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Introduction and basic notation.

A $\varkappa$-metrik on a topological space X is a function ρ which assigns to each $x \in X$ and each regular closed subset C of X a non-negative number $\rho(x,C)$ and satisfies some naturally conditions (Scepin [3]):

(S1) $\rho(x,C) = 0 \Longleftrightarrow x \in C$

(S2) $C_1 \subseteq C_2 \Longrightarrow \rho(x,C_1) \geq \rho(x,C_2)$

(S3) $\rho(x,C)$ is a continuous mapping for fixed C.

(S4) If $\{C_\alpha\}$ is a transfinite increasing sequence of regular closed sets, then $\rho(x,[\bigcup C_\alpha])=\inf \rho(x,C_\alpha)$.

For a space X we denote by $\tau(X)$ the set of open subsets of X.

Definition. A weak $\varkappa$-metrik on X is a function, defined on the set $X \times \tau(X)$, assuming non-negative real values and satisfying the following axioms:

(A1) $\rho(x,W)=0 \Longleftrightarrow x \in [W]$

(A2) $W_1 \subseteq W_2 \Longrightarrow \rho(x,W_1) \geq \rho(x,W_2)$

(A3) $\rho(x,W)$ is a continuous mapping for fixed W.

(A4) If $\{W_\alpha\}$ is a transfinite increasing sequence of open sets, then $\rho(x,\bigcup W_\alpha)=\inf \rho(x,W_\alpha)$.

For weak $\varkappa$-metrics all of the main results of Scepin about $\varkappa$-metrics are also valid. The proofs of the following theorems and also some others are completely analog to those of the corresponding results of Scepin stated for the case of $\varkappa$-metrics.

Theorem. A product space of weak $\varkappa$-metric spaces is weakly $\varkappa$-metrizable.

Theorem. Let X be a weak $\varkappa$-metric compact space. Then every mapping from X onto any compact space Y can be represented as a composition of two mapping g and h, f=g h, where h is an open mapping from X onto any compact space Z and wZ=wY.

In this paper we prove two main results.

Theorem 1. Let X be a weak $\varkappa$-metric space, $f:X \longrightarrow Y$ a continuous mapping, $X' \subseteq X$ such that $f|_{X'}$ is open and perfect and $f(X')=Y$. Then Y is weakly $\varkappa$-metrizable.

It is an open problem by Scepin [4] to show this fact for - metrics.

Theorem 2. Every limit space of a transfinite inverse system with weak $\varkappa$-metric spaces and continuous open perfect and surjective projection mappings is weakly $\varkappa$-metrizable.

To prove this theorem we construct a weak $\varkappa$-metric on the limit space. The assertion of Theorem 2 for $\varkappa$-metrics and compact spaces is already proved by Scepin [5]. But his proof does not contain such a construction and is very difficult. Our proof of Theorem 2 and also the proofs of other facts show that as a rule it is easier to work with weak $\varkappa$-metrics than with $\varkappa$-metrics. Therefore we introduced the new concept of weak $\varkappa$-metrics.

Theorem 2 implies

Corollary. A compact space is weakly $\varkappa$-metrizable iff it is $\varkappa$-metrizable.

It would be nice to find a construction which yields a $\varkappa$-metric induced by a weak $\varkappa$-metric. But we do not know (exceptly for compact spaces) if every weak $\varkappa$-metric space is $\varkappa$-metrizable.

We consider an ordinal to be the set of smaller ordinals. All topological spaces in the present paper are assumed to be completely regular and all mappings of topological spaces to be continuous. A transfinite inverse system $(X_\alpha, p_\alpha^\beta)_{\alpha<\beta<k}$ is said to be continuous if for every limit ordinal $\gamma<k$ the space X_γ is canonically homeomorphic to the limit space of $(X_\alpha, p_\alpha^\beta)_{\alpha<\beta<\gamma}$. If $\{A_t : t \in T\}$ is a family of subsets of the space X and $\mathcal{J}$ is a filter on T we set

$$\overline{\lim_{J}}\, A_t = \bigcap_{E\in\mathcal{J}} [\bigcup_{t\in E} A_t] .$$

See [1], [2] for unexplained notation.

1. Preparations for the main results.

Using transfinite induction one can easily verify

Lemma 1. If ρ is a weak $\varkappa$-metric on X and $\{W_\alpha : \alpha \in A\}$ is a family of open subsets of X, then

$$\rho(x, \bigcup_{\alpha\in A} W_\alpha) = \inf\{\rho(x, \bigcup_{\beta\in B} W_\beta) : B \subseteq A,\ B \text{ is finite}\}.$$

Lemma 2. Let ρ be a weak $\varkappa$-metric on X. For any closed subset $F \subseteq X$ and any point $x \in X$, let $e(x,F) = \rho(x, X\setminus F)$.

If $\{F_t: t\in T\}$ is a family of closed subsets of X and $x_t \xrightarrow[\mathfrak{J}]{} x$, where $\mathfrak{J}$ is an ultrafilter on T, then $\lim\limits_{\mathfrak{J}} e(x_t,F_t)\le e(x,\overline{\lim\limits_{\mathfrak{J}}} F_t)$.

Proof. We set $W_t=X\setminus F_t$ for $t\in T$, $U_E=\mathrm{Int}(\bigcap\limits_{t\in E} W_t)$ and

$V_E=[\bigcup\limits_{t\in E} F_t]$ for every $E\in\mathfrak{J}$. Hence, $V_E=X\setminus U_E$, $E\in J$, and

$\overline{\lim\limits_{\mathfrak{J}}} F_t=X\setminus \bigcup\limits_{E\in\mathfrak{J}} U_E$. Let a be a real number $<\lim\limits_{\mathfrak{J}} e(x_t,F_t)$. Then

$E_o=\{t\in T: e(x_t,F_t)>a\}\in\mathfrak{J}$. Since $\rho(x_t,W_t)=e(x_t,F_t)$ for every $t\in\Gamma$, (A2) implies $\rho(x_t,U_E)>a$, $E\in\mathfrak{J}$, $E\subseteq E_o$, $t\in E$. Hence, by (A3), $\rho(x,U_E)\ge a$ and in virtue of Lemma 1, $\rho(x,\bigcup\limits_{E\in\mathfrak{J}} U_E)\ge a$.

Consequently, $e(x,\overline{\lim\limits_{\mathfrak{J}}} F_t)\ge a$. //

Lemma 3. Let $\mathcal{F}$ be a family of closed subsets of X and $e: X\times\mathcal{F}\longrightarrow[0,\infty)$ be a function satisfying the following conditions:

(B1) $e(x,F)$ is continuous for fixed $F\in\mathcal{F}$.

(B2) $x\notin F \Longrightarrow e(x,F)=0$, where $x\in X$ and $F\in\mathcal{F}$.

(B3) For every point $x\in X$ and every neighbourhood Ox of x there exists an $F\in\mathcal{F}$, $F\subseteq Ox$, such that $e(x,F)>0$.

(B4) If $\{F_t:t\in T\}\subseteq\mathcal{F}$, $x_t\xrightarrow[\mathfrak{J}]{} x$, where $\mathfrak{J}$ is an ultrafilter on T and $e(x_t,F_t)>a>0$ for every $t\in T$, then there exists an $F\in\mathcal{F}$, $F\subseteq\overline{\lim\limits_{\mathfrak{J}}} F_t$, such that $e(x,F)\ge a$.

Then the formula

$$\rho(x,W)=\sup\{e(x,F):F\in\mathcal{F} \text{ and } F\cap W=\emptyset\}$$

defines a weak $\varkappa$-metric on X.

Remark. By Lemma 2 the conditions (B1)-(B4) are satisfied for the function $e(x,F)=\rho(x,W)$, where $W\in\tau(X)$ and ρ is a weak $\varkappa$-metric on X.

Proof of Lemma 3. Axioms (A1) and (A2) are clearly satisfied. Now we prove (A3). It is easy to see that $\{x: \rho(x,W)>a\}$ is an open set for every real number a. We show that $\{x: \rho(x,W)\ge a\}$ is a closed set. Let $x_t\xrightarrow[\mathfrak{J}]{} x$, where $\mathfrak{J}$ is an ultrafilter on T and $\rho(x_t,W)\ge a$ for every $t\in T$. Let ε be a positive real number. For every $t\in T$ there exists a set $F_t\in\mathcal{F}$ such that $e(x_t,F_t)>a-\varepsilon$ and $F_t\cap W=\emptyset$. (B4) implies that there exists an $F\in\mathcal{F}$ such that $F\subseteq\overline{\lim\limits_{\mathfrak{J}}} F_t$ and $e(x,F_\varepsilon)\ge a-\varepsilon$. Clearly, $F_\varepsilon\cap W=\emptyset$. Hence,

$\rho(x,W) \geq a$.

Proof of (A4): Let $\{W_\alpha\}_{\alpha<k}$ be a transfinite increasing sequence of open subsets of X and set $W = \bigcup_{\alpha<k} W_\alpha$, $a = \inf \rho(x,W_\alpha)$,

$B = \rho(x,W)$. Assume $b<a$. Then for every $\alpha<k$ there exists an $F_\alpha \in \mathcal{F}$ such that $F_\alpha \cap W_\alpha = \emptyset$ and $e(x,F_\alpha) > \frac{1}{2}(a+b)$. Let $\mathfrak{J}$ be an ultrafilter on k, $\mathfrak{J} \supseteq \{(\beta,k) : \beta<k\}$. By (B3) there exists an $F \in \mathcal{F}$ such that $e(x,F) \geq \frac{1}{2}(a+b)$ and $F \subseteq \overline{\lim_{\mathfrak{J}}} F$. It is easy to see that $F \cap W = \emptyset$. Hence, $\rho(x,W) \geq \frac{1}{2}(a+b)$; a contradiction. //

Lemma 4. a) If $f: X \longrightarrow Y$ is a perfect mapping, $\{A_t : t \in T\}$ a family of subsets of X and $\mathfrak{J}$ a filter on T, then $f(\overline{\lim_{\mathfrak{J}}} A_t) = \overline{\lim_{\mathfrak{J}}} fA_t$.

b) If $f: X \longrightarrow Y$ is an open and perfect mapping, $\{B_t : t \in T\}$ a family of subsets of Y and $\mathfrak{J}$ a filter on T, then $f^{-1}(\overline{\lim_{\mathfrak{J}}} B_t) = \overline{\lim_{\mathfrak{J}}} f^{-1}B_t$.

The easy proof is omitted.

2. Main results.

Theorem 1. Let ρ be a weak $\varkappa$-metric on X, $f: X \longrightarrow Y$, $X' \subseteq X$ such that $f|_{X'}$ is open and perfect and $f(X')=Y$. Then $\rho'(y,W) = \sup\{\rho(x,f^{-1}W) : x \in f^{-1}y \cap X'\}$, $y \in Y$ and $W \in \tau(Y)$, defines a weak $\varkappa$-metric on Y.

Proof. It is easy to check (A1) and (A2) for ρ'. Now we prove (A3). Let $y \in Y$, $W \in \tau(Y)$ and $\varepsilon > 0$. For every $x_0 \in f^{-1}y \cap X'$ there exists a neighbourhood Ox such that $|\rho(x,f^{-1}U) - \rho(x_0,f^{-1}U)| < \varepsilon$ for each $x \in Ox_0$. Since $f^{-1}y \cap X'$ is compact, there exist points $x_1,\dots,x_n \in f^{-1}y \cap X'$ such that $f^{-1}y \cap X' \subseteq \bigcup_{i=1}^{n} Ox_i$. $O = \bigcap_{i=1}^{n} f(Ox_i \cap X')$ is a neighbourhood of y_0. Let O' be a neighbourhood of y such that $O' \subseteq O$ and $f^{-1}O' \cap X' \subseteq \bigcup_{i=1}^{n} Ox_i$. It is easy to verivy that for every $y \in O'$ we have $|\rho'(y,W) - \rho'(y_0,W)| < \varepsilon$.

Proof of (A4): Let $\{W_\alpha\}_{\alpha<k}$ be a transfinite increasing sequence of open subsets of Y and let a be a real number $< \inf \rho'(y,W_\alpha)$. For each $\alpha<k$ let us define

$F_\alpha = \{x \in f^{-1}y \cap X' : \rho(x,f^{-1}W_\alpha) \geq a$.

For each α F_α is a nonempty compact set. Since $\{F_\alpha\}_{\alpha<k}$ is a

decreasing sequence, we conclude: $\bigcap F_\alpha \neq \emptyset$. Hence, there exists a point $x \in f^{-1}y \cap X'$ such that $\varrho(x, f^{-1}W_\alpha) \geq a$ for each $\alpha < k$. Consequently, $\varrho(x, f^{-1}(\bigcup W_\alpha)) \geq a$ and $\varrho'(y, \bigcup W_\alpha) \geq a$. //

Let $S = (X_\alpha, p_\alpha^\beta)_{\alpha<\beta<k}$ be a transfinite inverse continuous system with open perfect and surjective projection mappings. Let A be an arbitrary subset of $\varprojlim S$. Scepin [4] introduced the following notation

$$d(A) = \{ \alpha < k : p_{\alpha+1}A \neq (p_\alpha^{\alpha+1})^{-1} p_\alpha A \}.$$

One can easily verify

Lemma 5. $d([A]) \subseteq d(A)$.

Lemma 6 (Scepin [4]). Let $\{A_t : t \in T\}$ be a family of subsets of $\varprojlim S$. If $\mathfrak{J}$ is an ultrafilter on T, then

$$d(\overline{\lim_{\mathfrak{J}}} A_t) \subseteq \{\alpha : \{ t : \alpha \in d(A_t)\} \in \mathfrak{J}\}.$$

If there exists a natural number n_0 such that $|d(A_t)| \leq n_0$ for every t, then $|d(\overline{\lim_{\mathfrak{J}}} A_t)| \leq n_0$.

Proof. Assume that $E = \{ t : \alpha \notin d(A_t)\} \in \mathfrak{J}$, $\alpha < k$. Then $p_{\alpha+1}A_t = (p_\alpha^{\alpha+1})^{-1} p_\alpha A_t$ for every $t \in E$. In account of Lemma 4 we have $(p_\alpha^{\alpha+1})^{-1} p_\alpha (\overline{\lim_{\mathfrak{J}}} A_t) = (p_\alpha^{\alpha+1})^{-1} (\overline{\lim_{\mathfrak{J}}} p_\alpha A_t) = \overline{\lim_{\mathfrak{J}}} (p_\alpha^{\alpha+1})^{-1} p_\alpha A_t = \overline{\lim_{\mathfrak{J}}} p_{\alpha+1} A_t = p_{\alpha+1} \overline{\lim_{\mathfrak{J}}} A_t$, i.e. $\alpha \notin d(\overline{\lim_{\mathfrak{J}}} A_t)$. //

Lemma 7 (Scepin [3]). The family of open subsets W of $\varprojlim S$ such that $d(W)$ is finite is a base of $\lim S$.

Theorem 2. Let $S = (X_\alpha, p_\alpha^\beta)_{\alpha<\beta<k}$ be a transfinite inverse continuous system with open perfect and surjective projection mappings. If all X_α's are weak $\varkappa$-metric spaces, then the limit space of S is also weakly $\varkappa$-metrizable.

Proof. Let ϱ_α be a weak $\varkappa$-metric on X_α, $\varrho_\alpha \leq 1$, $\alpha < k$. For $\alpha < k$ define $e_\alpha(x, F) = \varrho_\alpha(x, X \setminus W)$, $x \in X$, F closed subset of X. Then e is a function, satisfying conditions (B1)-(B4). We denote by $\mathfrak{F}$ the set of all closed subsets F of $X = \varprojlim S$ such that $d(F)$ is finite. Define $e : X \times \mathfrak{F} \longrightarrow [0, \infty)$ by

$$e(x, F) = \frac{1}{|d(F)|} \min\{ e_{\alpha+1}(p_{\alpha+1}x, p_{\alpha+1}F) : \alpha \in d(U)\}.$$

We show that e satisfies conditions (B1)-(B4).

(B1) is clearly fulfilled. If $x \in F$, where $F \in \mathfrak{F}$, then there exists an $\alpha \in d(U)$ such that $p_{\alpha+1}x \notin p_{\alpha+1}F$. Hence, $e_{\alpha+1}(p_{\alpha+1}x, p_{\alpha+1}F) = 0$

and $e(x,F)=0$, i.e. (B2) is also valid.
Proof of (B3): Let $x \in X$ and Ox be a neighbourhood of x. By Lemma 7, there exists an open set $W \subseteq X$ such that $d(W)$ is finite and $x \in W$, $[W] \subseteq Ox$. Hence, by Lemma 5, $[W] \in \mathfrak{F}$. We put $\beta = \max d([W])+1$. Consequently, $p_\beta^{-1} p_\beta [W]=[W]$, $p_\beta(X \setminus [W]) = X_\beta \setminus p_\beta([W])$ and $p_\beta(W) \cap p_\beta(X \setminus [W]) = 0$. Thus, $p_\beta(x) \notin [p_\beta(X \setminus [W])$, $e_\beta(p_\beta(x), p_\beta[W]) > 0$, and $e(x,[W]) > 0$.
Now we prove (B4). Let $\{F_t : t \in T\} \subseteq \mathfrak{F}$, $x_t \xrightarrow{\mathfrak{J}} x$, where $\mathfrak{J}$ is an ultrafilter on T, and $e(x_t, F_t) > a > 0$ for each $t \in T$. Hence, $|d(F_t)| \leq \frac{1}{a}$ for every $t \in T$. Since $\mathfrak{J}$ is an ultrafilter, there exists a natural number n_0 such that $E_0 = \{t \in T : |d(F_t)| = n_0\} \in \mathfrak{J}$. By Lemma 6 we have $F = \overline{\lim_{\mathfrak{J}}}\, F_t \in \mathfrak{F}$ and $E_\alpha = \{t \in E : \alpha \in d(F_t)\} \in \mathfrak{J}$, $\alpha \in d(F)$. If $\alpha \in E_\alpha$, then $e_{\alpha+1}(p_{\alpha+1}(x_t), p_{\alpha+1}F_t) > n_0\, a$. Hence, by Lemma 2, $e_{\alpha+1}(p_{\alpha+1}x, \overline{\lim_{\mathfrak{J}}}\, p_{\alpha+1}F_t) \geq n_0\, a$. By Lemma 4, we have

$\overline{\lim_{\mathfrak{J}}}\, p_{\alpha+1} F_t = p_{\alpha+1}(\overline{\lim_{\mathfrak{J}}}\, F_t)$, i.e. $e_{\alpha+1}(p_{\alpha+1}x, p_{\alpha+1} F) \geq n_0\, a$ and $e(x,F) \geq a$. //

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