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REMARKS ON METRIZABLE LOCALES

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The notion of a locale is a generalization of that of a topological space, obtained by concentrating on the structure of open sets (for a basic information see, e.g., [8], for more detail the monograph [7]). In this paper we investigate some properties of metrizable locales (defined in [6]; see also [11]). In particular, we show that similarly as in the classical case, metrizable locales are always collectionwise normal, that they have one of the properties equivalent in the classical case with the paracompactness, and that the Bing and Nagata-Smirnov metrizability criteria hold. The proofs follow in a large extent the ideas of the corresponding classical ones (cf., e.g., [5], [9]). The notes in the last section concern preserving the metrizability in sublocales, sums and countable products of locales.

1. Preliminaries

1.1. A *locale* (see, e.g., [1], [7], [8]) is a complete lattice L satisfying the complete distributivity law

$$x \wedge \bigvee_{i \in J} y_i = \bigvee_{i \in J} (x \wedge y_i) .$$

The bottom of L will be denoted by 0, the top by e. Recall that the distributivity law implies also that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ while in general the join does not distribute over large meets (see, however, 1.13 below).

1.2. The *complement* $\bar{x}$ of an element x of a locale L is the largest $y \in L$ such that $y \wedge x = 0$ (thus, more formally, $\bar{x} = \bigvee\{y \mid y \in L,\ y \wedge x = 0\}$). We easily see that

$$\overline{\bigvee x_i} = \bigwedge \bar{x}_i .$$

1.3. One writes

$$x \lhd y$$

if $\bar{x} \vee y = e$. A locale L is *regular* if

$$\forall a \in L \qquad a = \bigvee\{x \mid x \lhd a\} .$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

One writes

$$x \lhd\!\lhd y$$

if there are $x_{ij} \in L$, $i=0,1,\dots$; $j=0,1,\dots,2^i$ such that

(1) $x_{00} = x$, $x_{01} = y$,
(2) $x_{ij} \lhd x_{i,j+1}$, and
(3) $x_{i+1,2j} = x_{i,j}$.

A locale is <u>completely regular</u> if

$$\forall a \in L \quad a = \bigvee \{ x \mid x \lhd\!\lhd a \}$$

(see, e.g., [2], [1]).

<u>1.4.</u> A <u>basis</u> of a locale is a subset $B \subset L$ such that $\forall a \in L \;\; \exists B' \subset B$ such that $a = \bigvee B'$.

<u>1.5.</u> A <u>cover</u> of a locale L is a subset $A \subset L$ such that

$$\bigvee A = e .$$

We say that a cover A is a refinement of a cover B (and write $A \prec B$) if for each $a \in A$ there is a $b \in B$ such that $a \leq b$.

For a cover A and an element $x \in L$ we put

$$Ax = \bigvee \{ a \mid a \in A,\ a \wedge x \neq 0 \} .$$

If $\mathcal{A}$ is a system of covers, we write

$$x \overset{\mathcal{A}}{\lhd} y$$

if there is an $A \in \mathcal{A}$ such that $Ax \leq y$. We put

$$L_{\mathcal{A}} = \{ x \in L \mid x = \bigvee \{ y \mid y \overset{\mathcal{A}}{\lhd} x \} \} .$$

Note that $x \overset{\mathcal{A}}{\lhd} y$ implies $x \lhd y$. Moreover (see [10]), $L = L_{\mathcal{A}}$ for a system of covers iff L is regular.

<u>1.6.</u> For a cover A put

$$A^* = \{ \bigvee B \mid B \subset A,\ (a,b \in B \Rightarrow a \wedge b \neq 0) \} .$$

We say that a system $\mathcal{A}$ of covers is a <u>uniformity basis</u> (briefly, a <u>u-basis</u>) on L if

$$\forall A \in \mathcal{A}\ \exists B \in \mathcal{A} \quad \text{such that } B^* \prec A.$$

By [11], $L = L_{\mathcal{A}}$ with a u-basis $\mathcal{A}$ iff L is completely regular.

We say that a locale L is <u>metrizable</u> if there is a countable u-basis $\mathcal{A}$ such that $L = L_{\mathcal{A}}$. (This is equivalent to the definition given in [6]; in the spatial case, i.e. in the case of a locale which is the lattice of open sets of a space, it coincides with the classical metrizability. Also in general it seems to be well motivated - see the following paragraph.)

<u>1.7.</u> A <u>pre-diameter</u> on a locale L is a function

$$d : L \to \mathbb{R}_+$$

($\mathbb{R}_+$ is the set of non-negative reals) such that

(1) $d(0) = 0$,
(2) $a \leq b \Rightarrow d(a) \leq d(b)$,

(3) $\forall \varepsilon > 0$, $\{a \mid d(a) < \varepsilon\}$ is a cover of L.

It is said to be a <u>star-diameter</u> if

for any $S \subset L$ such that $a,b \in L \Rightarrow a \wedge b \neq 0$,

$d(\bigvee S) \leqslant 2 \sup\{d(a) \mid a \in S\}$;

it is said to be a <u>metric diameter</u> if

(4) for a,b such that $a \wedge b \neq 0$

$d(a \vee b) \leqslant d(a) + d(b)$, and

(5) $\forall x \in L\ \ \forall \varepsilon > 0\ \ \exists a,b \leqslant x$ such that

$d(a), d(b) < \varepsilon$ and $d(a \vee b) > d(x) - \varepsilon$.

(cf. [11], §1). Every metric diameter is a star diameter (see [10], Lemma 5.1). In the spatial case, the bounded metric diameters are in a natural one-one correspondence with the bounded metrics on the space in question such that the induced topologies are weaker than the original one (see [11], Theorem 2.7).

For any star diameter d,

$$\mathcal{U}(d) = \{ \{x \mid d(x) < \tfrac{1}{n}\} \mid n=1,2,\dots \}$$

is a u-basis. More generally, if d_i $(i \in J)$ are star-diameters then

$$\mathcal{U}(\{d_i \mid i \in J\}) = \{ \{x \mid d_i(x) < \tfrac{1}{n}\} \mid n=1,2,\dots\ ;\ i \in J\}$$

is a u-basis. By [11] (Theorem 4.6),

$L = L_{\mathcal{A}}$ with a countable u-basis $\mathcal{A}$ iff $L = L_{\mathcal{U}(d)}$ for a metric diameter d.

(Which fact gives the formal definition of metrizability a more concrete contents.)

<u>1.8.</u> We say that a diameter d separates v from u in L if

(1) $d(v) = 0$ and $d(u) = 1$,

(2) if $x \wedge v \neq 0$ and $d(x) < 1$ then $x \leqslant u$.

By [10] (Theorem 4.11) there is a metric diameter separating v from u iff $v \lhd\!\lhd u$. (Moreover, one can always chose a d with $d(x) \leqslant 1$ for all x.)

<u>1.9.</u> The following is straightforward :

<u>Lemma</u> : Let d_i $(i \in J)$ be star-diameters on L such that $d_i(x) \leqslant 1$ for all i and x. Put $d = \sup d_i$. If all $\{x \mid d(x) < \varepsilon\}$ are covers, d is a star-diameter. □

<u>1.10.</u> In the sequel, countable systems of covers $\mathcal{A} = \{A_1, A_2, \dots, A_n, \dots\}$ will be considered. The symbols A_n will be used always in this sense. Furthermore, we put

$$\alpha_n x = \bigvee\{y \mid A_n y \leqslant x\}.$$

We have

<u>Lemma</u> : 1. $A_n \alpha_n x \leqslant x$.

2. $\alpha_n x \lhd x$ (or, equivalently, $\overline{\alpha_n x} \vee x = e$).

3. If $L = L_{\mathcal{A}}$ then $\bigvee_{n=1}^{\infty} \alpha_n x = x$ for all x.

Proof : 1 follows from the distributivity, 2 follows from 1 and from [10] (Proposition 2.2). 3 is an immediate consequence of the definition of $L_{\mathcal{A}}$. □

1.11. A system $\{x_i \mid i \in J\}$ of elements of L is said to be disjoint if

$$i \neq j \Rightarrow x_i \wedge x_j = 0 .$$

It is said to be discrete if there is a cover C such that

$$\forall c \in C, \quad c \wedge x_i \neq 0 \quad \text{for at most one} \quad i \in J.$$

It is said to be co-discrete if there is a cover C such that

$$\forall c \in C, \quad c \not\leq x_i \quad \text{for at most one} \quad i \in J.$$

A subset $B \subset L$ is said to be σ-discrete if $B = \bigcup_{n=1}^{\infty} B_n$ with B_n discrete.

1.12. We have $c \not\leq \overline{x}_i$ iff $c \wedge x_i \neq 0$. Thus, we make an

Observation : $\{x_i\}$ is discrete iff $\{\overline{x}_i\}$ is co-discrete. □

1.13. Lemma : Let $\{x_i\}_J$ be co-discrete. Then, for any y,

$$y \vee (\bigwedge_J x_i) = \bigwedge_J (y \vee x_i) .$$

Proof : It suffices to show that $c \wedge (y \vee \bigwedge x_i) \geqslant c \wedge (\bigwedge (y \vee x_i))$ for all $c \in C$ where C is a cover (indeed, this implies that $y \vee \bigwedge x_i \geqslant \bigwedge (y \vee x_i)$ and the $\leqslant$-inequality holds anyway).

Take the C from the definition of co-discrete. For $c \in C$ we have $c \wedge \bigwedge x_i = \bigwedge (c \wedge x_i) = c \wedge x_{i(c)}$ for a suitable $i(c) \in J$. Thus,

$$c \wedge (y \vee \bigwedge x_i) = (c \wedge y) \vee (c \wedge x_{i(c)}) = c \wedge (y \vee x_{i(c)}) \geqslant$$
$$\geqslant c \wedge \bigwedge (y \vee x_i) . \quad \square$$

1.14. Lemma : Let $\{x_i\}_J$ be discrete and $x_i \lhd y$ for all $i \in J$. Then $\bigvee x_i \lhd y$.

Proof : By 1.2, 1.12 and 1.13 we obtain

$$\overline{\bigvee x_i} \vee y = (\bigwedge \overline{x}_i) \vee y = \bigwedge (\overline{x}_i \vee y) = e . \quad \square$$

2. Normality and collectionwise normality

2.1. A locale L is said to be normal if for any $x, y \in L$ such that $x \vee y = e$ there are $a, b \in L$ such that

$$a \vee y = e = x \vee b \quad \text{and} \quad a \wedge b = 0.$$

It is said to be collectionwise normal if for each co-discrete system $\{x_i\}_J$ there is a discrete $\{y_i\}_J$ such that

$$x_i \vee y_i = e \quad \text{for all} \quad i \in J$$

(cf. [2], [5]).

2.2. Proposition : In normal locales we have the implication

$$x \lhd y \Rightarrow x \lhd\!\!\lhd y .$$

Consequently, a normal regular locale is completely regular.

Proof : It suffices to show that whenever $x \lhd y$, there is a z such that $x \lhd z \lhd y$. Let $x \lhd y$. Thus, $\bar{x} \vee y = e$ and hence we have u, z such that $\bar{x} \vee z = e$, $u \vee y = e$ and $u \wedge z = 0$. Thus, $u \leq \bar{z}$ and hence also $\bar{z} \vee y = e$. □

2.3. Lemma : Let there exist sequences x_n, y_n such that

$$x_n \lhd x \quad , \quad y_n \lhd y$$

and

$$(\bigvee x_n) \vee y = e = x \vee (\bigvee y_n) .$$

Then there are a, b such that $a \vee y = e = x \vee b$ and $a \wedge b = 0$.

Proof : Put

$$a_n = x_n \wedge \bigwedge_{k=1}^{n} \bar{y}_k , \quad b_n = y_n \wedge \bigwedge_{k=1}^{n} \bar{x}_k , \quad a = \bigvee a_n ,$$
$$b = \bigvee b_n .$$

We have $a_n \vee y = (x_n \vee y) \wedge \bigwedge^n (\bar{y}_k \vee y) = x_n \vee y$ and hence $a \vee y = = \bigvee x_n \vee y = e$, and similarly $x \vee b = e$. Obviously, $a_n \wedge b_k = 0$ and hence $a \wedge b = \bigvee_{n,k} (a_n \wedge b_k) = 0$. □

2.4. Theorem : 1. Each regular Lindelöf locale is normal.

2. Each L such that $L = L_{\mathcal{A}}$ for a countable system of covers is normal.

3. Each regular locale with a σ-discrete basis is normal.

Proof : The statements follow from 2.3 : Let $x \vee y = e$.

1 : Consider countable subcovers $\{x_n\} \cup \{y\}$ of $\{u \mid u \lhd x\} \cup \{y\}$ and $\{y_n\} \cup \{x\}$ of $\{u \mid u \lhd y\} \cup \{x\}$.

2 : Put $x_n = \alpha_n x$, $y_n = \alpha_n y$ (see 1.10).

3 : Let B_n be discrete, $\bigcup B_n$ a basis. Put $x_n = \bigvee \{b \mid b \in B_n, b \lhd x\}$, and similarly y_n. By 1.14, $x_n \lhd x$, $y_n \lhd y$, by the regularity $\bigvee x_n = x$, $\bigvee y_n = y$. □

2.5. Theorem : Let $L = L_{\mathcal{A}}$ with a countable $\mathcal{A}$. Then L is collectionwise normal.

Proof : I. First we will prove a weaker statement

(*) for each co-discrete $\{x_i\}_J$ there is a disjoint $\{y_i\}_J$ such that $\forall i \;\; x_i \vee y_i = e$.

Let C be the cover from the definition of co-discreteness. Put

$$v_i = \bigvee \{c \mid c \in C, \; \forall j \neq i \;\; c \leq x_j\} .$$

Obviously

(1) $\forall j \neq i , \;\; v_i \leq x_j$

and, since $\bigvee C = \bigvee \{c \mid c \leq x_i\} \cup \bigvee \{c \mid c \nleq x_i\}$ and the second summand is $\leq v_i$,

(2) $\forall i , \;\; x_i \vee v_i = e$.

Put

$$u_{in} = \alpha_n v_i \wedge \bigwedge_{k=1}^{n} \overline{\alpha_k x_i} , \quad u_i = \bigvee_{n=1}^{\infty} u_{in} .$$

We have

$$x \vee u_i = \bigvee_n (x_i \vee u_{in}) = \bigvee_n ((x_i \vee \alpha_n v_i) \wedge \bigwedge_{k=1}^{n} (x_i \vee \overline{\alpha_k x_i})) =$$

$$= \bigvee_n (x_i \vee \alpha_n v_i) = x_i \vee \bigvee_n \alpha_n v_i = x_i \vee v_i = e$$

(see 1.10).

Let $i \neq j$, $k \leq n$. Then $u_{in} \wedge \alpha_k v_j \leq \overline{\alpha_k x_i} \wedge \alpha_k v_j = 0$ (as $v_j \leq x_i$, we have $\alpha_k v_j \leq \alpha_k x_i$ and hence $\alpha_k x_i \leq \alpha_k v_j$) so that $u_{in} \wedge u_{jk} = 0$. Consequently, $u_i \wedge u_j = 0$.

II. Take the u_i from I and denote

$$D' = \{d \in L \mid d \wedge u_i \neq 0 \text{ for at most one } i\} .$$

In particular, $u_i \in D'$ and hence $u_i \leq \bigvee D'$ so that (see 1.13)

$$\bigvee D' \vee \bigwedge_i x_i = \bigwedge_i (\bigvee D' \vee x_i) \geq \bigwedge_i (u_i \vee x_i) = e .$$

Since L is normal (see 2.4.2), we have $a, b \in L$ such that

$$a \vee (\bigwedge x_i) = e , \quad \bigvee D' \vee b = e \quad \text{and} \quad a \wedge b = 0 .$$

Put

$$y_i = u_i \wedge a , \quad D = D' \cup \{b\} .$$

D is a cover and $d \in D$ meets at most one y_i. Finally,

$$x_i \vee y_i = (x_i \vee u_i) \wedge (x_i \vee a) = x_i \vee a \geq (\bigwedge x_i) \vee a = e. \ \square$$

3. -discrete refinements of covers of $L = L_{\mathcal{A}}$ with countable $\mathcal{A}$

<u>3.1. Construction</u> : Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be such that $L = L_{\mathcal{A}}$. Let $B = \{b_i\}_{i \in J}$ be an arbitrary cover of L. Consider a fixed well-ordering $<$ on the set of indices J. For $i \in J$ and n natural put

$$c_{in} = \bigvee_{j<i} b_j \vee \overline{\alpha_n b_i} .$$

<u>3.2. Lemma</u> : For each n , $\{c_{in}\}_{i \in J}$ is a co-discrete system.

<u>Proof</u> : Consider the cover A_n . For $a \in A_n$ let i be the least index such that $a \nleq c_{in}$. Thus, in particular, $a \nleq \overline{\alpha_n b_i}$, hence $a \wedge \alpha_n b_i \neq 0$ so that $a \leq b_i$ (see 1.10) and hence $a \leq c_{jn}$ for all $j > i$ (and $a \leq c_{jn}$ for $j < i$ by the definition of i). $\square$

<u>3.3. Construction continued</u> : By 3.2 and 2.5 there are discrete systems $\{d_{in}\}_{i \in J}$ such that

$$c_{in} \vee d_{in} = e \quad \text{for all } i .$$

Put

$$d^*_{in} = d_{in} \wedge b_i .$$

<u>3.4. Lemma</u> : For every i ,

$$\bigvee_{j\leq i}(\bigvee_{n=1}^{\infty} d_{jn}^{*}) \geqslant \bigvee_{j\leq i} b_j .$$

Proof : I. Let i be the first element in $(J,<)$. Thus, we have to show that $\bigvee d_{in}^{*} \geqslant b_i$ which will follow from $\bigvee d_{in} \geqslant b_i$. We have $d_{in}\vee\overline{\alpha_n b_i} = e$ and hence $d_{in} \geqslant \alpha_n b_i$. Thus

$$\bigvee_n d_{in} \geqslant \bigvee_n \alpha_n b_i = b_i \quad \text{(see 1.10).}$$

II. Let the statement hold for $j<i$. We have

$$\bigvee_{j\leq i}(\bigvee_n d_{jn}^{*}) = \bigvee_{j<i}(\bigvee_n d_{jn}^{*})\vee\bigvee_n d_{in}^{*} \geqslant \bigvee_{j<i} b_j \vee \bigvee_n d_{in}^{*}$$

so that it suffices to show that

$$(1) \qquad \bigvee_{j<i} b_j \vee \bigvee_n d_{in} \geqslant b_i .$$

We have $e = c_{in}\vee d_{in} = (d_{in}\vee\bigvee_{j<i} b_j)\vee\overline{\alpha_n b_i}$ so that

$$\alpha_n b_i \leqslant d_{in}\vee\bigvee_{j<i} b_j$$

and hence, finally, we obtain (1) using 1.10. □

3.5. Theorem : Let $L = L_{\mathcal{A}}$ for a countable $\mathcal{A}$. Then each cover of L has a σ-discrete refinement.

Proof : Notation from 3.1 and 3.3. The system $D = \{d_{in}^{*}\}_{i,n}$ is σ-discrete and $d_{in}^{*} \leqslant b_i$. Thus, it suffices to prove that D is a cover. By 3.4 we have

$$\bigvee D = \bigvee_i(\bigvee_n d_{in}^{*}) = \bigvee_i(\bigvee_{j\leq i}(\bigvee_n d_{in}^{*}))\geqslant\bigvee_i\bigvee_{j\leq i} b_j = \bigvee B = e . \square$$

3.6. Remark : In the spatial case it is well-known that the existence of σ-discrete refinements of all covers is equivalent to the paracompactness (see, e.g., [9]). In the case of general locales this question seems to be open (cf. [3]). It may also be so that the two properties do not coincide in general while still being equivalent for the case of $L = L_{\mathcal{A}}$ with countable $\mathcal{A}$. So far, Theorem 3.5 is all we are able to tell on the question of paracompactness of metrizable locales.

4. Bing and Nagata-Smirnov metrization theorems

4.1. Lemma : Let $L = L_{\mathcal{A}}$ and let there be given for each $A\in\mathcal{A}$ a refinement $B\in\mathcal{B}$. Then $\bigcup\mathcal{B}$ is a basis of L.

Proof : Obviously $x \overset{\mathcal{A}}{\lhd} y \Rightarrow x \overset{\mathcal{B}}{\lhd} y$ and hence $L = L_{\mathcal{B}}$. Thus, it suffices to prove that $\bigcup\mathcal{A}$ is a basis. Take an $x\in L$ and put $\mathcal{A}(x) = \{u \mid u \overset{\mathcal{A}}{\lhd} x\}$. For $u\in\mathcal{A}(x)$ chose an $A\in\mathcal{A}$ such that $Au\leqslant x$ and put $A_u = \{a \mid a\in A, a\wedge u \neq 0\}$. We have $u\leqslant\bigvee A_u\leqslant x$ and hence $x = \bigvee C$ where $C = \bigcup\{A_u \mid u\in\mathcal{A}(x)\}$. □

<u>4.2. Theorem</u> : If $L = L_{\mathcal{A}}$ with a countable $\mathcal{A}$ then L has a σ-discrete basis.

<u>Proof</u> : For $A_n \in \mathcal{A}$ consider a σ-discrete refinement B_n (recall Theorem 3.5). By 4.1 we obtain a σ-discrete basis by putting $B = \bigcup B_n$. □

<u>4.3. Theorem</u> : The following statements are equivalent :

(i) L is metrizable

(ii) $L = L_{\mathcal{A}}$ for a countable $\mathcal{A}$.

(iii) L is regular and has a σ-discrete basis.

<u>Proof</u> : (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) follows from [10](Theorem 2,8) and Theorem 4.2.

(iii) ⇒ (i) : We have a basis $B = \bigcup B_n$ with $B_n = \{b_{ni} \mid i \in J(n)\}$ discrete. Since L is regular, each x equals $\bigvee\{b \mid b \in B,\ b \lhd x\}$. Consequently,

$$b_{ni} = \bigvee_{k \in \mathbb{N}} c_{nki} \quad \text{where} \quad c_{nki} = \bigvee\{c \mid c \in B_k,\ c \lhd b_{ni}\}.$$

($\mathbb{N}$ is the set of natural numbers.) By 1.14, $c_{nki} \lhd b_{ni}$ and hence, by 2.4.3 and 2.2,

$$c_{nki} \lhd\!\!\lhd b_{ni} .$$

By [10](Proposition 4.8 and Construction 4.3) there are metric diameters d_{nki} separating c_{nki} from b_{ni} . Put

$$d_{nk} = \sup d_{nki} .$$

Take a cover C such that each $c \in C$ meets at most one $b_{ni} = b_{n,i(c)}$. Take an $\varepsilon > 0$. We can write

$$c = \bigvee D_c \quad \text{with} \quad d_{n,k,i(c)}(u) < \varepsilon \text{ for all } u \in D_c$$

(put $D_c = \{c \wedge v \mid d_{n,k,i(c)}(v) < \varepsilon\}$). For $u \in D_c$ and $j \neq i(c)$ we obviously have $d_{nkj}(u) = 0$ and hence $d_{nk}(u) < \varepsilon$. As $\bigcup\{D_c \mid c \in C\}$ is a cover, $\{u \mid d_{nk}(u) < \varepsilon\}$ is one and the assumption of 1.9 is satisfied. Thus,

each d_{nk} is a star-diameter.

We easily see that

for $x \leq b_{ni}$ one has $d_{nk}(x) = d_{nki}(x)$.

Consider the system

$$\mathcal{V} = \mathcal{V}(\{d_{nk} \mid n,k \in \mathbb{N}\})$$

(recall 1.7). Now, by [11] (theorem 4.6) it suffices to show that $L = L_{\mathcal{V}}$. We have $c_{nki} \overset{\mathcal{V}}{\lhd} b_{ni}$ and hence $b_{ni} = \bigvee\{y \mid y \overset{\mathcal{V}}{\lhd} b_{ni}\}$. Finally, for a general x we have

$$x = \bigvee\{b \mid b \in B,\ b \leq x\} = \bigvee\{\bigvee\{y \mid y \overset{\mathcal{V}}{\lhd} b\} \mid b \in B,\ b \leq x\} \leq$$

$$\leq \bigvee\{y \mid y \overset{\mathcal{V}}{\lhd} x\} \leq x . \ \square$$

4.4. Remark : The equivalence (i)$\Leftrightarrow$(ii) in 4.3 is a generalization of the Bing metrization criterion, the equivalence (i)$\Leftrightarrow$(iii) is a generalization of the Nagata-Smirnov metrization theorem (see [5], pp.408 and 351 respectively).

5. Sublocales, products and coproducts of metrizable locales

5.1. So far we have been concerned with individual locales only. Now, however, we will have to deal with morphisms between them. This forces us to be more particular about the terminology. The category of complete lattices satisfying the distributivity law $x \wedge (\bigvee y_i) = \bigvee (x \wedge y_i)$ and the mappings between them preserving finite meets and general joins is usually referred to as the category of frames and frame homomorphisms. The category of locales is its dual; thus, representing a topological space by the locale of open sets, and a continuous mapping $f:X \to Y$ as the frame homomorphism sending U to $f^{-1}(U)$, we obtain a covariant embedding into the category of locales instead of the contravariant one into the category of frames. A sublocale L' of a locale L is represented by a frame homomorphism of L onto L'. A product of locales is represented as a coproduct of frames and vice versa.

5.2. Proposition : Let $\varphi: L \to L'$ be a frame homomorphism. If A is a cover of L, $\varphi(A)$ is a cover of L' and we have

$$\forall x \quad \varphi(A)\varphi(x) \leq \varphi(Ax).$$

Consequently, if $\mathcal{A}$ is a system of covers and if we put $\mathcal{A}' = \{\varphi(A) \mid A \in \mathcal{A}\}$, we have the following implication :

$$x \overset{\mathcal{A}}{\lhd} y \text{ in } L \Rightarrow \varphi(x) \overset{\mathcal{A}'}{\lhd} \varphi(y) \text{ in } L'.$$

Proof : We have $\bigvee \varphi(A) = \varphi(\bigvee A) = \varphi(e) = e$. Now, let $\varphi(a) \wedge \varphi(x) \neq 0$ for an $a \in A$. Then $\varphi(a \wedge x) \neq 0$ and hence $a \wedge x \neq 0$ so that $\varphi(a) \leq \varphi(Ax)$. $\square$

5.3. Proposition : Let a frame morphism $\varphi: L \to L'$ be onto, let $L = L_{\mathcal{A}}$. Then, in the notation of 5.2, $L' = L'_{\mathcal{A}'}$.

Proof : Take an $x' \in L'$; we have an $x \in L$ such that $x' = \varphi(x)$. Since $L = L_{\mathcal{A}}$, $x = \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x\}$ so that by 5.2

$$x' = \bigvee\{\varphi(y) \mid y \overset{\mathcal{A}}{\lhd} x\} \leq \bigvee\{\varphi(y) \mid \varphi(y) \overset{\mathcal{A}'}{\lhd} x'\} \leq \bigvee\{z \mid z \overset{\mathcal{A}'}{\lhd} x'\} \leq x'. \square$$

5.4. Corollary : A sublocale of a metrizable locale is metrizable. $\square$

5.5. Proposition : Coproduct of any system of metrizable locales is metrizable.

Proof : Let $L_i = L_{i\mathcal{A}_i}$ with $\mathcal{A}_i = \{A_{i1}, A_{i2}, \dots\}$. The coproduct L of the localee L_i is the product $\bigtimes L_i$ of the corresponing frames. For $x \in L_i$ put $x_i^* = (y_j)_{j\in J}$ where $y_j = 0$ for $j \neq i$ and $y_i = x$. Put

$$A_{in}^* = \{x_i^* \mid x \in A_{in}\} .$$

Thus, $\bigvee A_{in}^* = e_i^*$ and, since $\bigvee_J e_i^* = e$ in L,

$$A_n^* = \bigcup_{i\in J} A_{in}^*$$

is a cover of L. Put $\mathcal{A} = \{ A_1^*, A_2^*, \dots \}$.

For $y \overset{\mathcal{A}_i}{\lhd} x$ in L_i we have $y_i^* \overset{\mathcal{A}}{\lhd} x_i^*$ (indeed, if $A_{in} y \leq x$, we have $A_{in}^* y_i^* \leq x_i^*$). Thus, for each $x \in L_i$, $x_i^* = \bigvee\{u \mid u \overset{\mathcal{A}}{\lhd} x_i^*\}$ and since $\{x_i^* \mid x \in L_i, i \in J\}$ is a basis of L, the statement follows. □

<u>5.6. Theorem</u> : Product L of at most countably many metrizable locales is metrizable.

<u>Proof</u> : Let $L_i = L_{i\mathcal{A}_i}$, $i=1,2,\dots$; $\mathcal{A}_i = \{A_{i1}, A_{i2}, \dots\}$. We have (in the frame language) $L = \bigoplus_{i=1}^{\infty} L_i$ and frame homomorphisms $\iota_i : L_i \longrightarrow L$ such that the elements of the form

$$\bigwedge_{k=1}^{m} \iota_{i_k}(x_k) \quad \text{with } x_k \in L_{i_k}$$

constitute a basis of L. (For a handy description of the product of locales - coproduct of frames - see e.g. [4]). Put

$$A_{in}^* = \{\iota_i(a) \mid a \in A_{in}\} = \iota_i(A_{in}) ,$$

$$A(i_1,\dots,i_m;n_1,\dots,n_m) = A_{i_1n_1}^* \wedge A_{i_2n_2}^* \wedge \dots \wedge A_{i_mn_m}^* ,$$

$$\mathcal{A} = \{A(i_1,\dots,i_m;n_1,\dots,n_m) \mid m,i_1,\dots,i_m,n_1,\dots,n_m = 1,2,\dots\}.$$

Obviously, $\mathcal{A}$ is a countable system of covers of L.

Now, let

$$y_k \overset{\mathcal{A}_{i_k}}{\lhd} x_k$$

for $k=1,2,\dots,m$. We have $A_{i_kn_k} y_k \leq x_k$ for a suitable n_k .

Hence, by [10] (Proposition 1.7.2) and 5.2,

$$A(i_1,\dots,n_m)(\bigwedge_{k=1}^{m} \iota_{i_k}(y_k)) \leq \bigwedge_{k=1}^{m} A_{i_kn_k}^* \iota_k(y_k) \leq \bigwedge_{k=1}^{m} \iota_{i_k}(x_k)$$

so that

$$\bigwedge_{k=1}^{m} \iota_{i_k}(y_k) \overset{\mathcal{A}}{\lhd} \bigwedge_{k=1}^{m} \iota_{i_k}(x_k) .$$

Now we have

$$\bigvee\{\bigwedge_{k=1}^{m} \iota_{i_k}(y_k) \mid y_k \overset{\mathcal{A}_k}{\lhd} x_k,\ y_k \in L_{i_k}\} = \bigvee_{y_1 \overset{\mathcal{A}_1}{\lhd} x_1} \cdots \bigvee_{y_m \overset{\mathcal{A}_m}{\lhd} x_m} \bigwedge_{k=1}^{m} \iota_{i_k}(y_k) =$$

$$= \bigvee_{y_1 \overset{\mathcal{A}_1}{\lhd} x_1} \cdots \bigvee_{y_{m-1} \overset{\mathcal{A}_{m-1}}{\lhd} x_{m-1}} (\bigwedge_{k=1}^{m-1} \iota_{i_k}(y_k) \wedge \bigvee_{y_m \overset{\mathcal{A}_m}{\lhd} x_m} \iota_{i_m}(y_m)) =$$

$$= (\bigvee_{y_1 \overset{\mathcal{A}_1}{\lhd} x_1} \cdots \bigvee_{y_{m-1} \overset{\mathcal{A}_{m-1}}{\lhd} x_{m-1}} \bigwedge_{k=1}^{m-1} \iota_{i_k}(y_k)) \wedge \iota_{i_m}(x_m) = \ldots =$$

$$= \bigwedge_{k=1}^{m} \iota_{i_k}(x_k)$$

so that

$$\bigwedge_{k=1}^{m} \iota_{i_k}(x_k) \leq \bigvee\{\bigwedge_{k=1}^{m} \iota_{i_k}(y_k) \mid \bigwedge_{k=1}^{m} \iota_{i_k}(y_k) \overset{\mathcal{A}}{\lhd} \bigwedge_{k=1}^{m} \iota_{i_k}(x_k)\} \leq$$

$$\leq \bigvee\{z \mid z \overset{\mathcal{A}}{\lhd} \bigwedge_{k=1}^{m} \iota_{i_k}(x_k)\} \leq \bigwedge_{k=1}^{m} \iota_{i_k}(x_k)$$

and since the elements

$$\bigwedge \iota_{i_k}(x_k)$$

constitue a basis of L, we obtain that $L = L_{\mathcal{A}}$. □

REFERENCES :

[1] BANASCHEWSKI B. and MULVEY C.J. "Stone-Čech compactifications of locales,I", Houston J.Math.6(1980), pp.301-312

[2] DOWKER C.H. and PAPERT STRAUSS D. "Separation axioms for frames", Topics in Topology, Coll.Math.Soc.J.Bolyai, 8, North-Holland/American Elsevier, 1974, pp.223-239

[3] DOWKER C.H. and PAPERT STRAUSS D. "Paracompact frames and closed maps", Symposia Mathematica, Vol.XVI, Academic Press, 1975, pp.93-116

[4] DOWKER C.H. and STRAUSS D. "Sums in the category of frames", Houston J.Math.3(1977), pp.17-32

[5] ENGELKING R. "General Topology", Monografie Matematyczne, Tom 60, Warszawa 1977

[6] ISBELL J.R. "Atomless parts of spaces", Math.Scand.31(1972), pp.5-32

[7] JOHNSTONE P.T. "Stone spaces", Cambridge studies in advanced

mathematics 3, Cambridge University Press 1982

[8] JOHNSTONE P.T. "The point of pointless topology" Bull.of the AMS (New Séries), 8(1983), pp.41-53

[9] KELLEY J.L. "General Topology", Van Nostrand 1955

[10] PULTR A. "Pointless uniformities I.Complete regularity", Comment.Math.Univ.Carolinae 25,1(1984), pp.91-104

[11] PULTR A. "Pointless uniformities II.(Dia)metrization", Comment.Math.Univ.Carolinae 25.1(1984), pp.105-120

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