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COMPACTNESS AND WEAK COMPACTNESS OF GRADIENT MAPS

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Let X be a real normed space $E \subset X$ a subset of X. A mapping $F : E \to X^*$ is said to be a gradient map if there exists a functional $f : E \to R$ having the Gâteaux (or Fréchet) derivative f' on E such that $F(u) = f'(u)$ for each $u \in E$. Using the James [3] deep criteria of the compactness of the sets and the results from the theory of locally convex spaces, we establish some further results concerning the compactness and weak compactness of gradient maps. Recall that the compactness and continuity properties of gradient mappings were considered by Ando [1], Daniel [2], Kadec [4], De Lamadrid [5], Palmer [6], Restrepo [7], Rothe [8], Vajnberg [9] and others.

NOTATIONS AND DEFINITIONS

Let X be a real normed linear spaces, X^* its dual, $\langle\ ,\ \rangle$ the pairing between X^* and X. We denote by $\sigma(X,X^*)$, $\sigma(X^*,X)$ the weak and weak* topologies in X, X^*, respectively. Let Γ be a linear subspace of X^*, which is total over X. We define the Γ-topology of X (or the weak topology in X induced by Γ) as the topology with the fundamental system of neighborhoods of 0 consisting of all sets of the form

$$V_{K,\varepsilon} = \{ u \in X : |\langle u^*, u\rangle| < \varepsilon \quad \text{for all} \quad u^* \in K \},$$

where K is a finite subset of Γ and $\varepsilon > 0$. This topology is a Hausdorff locally convex topology on X, which is weaker than the $\sigma(X,X^*)$-topology on X. A Banach space X is said to be :

(i) smooth, if the norm of X is Gâteaux differentiable on

$$S_1(0) = \{ u \in X : \|u\| = 1 \} ;$$

(ii) a dual Banach space if there exists a Banach space Z such that $X = Z^*$ in the sense of topology and the norm. By the weak* topology in X we always mean the $\sigma(Z^*,Z)$-topology of Z^*, where

$X = Z^*$. The closed (bounded) balls of X are $\sigma(Z^*,Z)$-compact. The examples of dual Banach spaces : l_1, l_∞, L_∞, reflexive Banach spaces, the Orlicz space $L_M(G)$, where an N-function satisfies Δ_2 -condition. A set valued mapping $J : X \to 2^{X^*}$ is called a (normalized) duality mapping, if for each $u \in X$

$$J(u) = \{ u^* \in X^* : \langle u^*, u\rangle = \|u\|^2 , \quad \|u^*\| = \|u\| \} .$$

For each $u \in X$, $J(u)$ is non-empty convex weakly* compact subset of X^*. By $f'(u)$ we denote the Gâteaux or Fréchet derivative of $f : X \to R$ at u. From the context it will be always clear which kind of derivative is considered. Let $G \subset X$ be a subset of X. We shall say that $f : G \to R$ is uniformly Fréchet differentiable on G if f has the Fréchet derivative f' on G and $\|h\|^{-1} \| f(u+h) - f(u) - f'(u)h \| \to 0$ uniformly with respect to $u \in G$ as $\|h\| \to 0$ for all $h \in X$ such that $u + h \in G$. We shall say that a subset M of X contains a set E of X properly, if $E \subset M$ and $\operatorname{dist}(\partial E, \partial M) > 0$, where ∂E denotes the boundary of E in X.

Let E, M be subsets of X such that M is open and E is contained in M properly. Let $f : M \to R$ be a functional having the Gâteaux derivative $f'(u)$ on E, Γ be a linear subspace of X^* which is total over X. We shall say that a functional f has a property (P_β) on E it there exists a $\sigma(X, \Gamma)$-continuous functional $g : M \to R$ which is uniformly Fréchet differentiable on bounded subsets of E such that the following condition is satisfied: if $(v_\beta) \subset E$, $(u_\beta) \subset X$ are nets, $v_\beta \to v_0 \in E$, $u_\beta \to 0$ converges both in the $\sigma(X, \Gamma)$-topology, then there exists an index β_0 such that $|\langle f'(v_\beta), u_\beta\rangle| \leqq |\langle g'(v_\beta), u_\beta\rangle|$ for $\beta \geqq \beta_0$. J. Daneš pointed out that the last condition is satisfied iff there exists a functional $\varphi : E \to R$ such that $|\varphi(u)| \leqq 1$ and $f'(u) = \varphi(u)\, g'(u)$ for each $u \in E$. Note that φ need not be either $\sigma(X, \Gamma)$-continuous, or uniformly Fréchet differentiable on E. A mapping $F : E \to X^*$, where $E \subset X$, is said to be compact (weakly compact) on E if for each bounded set $B \subset E$ the set $F(B)$ is relatively compact (relatively weakly compact) in X. For a set $E \subset X$ we set $F = f'$, where $E \ni u \to f'(u) \in X^*$ and f' denotes the Gâteaux (or Fréchet) derivative of f on E.

RESULTS

THEOREM 1. Let X be a Banach space, $\Gamma \subset X^*$ a closed separable total subspace over X, E a convex balanced $\sigma(X, \Gamma)$-compact subset of X, M an open subset of X which contains E properly, $f : M \to R$ a functional having the Gâteaux derivative f' on E and satisfying the condition (P_β) on E. Assume that $F(E) \subset \Gamma$ and $J(u) \cap \Gamma \neq \emptyset$ for each $u \in X$.

Then F is compact on E. Moreover, f is Lipschitzian on each closed convex subset of E.

THEOREM 2. Let X be a separable reflexive Banach space, $\emptyset \neq E \subset X$ a convex closed bounded subset of X, M an open subset of X which contains E properly, $f : M \to R$ a functional having the Gâteaux derivative f' on E. Suppose that there exists a sequentially weakly continuous functional $g : M \to R$ such that either

(i) M is convex, g is convex on M and has the Gâteaux derivative g' on E; or

(ii) g is uniformly Fréchet differentiable on E. If $\|f'(u)\| \leqq \|g'(u)\|$ for each $u \in E$, then F is compact on E.

THEOREM 3. Under the assumptions of Theorem 2 assume only that X is reflexive Banach space.

Then F is weakly compact on E.

PROPOSITION 1. Let X be a dual Banach space, M an open subset of X, E a closed bounded subset of X properly contained in M, $f : M \to R$ a weakly* continuous functional on M.

Then the following statements are valid:

(i) If f is uniformly Fréchet differentiable on E, $F : E \to Z$, then F is continuous on E from the weak* topology of X to the strong topology of Z. Moreover, if Z is separable, then F is compact on E.

(ii) Suppose M is convex, Z^* is separable, f is continuous and convex on X and has the Gâteaux derivative f' on E. If F maps E into Z, then F is compact on E.

THEOREM 4. Let X be a separable complete semireflexive Hausdorff locally convex space, M an open convex subset of X, E a closed convex bounded subset of X such that $E \subset M$ and $\partial M \cap \partial E = \emptyset$. If $f : M \to R$ is convex weakly continuous functional having the Gâteaux derivative f' on E, then F is weakly compact on E.

REMARK. Let X, Y be Hausdorff locally convex spaces, $A : X \to Y$ a linear mapping. Then the graph $G(A)$ of A is closed in $X \times Y$ if and only if the closed linear subspace

$\Gamma = \{u^* \in Y^* : x \to \langle u^*, Ax\rangle$ is continuous on $X\}$ is total over Y ; i.e. Γ is weak* dense in Y^*. Setting $X = Y$ we have the examples of the closed total subspaces Γ over X.

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