T from $H(Y)$ to $B(Y)$ coincide.

We shall define the following two sets of functions. Denote by K_T (resp. K_T^o) the set of all functions $g \in C(Y)$ for which the following condition holds: Whenever $\{T_n\}$ is a sequence of increasing (resp. positive linear) mappings from $C(Y)$ into $D(V)$ such that $\lim T_n h = Th$ uniformly on V for every $h \in H(Y)$, then $\lim T_n g = T_o g$ uniformly on compact subsets of the space V.

3.6 Theorem. For $f \in B(Y)$, the following conditions are equivalent:

(i) $f \in U_T$;

(ii) $f \in U_T^o$;

(iii) the set $\{y \in Y;\ \check{f}(y) \neq \hat{f}(y)\}$ is T-negligible.

Moreover, $K_T = K_T^o = U_T \cap C(Y) = U_T^o \cap C(Y)$.

Proof. Note that $T_o(\hat{f} - \check{f}) = 0$ if and only if (iii) holds. It follows from 3.1, 3.3 and Lemma 3.2 that one can apply results of Sec. 2 for $B = B(Y)$, $H = H(Y)$, $L = L(Y)$ and $D = D(V)$. Thus the conditions (i),(ii) and (iii) are equivalent by Theorem 2.6.

As we know, $U_T \cap C(Y) = U_T^o \cap C(Y)$ and, obviously, $K_T \subset K_T^o$. If $f \in C(Y) \setminus U_T^o$, then there are positive linear extensions S_1 and S_2 of the mapping T from $H(Y)$ to $B(Y)$ such that $S_1 f \neq S_2 f$. Putting $T_{2n} = S_1$, $T_{2n+1} = S_2$, we have $T_n h = Th$ whenever $h \in H(Y)$ and $n \in \mathbb{N}$, but $\{T_n f\}$ does not even converge on V pointwise. Thus $K_T^o \subset U_T^o \cap C(Y)$.

Suppose finally that $f \in C(Y) \cap U_T^o$. We are going to show that $f \in K_T$. Note that the set $\{y \in Y;\ \check{f}(y) \neq \hat{f}(y)\}$ is T-negligible by the implication (ii) $\Longrightarrow$ (iii).

Let $C \subset V$ be a compact set and $\varepsilon > 0$. The set $Q = \{y \in Y;\ \check{f}(y) = \hat{f}(y)\}$ is a Borel set having T-negligible complement. By Lemma 3.4 there is a compact set $K \subset Q$ such that $\mu_y(Y \setminus K) < \varepsilon$ whenever $y \in C$. Since $\check{f} = \hat{f}$ everywhere on K, for each $z \in K$, there are functions $h_z', h_z'' \in H(Y)$ such that $h_z' \geqq f \geqq h_z''$ on Y and $f(z) + \varepsilon > h_z'(z) \geqq h_z''(z) > f(z) - \varepsilon$. By continuity, there is a neighbourhood W_z of z such that

$$f(x) + \varepsilon > h_z'(x) \geqq h_z''(x) > f(x) - \varepsilon, \qquad x \in W_z.$$

Since K is compact, there are $z_1, \dots, z_k \in K$ such that $K \subset W_{z_1} \cup \dots \cup W_{z_k}$. Write h_j' and h_j'' instead of h_{z_j}' and h_{z_j}'', respectively, and put

$$h' = \inf(h_1', \dots, h_k'), \quad h'' = \sup(h_1'', \dots, h_k'').$$

Then h', $(-h'') \in \hat{H}(Y)$, $h' \geqq f \geqq h''$ and $h' - h'' < \varepsilon$ on K.

Since T_o is a Riesz homomorphism and T_o is an extension of T, we have

$$T_o h' = Th_1' \wedge \dots \wedge Th_k', \quad T_o h'' = Th_1'' \wedge \dots \wedge Th_k''.$$

Recall that, by hypothesis, there is a function $d \in D(V)$ such that $\inf d(V) > 0$.

Suppose that $\{T_n\}$ is a sequence of increasing mappings from C(Y) into D(V) such that $T_n h \to Th$ uniformly on V whenever $h \in H(Y)$. We shall show that $T_n f \to T_o f$ uniformly on C.

It is easy to see that there is $n_o \in \mathbb{N}$ such that

$$Th_j' - \varepsilon d \leqq T_n h_j' \leqq Th_j' + \varepsilon d, \quad Th_j'' - \varepsilon d \leqq T_n h_j'' \leqq Th_j'' + \varepsilon d,$$

whenever $n \geqq n_o$ and $j = 1, \dots, k$. Thus

$$Th_j'' - \varepsilon d \leqq T_n h_j'' \leqq T_n f \leqq T_n h_j' \leqq Th_j' + \varepsilon d.$$

The inequalities

$$T_n f - \varepsilon d \leqq Th_j' \text{ and } T_n f + \varepsilon d \geqq Th_j''$$

imply that

$$T_n f - \varepsilon d \leqq Th_1' \wedge \dots \wedge Th_k' = T_o h',$$

$$T_n f + \varepsilon d \geqq Th_1'' \vee \dots \vee Th_k'' = T_o h''.$$

Thus we have

$$T_o h'' - \varepsilon d \leqq T_n f \leqq T_o h' + \varepsilon d$$

on V, whenever $n \geqq n_o$.

Put $\alpha = \sup \{\mu_y(Y);\ y \in C\}$, $\beta = \sup (|f| + |h'| + |h''|)(Y)$, $\delta = \sup d(C)$. Obviously, $\alpha, \beta, \delta \in \mathbb{R}$. Recall that, for $y \in C$,

$$T_o f(y) = \int f \, d\mu_y = \int_K f \, d\mu_y + \int_{Y\setminus K} f \, d\mu_y,$$

$$\left| \int_{Y\setminus K} f \, d\mu_y \right| \leqq \int_{Y\setminus K} |f| \, d\mu_y \leqq \beta \, \mu_y(Y\setminus K) < \beta\varepsilon,$$

and, similarly, for the functions h', h''. Clearly,

$$\int h'' \, d\mu_y - \beta\varepsilon \leqq \int_K h'' \, d\mu_y \leqq \int_K f \, d\mu_y \leqq \int_K h' d\mu_y \leqq$$

$$\leqq \int h' d\mu_y + \beta\varepsilon,$$

so that

$$T_o h''(y) - 2\beta\varepsilon \leqq T_o f(y) \leqq T_o h'(y) + 2\beta\varepsilon .$$

We showed that

$$|T_n f(y) - T_o f(y)| \leqq T_o(h' - h'')(y) + \varepsilon(2\beta + \delta),$$

whenever $n \geqq n_o$ and $y \in C$. Since $0 \leqq h' - h'' < \varepsilon$ on K we get

$$T_o(h' - h'')(y) = \int (h' - h'') d\mu_y = \int_K (h' - h'') d\mu_y +$$

$$+ \int_{Y\setminus K} (h' - h'') d\mu_y \leqq \varepsilon \mu_y(K) + 2\beta \mu_y(Y\setminus K) < \varepsilon\alpha + 2\beta\varepsilon .$$

We conclude that

$$|T_n f(y) - T_o f(y)| \leqq \varepsilon(\alpha + 4\beta + \delta),$$

whenever $n \geqq n_o$ and $y \in C$. It follows that $T_n f \longrightarrow T_o f$ uniformly on C, thus $f \in K_T$.

3.7 Let $y \in Y$ and M_y be the system of all $H(Y)$ - representing measures. So $\nu \in M_y$ means that ν is a positive Radon measure on Y and $\nu(h) = h(y)$ for every function $h \in H(Y)$. Clearly, the Dirac measure ε_y concentrated at the point y belongs to M_y.

Let us apply Theorem 2.2 to the case $B = C(Y)$, $D = \mathbb{R}$, $H = H(Y)$ and the mapping $T: H(Y) \longrightarrow \mathbb{R}$ defined by $Th = h(y)$. Then $\check{T}f = \check{f}(y)$, $\hat{T}f = \hat{f}(y)$ and we conclude that the following statements are equivalent:

(i) $M_y = \{\varepsilon_y\}$;

(ii) $\check{f}(y) = \hat{f}(y)$ for every function $f \in C(Y)$.

Recall that $\{y \in Y;\ M_y = \{\varepsilon_y\}\}$ is called the Choquet boundary of Y with respect to $H(Y)$ (cf. e.g. [33],[4]) and is denoted by $Ch_{H(Y)}Y$. It follows from 3.3 that the Choquet boundary is a G_δ set.

Let us agree to denote by $d(f)$ the set of all points at which a function $f \in B(Y)$ is discontinuous.

3.8 Proposition. For every $f \in B(Y)$,

$$d(f) \subset \{y \in Y;\ \check{f}(y) \neq \hat{f}(y)\} \subset d(f) \cup (Y \setminus Ch_{H(Y)}Y).$$

Proof. Recall that $\check{f} \leqq f \leqq \hat{f}$ and the functions $\check{f}$ and $(-\hat{f})$ are lower semicontinuous. Consequently, if $y \in Y$ is such that $\check{f}(y) = \hat{f}(y)$, this common value equals $f(y)$ and f is continuous at y. The first inclusion follows.

In order to complete the proof, it is sufficient to verify the following assertion: If $y \in Ch_{H(Y)}Y$ and $f \in B(Y)$ is continuous at y, then $\check{f}(y) = \hat{f}(y)$.

Given $\varepsilon > 0$, there is a neighbourhood W of y such that $f(y) - \varepsilon \leqq f(z) \leqq f(y) + \varepsilon$ whenever $z \in W$. Let $|f| \leqq \alpha$ on Y. By Tietze's extension theorem, there are functions $f_1, f_2 \in C(Y)$ such that $f_1 = \alpha + \varepsilon$, $f_2 = -\alpha - \varepsilon$ on $Y \setminus W$, $f_1(y) = f(y) + \varepsilon$, $f_2(y) = f(y) - \varepsilon$ and $f(y) + \varepsilon \leqq f_1 \leqq \alpha + \varepsilon$, $-\alpha - \varepsilon \leqq f_2 \leqq f(y) - \varepsilon$ on Y. Then $f_1 \geqq f \geqq f_2$. Since $y \in Ch_{H(Y)}Y$, we have $\check{f}_2(y) = f_2(y)$, $\hat{f}_1(y) = f_1(y)$. Consequently, there are $h_1, h_2 \in H(Y)$ such that $h_1 \geqq f_1$, $h_2 \leqq f_2$ and $h_1(y) \leqq f_1(y) + \varepsilon$, $h_2(y) \geqq f_2(y) - \varepsilon$. We conclude that $h_1 \geqq f \geqq h_2$ and $h_1(y) - h_2(y) \leqq 4\varepsilon$. Hence $\check{f}(y) = \hat{f}(y)$.

3.9 A space $H(Y)$ is said to be a Korovkin space with respect to increasing mappings (resp. with respect to positive linear mappings), if $K_T = C(Y)$ (resp. $K_T^o = C(Y)$).

With applications to potential theory in mind, H(Y) is said to be a <u>K-space</u> (or a <u>Keldyš space</u>), provided $U_T \supset C(Y)$ (resp. $U_T^o \supset C(T)$).

<u>3.10 Theorem.</u> The following statements are equivalent:

(i) H(Y) is a Korovkin space with respect to increasing mappings;

(ii) H(Y) is a Korovkin space with respect to positive linear mappings;

(iii) H(Y) is a K-space;

(iv) H(Y) is a Keldyš space;

(v) $\{y \in Y;\ \check{f}(y) \neq \hat{f}(y)\}$ is T-negligible for every $f \in C(Y)$;

(vi) $Y \setminus Ch_{H(Y)}Y$ is T-negligible;

(vii) $U_T = \{f \in B(Y);\ d(f)$ is T-negligible$\}$;

(viii) $U_T^o = \{f \in B(Y);\ d(f)$ is T-negligible$\}$.

Proof. The equivalence of conditions (i) - (v) follows immediately from Theorem 3.6. The conditions (v) and (vi) are equivalent by 3.7. If $f \in B(Y)$ and $f \in U_T$, then d(f) is T-negligible by Theorem 3.6 and Proposition 3.8.

Suppose that (vi) holds, $f \in B(Y)$ and d(f) is T-negligible. The second inclusion of Proposition 3.8 and implication (iii) $\Longrightarrow$ $\Longrightarrow$ (i) of Theorem 3.6 show that $f \in U_T$. Thus (vii) holds and (vii) and (viii) are equivalent by Theorem 3.6.

Assume finally (viii). Clearly, $d(f) = \emptyset$ for every $f \in C(Y)$, thus $C(Y) \subset U_T^o$ and $K_T = C(Y)$ by Theorem 3.6. Consequently, (i) is established and the proof is complete.

<u>3.11 Notes and comments.</u> Theorems 3.6 and 3.10 show, in an abstract context, a relation between theorems of Keldyš and Korovkin type. The question of such a relation was raised by Prof. H. Bauer on the occasion of the conference "Funktionenräume und Funktionenalgebren", Oberwolfach, 1978.

Korovkin type theorems have been intensively studied during the last decades; see e.g. [5],[11],[4],[3],[1].

Proposition 3.8 turns out to be useful in investigations of Keldyš type theorems for discontinuous functions. In a less general form it appears in [27].

The main results of Sec. 3 were announced in [29],[31].

4. Theorems of Keldyš and Korovkin type in harmonic spaces

4.1 Suppose that X is a $\mathfrak{P}$-harmonic space with countable base in the sense of the axiomatic potential theory developed in [9]. The symbol $\mathcal{H}$ stands for the corresponding sheaf of harmonic functions.

Let U be a nonempty relatively compact open subset of X. The set of irregular points of U is denoted by U_i. Define, as in the introduction,

$$H(U) = \{h \in C(\overline{U});\ h_{|U} \in \mathcal{H}(U)\}, \qquad H(\partial U) = H(U)_{|\partial U}.$$

Recall that $f \in H(\partial U)$, if and only if there is a solution of the classical Dirichlet problem for the boundary condition f.

The complement of U is denoted by CU and for $x \in X$, the symbol ε_x^{CU} means the balayage of ε_x on CU.

A Borel set $Q \subset \partial U$ is said to be negligible, if $\varepsilon_x^{CU}(Q) = 0$ for every $x \in U$.

A set $U \subset X$ is said to be admissible, if U is nonempty, relatively compact and open and the space H(U) contains a strictly positive function and linearly separates the points of $\overline{U}$.

The following important assertion is a consequence of results of [6], p. 97.

4.2 Proposition. Let U be an admissible set. Then the following conditions are equivalent:

(i) U_i is negligible;

(ii) $\partial U \setminus Ch_{H(\partial U)}\overline{U}$ is negligible.

4.3 Let U be a nonempty relatively compact open set in X. A mapping $A: C(\partial U) \longrightarrow \mathcal{H}(U)$ is said to be a K-operator (on U), if A is an increasing mapping and $A(h_{|\partial U}) = h_{|U}$, whenever $h \in H(U)$. If, moreover, A is linear, then A is called a Keldyš operator.

Given a resolutive function f on ∂U, $H^U f$ stands for the PWB-solution of the generalized Dirichlet problem; see [9], p. 18. By [9], pp. 18, 50, the mapping

$$A: f \longmapsto H^U f, \qquad f \in C(\partial U),$$

is a Keldyš operator.

A nonempty relatively compact open set $U \subset X$ is said to be a K-set (resp. a Keldyš set), provided there is exactly one K-operator (resp. Keldyš operator) on U.

Obviously every K-set is a Keldyš set.

4.4 Theorem. If U is a Keldyš set, then U_i is negligible.
Proof. See [24]; cf. also [22].

4.5 Suppose that U is an admissible set. Then $H(\partial U)$ is a majorizing subspace of $B(\partial U)$, the space of all bounded functions on ∂U.

We shall apply the results of Sec. 3 to the following situation: $Y = \partial U$, $H(Y) = H(\partial U)$, $V = U$ and $D(U) = \mathcal{H}^+(U) - \mathcal{H}^+(U)$, the space of differences of positive harmonic functions on U. By [9], p. 38, D(U) is a Dedekind complete Riesz space. The corresponding lattice operations are denoted again by $\vee$ and $\wedge$. Recall that $L(\partial U)$ is the space of differences of bounded lower semicontinuous functions.

For $h \in H(U)$ define

$$T(h_{|\partial U}) = h_{|U}.$$

Then $h_{|U} \in D(U)$ and $T: H(\partial U) \longrightarrow D(U)$ is a positive linear mapping by the minimum principle [9], p. 26. Recall that by [9], p. 51, every function $f \in L(\partial U)$ is resolutive and $H^U f \in D(U)$.

It follows from [9], p. 50, that the mapping

$$T_0 : f \longmapsto H^U f, \qquad f \in L(\partial U),$$

satisfies conditions (a),(b) and (c') of Lemma 3.2. The corresponding system M_T is, of course, $\{\varepsilon_x^{CU};\ x \in U\}$. Consequently, T-negligible simply means negligible in the sense of 4.1.

Since U is supposed to be admissible, there is a strictly positive function $h_0 \in H(\partial U)$ such that $\inf\,(Th_0)(U) > 0$. We see that all assumptions of 3.1 are satisfied.

Comparing definitions 3.9 and 4.3, we notice that $H(\partial U)$ is a K-space (resp. Keldyš space), if and only if U is a K-set (resp. Keldyš set).

Recall the definition of $\check{f}(y)$, $\hat{f}(y)$ for $f \in B(\partial U)$ and $y \in \partial U$:

$\check{f}(y) = \sup\{h(y);\ h \leqq f,\ h \in H(\partial U)\}$, $\hat{f}(y) = \inf\{h(y),\ h \geqq f,\ h \in H(\partial U)\}$.

Define

$$S(U) = \{s \in C(\overline{U});\ s_{|U} \text{ superharmonic on } U\}.$$

4.6 Proposition. Let U be an admissible set and $f \in C(\partial U)$. Then the following conditions are equivalent:

(i) If A_1, A_2 are K-operators on U, then $A_1 f = A_2 f$.

(ii) If A_1, A_2 are Keldyš operators on U, then $A_1 f = A_2 f$.

(iii) The complement of the set

$$\{y \in \partial U;\ \check{f}(y) = \hat{f}(y)\} \text{ is negligible.}$$

Proof. This follows as a special case of Theorem 3.6.

4.7 Theorem. Let U be an admissible set. The following conditions are equivalent:

(i) U is a K-set;

(ii) U is a Keldyš set;

(iii) U_i is negligible;

(iv) $H(\partial U)$ is a Korovkin space with respect to increasing mappings;

(v) $H(\partial U)$ is a Korovkin space with respect to positive linear mappings;

(vi) $U_T = \{f \in B(\partial U);\ d(f) \text{ is negligible}\}$;

(vii) $U_T^o = \{f \in B(\partial U);\ d(f) \text{ is negligible}\}$;

(viii) the complement of the set $\{y \in \partial U;\ \check{f}(y) = \hat{f}(y)\}$ is negligible, whenever $f \in C(\partial U)$;

(ix) $H^U f = \bigwedge \{h_{|U};\ h \in H(U),\ h_{|\partial U} \geqq f\}$, whenever $f \in C(\partial U)$;

(x) $H^U f = \bigvee \{h_{|U};\ h \in H(U),\ h_{|\partial U} \leqq f\}$, whenever $f \in C(\partial U)$;

(xi) $H^U f = \inf \{s_{|U};\ s \in S(U),\ s_{|\partial U} \geqq f\}$, whenever $f \in C(\partial U)$;

(xii) $H^U f = \sup \{t_{|U};\ -t \in S(U),\ t_{|\partial U} \leqq f\}$, whenever $f \in C(\partial U)$.

Proof. The equivalence of conditions (i) - (viii) follows from Theorem 3.10 and Proposition 4.2.

Conditions (ix) and (x) and also conditions (xi) and (xii) are obviously equivalent.

For $f \in C(\partial U)$ define

$$A_1 f = \bigwedge \{ h_{|U};\ h \in H(U),\ h_{|\partial U} \geqq f \},$$

$$A_2 f = \inf \{ s_{|U};\ s \in S(U),\ s_{|\partial U} \geqq f \}.$$

Clearly, A_1 is a K-operator and A_2 is an increasing mapping satisfying $A_2(h_{|\partial U}) = h_{|U}$ for every $h \in H(U)$ (this follows from the minimum principle, see [9], p. 26). It is not, however, evident that $A_2 f \in \mathcal{H}(U)$ whenever $f \in C(\partial U)$.

Fix $f \in C(\partial U)$. To show that $A_2 f \in \mathcal{H}(U)$, it is sufficient to prove that

$$\{ s_{|U};\ s \in S(U);\ s_{|\partial U} \geqq f \}$$

is a Perron set; cf. [9], pp. 37, 38. In view of [9], pp. 33, 37, 38, this is true provided for every $x \in U$ there is a regular set V such that $x \in V \subset \overline{V} \subset U$. But U is admissible, which implies by [9], p. 65, that every point of U even possesses a fundamental system of regular neighbourhoods.

We conclude that A_1, A_2 are K-operators and $A_2 f \leqq A_1 f$ whenever $f \in C(\partial U)$. It follows easily that (ix) $\Longrightarrow$ (xi). Thus if (i) holds, then $A_1 f = H^U f = A_2 f$ for any $f \in C(\partial U)$.
It remains to prove the implication (xi) $\Longrightarrow$ (i).

Suppose (xi). Then (xii) holds and by [9], pp. 164, 165, we have

$$(\varepsilon_x^{CU})^{CU}(s) \leqq \varepsilon_x^{CU}(s) \leqq s(x)$$

whenever $s \in S(U)$ and $x \in U$ (cf. [27]). If $f \in C(\partial U)$, $s, -t \in S(U)$, $t_{|\partial U} \leqq f \leqq s_{|\partial U}$ and $x \in U$, then

$$t(x) \leqq (\varepsilon_x^{CU})^{CU}(t) \leqq (\varepsilon_x^{CU})^{CU}(f) \leqq (\varepsilon_x^{CU})^{CU}(s) \leqq s(x).$$

By (xi) and (xii), $(\varepsilon_x^{CU})^{CU}(f) = \varepsilon_x^{CU}(f)$. We conclude that

$(\varepsilon_x^{CU})^{CU} = \varepsilon_x^{CU}$ for every $x \in U$, which implies (iii) (cf. [24] or

[22]). But (iii) implies (i) and the proof is complete.

<u>4.8 Notes and comments.</u> The use of the method of envelopes (like $\check{f}$ and $\hat{f}$) in connection with the Keldyš theorem goes back to [7]; cf. also [34] and [27].

The question of uniqueness of a reasonable generalization of the classical Dirichlet problem was raised by A.F. Monna fifty years ago. For the development of this problem in the context of classical potential theory, see the references and discussion in [27] and [28]. In [28], an elementary proof of the Keldyš theorem is given. The following remarks are related to Theorem 4.7 from the point of view of abstract potential theory.

For a Brelot harmonic space satisfying the domination axiom, condition (iii) is automatically satisfied. The validity of (i) in this case was proved in [8]. As pointed out in [12], the condition (iii) is no longer true in potential theory associated with the heat equation; cf. also [20].

As observed in [22], in this situation (ii) fails in general. For Bauer's axiomatics, under additional hypotheses, implications (viii) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are also proved and, as mentioned there, the converse implications follow from [6]. For operators A satisfying $A(s_{|\partial U}) \leqq s_{|U}$ for every $s \in S(U)$, equivalence of conditions (iii),(ii),(xi) and (xii) is proved in [23]; see also [16],[17]. The conditions (i),(ii),(viii),(x) are shown to be equivalent in [34].

In the above mentioned papers, only extensions from $H(\partial U)$ to $C(\partial U)$ are considered. Discontinuous boundary conditions have been investigated in [27] where related results and further information can be found.

Results of Sec. 4 show how an abstract approach from Sec. 2 and 3 enables the establishment of old as well as new results about theorems of Keldyš and Korovkin type.

We remark that Theorem 4.7 fails provided U is supposed to be relatively compact open, but not necessarily admissible. This is shown in [32].

Note also that $\mathcal{H}^+(U) - \mathcal{H}^+(U)$ can be shown to be a weakly σ-distributive super Dedekind complete Riesz space; cf. [30]. For terminology see [25],[13]. Significance of the condition of weak σ-distributivity for the theory of measures and integrals with values in ordered spaces is explained in [36],[37].

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