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n-tilability of acyclic polyominoes

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n-TILABILITY OF ACYCLIC POLYOMINOES

Igor Kříž

The main result of this paper is to present a polynomial algorithm for deciding, whether a given acyclic polyomino (roughly speaking, a connected finite configuration on infinite chessboard without holes) is tilable by $1\times n$- and $n\times 1$-rectangles.(n fixed)

The results on such tilings ([1,2,3,4,5]) so far known are based on global characteristics of some simple polyominoes. In our method we use a local analysis of the structure, which makes the general result possible.

From further results included let us name for instance the connectedness theorem (2.3.2.) or the theorem on the tilings of the complement of a subpolyomino (2.4.3.).

1. Preliminaries

1.1. We will use the symbol $\overline{AB}, \overrightarrow{AB}, \overleftrightarrow{AB}$, respectively, for the segment, half-line, line, respectively, determined by the points A,B of the Euclidian plane E_2. The points A,B are called the nodes of the segment $\overline{AB}$. An oriented segment is a couple (u,O(u)), where u is a segment and O(u) one of its nodes (the origin); the other node T(u) will be called the terminal. For two parallel oriented segments we distinguish coherent or reverse orientations in the obvious way.

If u and v are segments, $u\supseteq v$, we say u is an extension of v. The length of a segment u will be denoted by $|u|$.

For subsets $M\subset E_2$, $]M[$ is the interior, $[M]$ is the closure and ∂M is the boundary of M. The cardinality of a finite M will be denoted by $\#(M)$.

The plane will be endoved with a fixed coordinate system. The lattice (integral) points provide the plane with the obvious structure of a CW-complex K. Its closed 2-cells (the 1×1 squares) will be called simply cells. Refering to 1-cells in the sequel, we mean, of course, the 1-cells of K. The system of 1-cells obviously decom-

This paper is in final form and no version of it will be submitted for publication elsewhere.

poses into two classes (the vertical & the horizontal ones); these will be refered to as the K-directions. When speaking of a direction of a line or half-line, we mean the direction of the segments included.

The cell with the vertices $(i,j),(i,j+1),(i+1,j),(i+1,j+1)$, will be denoted by $\langle i,j\rangle$. The translations of the plane given by the formulas $(x,y)\mapsto(x,y+1)$ resp. $(x,y)\mapsto(x+1,y)$ are denoted by σ resp. τ.

An oriented segment u is said to be right perpendicular to an oriented segment v, if for $T(u) - O(u) = (x_1,y_1)$, $T(v) - O(v) = (x_2, y_2)$ it holds

$$\det\begin{pmatrix} x_1, & y_1 \\ x_2, & y_2 \end{pmatrix} > 0.$$

(Realize the obvious geometrical meaning of this, somewhat clumsy, definition.) Left perpendicular is an inverse relation to right perpendicular.

We say that the oriented segment $((0,0)(1,0),(0,0))$ is right incident with the cell $\langle 0,0\rangle$ and use this expression for all the configurations obtained from the mentioned one by translations and rotations.

1.2. A polyomino P is any finite regular subcomplex of K (i.e., we have $[\,]P[\,] = P$). Its volume is the number of its 2-cells. An I--component of P is the closure of a component of $]P[$. P is said to be acyclic, if both $]P[$ and $E_2 \setminus]P[$ are connected subsets of the plane. In the sequel, we will use the term rectangle for those rectangles, which are polyomina.

Given a polyomino P, then each 1-cell of ∂P will be oriented once for ever so that it is right-incident with a cell of P. This will be refered to as the standard orientation. A side of P is any segment $a \subset \partial P$ such that it is a subcomplex of K, its 1-cells are coherently oriented and it is maximal with respect to this property (see fig.1).

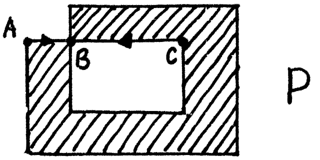


fig.1

$\overline{AB}$, $\overline{BC}$ are sides, $\overline{AC}$ is not. We can define the standard orientation of a side to be coherent with the orientation of its 1-cells.

We define a function $succ_P : M \to M$, where M is the set of all the sides of P, putting

(i) $T(s) = O(succ_P(s))$ for $s \in M$

(ii) If s' satisfies $T(s) = O(s')$ and s is right perpendicular to s', then $s' = succ_P(s)$.

1.2.1. By Jordan Theorem, we immediately obtain

Lemma: P is acyclic iff $succ_P$ is a cyclic permutation. □

(∂P is not necessarily a Jordan curve – for this we would have to assume a connected $E_2 \setminus P$. But, it behaves, in an obvious sense, almost as one: The exterior and the interior is canonically defined; moreover, $P = \mathrm{int}\,\partial P$.)

1.3. A side of P is said to be an edge of P, if $succ_P(s)$ is left perpendicular to s and $succ_P^{-1}(s)$ is right perpendicular to s (see fig.2 and compare it with the situation of the edges t_1, t_2).

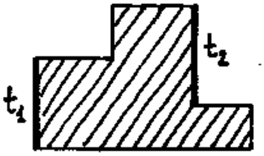


fig.2

t_1 is an edge, t_2 is not.

1.3.1. Lemma: In every polyomino P there is at least one edge in any direction and orientation.

Proof: Consider the least rectangle C containing P. It is easy to see that each of the sides of C contains an appropriate edge of P. □

An edge h of P is said to be left regular (resp. right regular) if there is a $k \in \mathbb{N}_0$ such that $succ_P^{2k+1}(h)$ (resp. $succ_P^{-2k-1}(h)$) is an edge left (resp. right) perpendicular to h, while for $i = 0, \ldots, 2k+1$ are $succ_P^{i}(h)$, $succ_P^{i+2}(h)$ parallel and coherently oriented (resp. $succ_P^{-i}(h)$, $succ_P^{-i-2}(h)$ are parallel and coherently oriented).

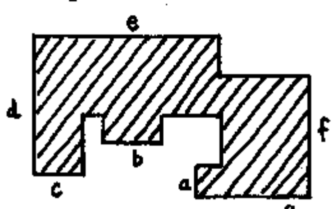


fig.3

left regular: c,d,e,f,g; right regular: a,d,e,f,g; neither: b.

1.3.2. Lemma: In every polyomino P there is at least one left regular and at least one right regular edge in any direction and orientation.
Proof will be done for left regular by induction on $n = |P|$. (the volume of P) The statement is obvious for $n = 1$. Let the statement hold for $n < k$. Choose a direction and an orientation. There is at least one corresponding edge a. Let α be the cell of P incident with O(a). We will distinguish two cases:

1) $|a| = 1$. In $[P \setminus \alpha]$ there is a left regular edge h of our direction and orientation. Let 2k+1 be the least number such that $succ^{2k+1}_{[P\setminus\alpha]}(h)$ is an edge left perpendicular to h. Obviously, α is not incident with any of the sides $succ^{2m+1}_{[P\setminus\alpha]}(h)$, $0 \leq m \leq k$. Thus, either it isn't incident with any of the sides $succ^{2m}_{[P\setminus\alpha]}(h)$, too, and then h is left regular, or it is, and then a is left regular.

2) $|a| > 1$. Consider C, the I-component of $[P \setminus \alpha]$ containing a 1-cell of a. By the induction hypothesis, C has a left regular edge h parallel to a and coherently oriented. If h is not incident with a, then it is left regular in P. Otherwise, a is left regular. □

Let C be an edge of P and let n be the smallest number such that $succ^{n}_{P}(c)$ (resp. $succ^{-n}_{P}(c)$) is an edge left (resp. right) perpendicular to c. The closure of the interior of $M = \{A \mid \exists\, i \in \{0,\dots,n\} \exists\, B \in succ^{i}_{P}(c)\colon \bar{AB} \subset P,\ \bar{AB} \perp c$ (resp. of $M = \{A \mid \exists\, i \in \{0,\dots,n\} \exists\, B \in succ^{-i}_{P}(c)\colon \bar{AB} \subset P,\ \bar{AB} \perp c\}$) will be called the left (resp. right) semisector of P over c. The intersection of both semisectors will be called the sector of P over c.

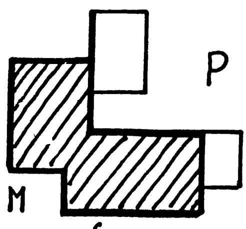


fig.4

(We have to take $[]M[]$, since M is not necessarily a polyomino, see fig.4.)

1.4. Let s be a direction. A subset M of E_2 will be called s-convex, if $p \cap M$ is connected for any line p of direction s. A polyomino P is said to be K-convex if it is s-convex for the both K-directions s.

1.4.1. Lemma: An acyclic polyomino P is s-convex iff no two of its s-edges have coherent orientations.

Proof: Let us have edges c_1, c_2 of the same orientation and the same direction s. Suppose P is s-convex. Then P has to lie in a halfplane determined by both c_1, c_2. Thus, the edges c_i lie in a common line and hence P is not s-convex, which is a contradiction.

On the other hand, let P be not s-convex. Since P is acyclic, we see easily that there is a line p in the direction s, dividing $]P[$ into three components at least. Thus, at least two of them, say P_1', P_2', share a half-plane determined by p. Put $P_i = [P_i']$. We have polyomina P_i such that all the 1-cells of $P_i \cap p$ have the same orientation. Thus, according tp 1.3.1., there are edges h_i of P_i in the direction s with the opposite orientation. This concludes the proof, since h_i are obviously edges of P. □

1.4.2. Lemma: Let c be an edge of P. Then the left (resp. right) semisector I of P over c is K-convex, iff c is left (resp. right) regular and there are no two edges of I parallel to c with opposite orientation to that of c.

Proof: The condition is obviously necessary and we see easily the sufficiency from 1.4.1. □

1.4.3. Theorem: Let P be acyclic. Then for every direction s there is an edge c of P in the direction s such that one of the semisectors of P over c is K-convex.

Proof will be done by contradiction constructing an infinite system of left (right) regular edges c_k and corresponding semisectors I_k of P. Put $I_{-1} = \emptyset$. By 1.3.2., there exists a left regular edge c of P in the direction s and an arbitrarily chosen orientation. Put $c_0 = c$ and let I_0 be the corresponding left semisector of P. Now, let us have c_i, I_i for $i < k$. Let, say, c_{k-1} be left regular. Then if I_{k-1} is not K-convex, we have, by 1.4.2., two parallel edges of I_{k-1} with the orientation opposite to that of c_{k-1}. Thus, at least one of those is not incident with I_{k-2}. Obviously, for some sides a_1, a_2 of P we have $a_1 \subset \operatorname{succ}_{I_{k-1}}(a)$, $a_2 \subset \operatorname{succ}^{-1}_{I_{k-1}}(a)$. Denote by $\bar{a}_i$ the maximal (oriented) extension of a_i such that $O(\bar{a}_1) = O(a_1)$, $T(\bar{a}_2) = T(a_2)$ and $\bar{a}_1, \bar{a}_2 \subset P$. Further, denote by C_i the closure of that component of $]P \setminus \bar{a}_i[$, which is incident with a. Since P is acyclic, we have by 1.4.2. $]C_1 \cap I_{k-2}[= \emptyset$ or $]C_2 \setminus I_{k-2}[= \emptyset$. Let, for instance, $]C_2 \setminus I_{k-2}[= \emptyset$. Then, according to 1.3.2., the polyomino C_2 has a left regular edge $c \parallel c_{k-1}$ and inverse oriented. Thus, c is also a left regular edge of P. Put $c_k = c$ and denote by I_k the left semisector (right in the case $]C_1 \setminus I_{k-2}[= \emptyset$) of P over c. According to the acyclity of P, we have $]I_k \cap I_\ell[= \emptyset$ for $|k-\ell| > 1$. □

Remark: The acyclicity is essential (see fig. 5)

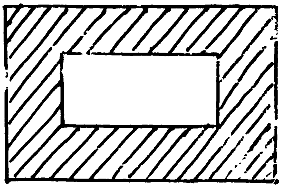


fig.5

2. Tilings

2.1.1. Put $C_{k,\ell} = \{(x,y) \mid 0 \leq x \leq k-1,\ 0 \leq y \leq \ell-1\}$. An n-tiling of a polyomino P is a set M of rectangles congruent with $C_{1,n}$ (or, equivalently, with $C_{n,1}$) such that $\bigcup M = P$ and for $a,b \in M$ we have $]a[\cap]b[= \emptyset$. If there is an n-tiling of P, we say that P is n-tilable.

Special tilings: We will use the notation $M_k^n = \{\sigma^i C_{n,1} \mid i = 0,\dots,k-1\}$, $N_k^n = \{\tau^i C_{1,n} \mid i = 0,\dots,k-1\}$. Let P_m, $m = 1,\dots,k$ be n-tilings and let i_m, j_m be the smallest natural numbers with $\bigcup P_m \subset C_{j_m,i_m}$ (supposed they exist) We will use the following notation

$$\sum_{i=1}^{k} P_i = P_1 + P_2 + \dots + P_k =_{df} \bigcup_{m=1}^{k} \sigma^{\sum_{j=1}^{m} i_j} P$$

$$k.P_i = \sum_{j=1}^{k} P_i, \quad o.P_i = \emptyset.$$

Further, for $\ell = \alpha n + \beta$, $\beta = 0,\dots,n-1$, $\alpha \in \mathbb{N}_0$, $k_1,k_2 < n$ put

$$P_i^n(\ell;k_1,k_2) = iN_{k_1}^n + M_\beta^n + (\alpha - i)N_{k_2}^n$$

(see fig.6)

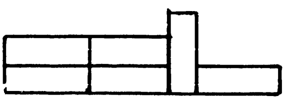
$P_1^3(10;2,1)$ 

fig.6

For an edge c of a polyomino P put $k_1 = \min(|succ_P(c)|,n)$, $k_2 = \min(|succ_P^{-1}(c)|,n)$. Then there is a unique congruence $\varphi : E_2 \to E_2$ mapping the segments $\overline{(0,0)(0,|c|)}$, $\overline{(0,0)(k_1,0)}$, $\overline{(0,|c|)(k_2,|c|)}$ to c, $succ_P(c)$, $succ_P^{-1}(c)$, respectively. Put

$$P_i^n(c,P) = \varphi(P_i^n(|c|;k_1,k_2)), \quad \Phi^n(c,P) = \varphi(M_{|c|}^n).$$

(As a rule, we will be able to assume φ = Id without loss of generality. Then we have $O(c) = (0,|c|)$, $T(c) = (0,0)$ and the edge c will be refered to as an edge in the normal position.) Moreover, we will use the notation

$$P_i^{n'}(c,P) = P^n_{\lfloor |c|/n \rfloor - i}(c,P),$$

where $\lfloor |c|/n \rfloor$ is the low integral part. Let us note that $\bigcup P_i^n(c,P)$, $\bigcup \Phi^n(c,P)$, $\bigcup P_i^{n'}(c,P)$ will be in typical cases subsets of P, although this is not the general case..

2.1.2. An equivalence of n-tilings. Write $A\, \mathcal{R}_M B$ for A,B n-tilings, $M \subset E_2$, if there exist $i,j \in \mathbb{Z}$ such that $A \setminus \sigma^i \tau^j N_n^n = B \setminus \sigma^i \tau^j M_n^n$, while $\bigcup \sigma^i \tau^j N_n^n \subset M$, and denote by $\sim_M$ the least equivalence containing $\mathcal{R}_M$. To the relation $\sim_M$ we will refer as to the M-equivalence. In the case of $M = E_2$ we will speak simply of the equivalence and write $A \sim B$.

2.2.

2.2.1. Theorem: Let c be an edge of P. If the sector I of P over c is K-convex (which is the same as being convex in the direction of c), then each n-tiling of P is I-equivalent with an n-tiling containing $P_i^n(c,P)$ for an i.

Proof: An induction according to the length $|c|$ of c. If $|c| < n$, the statement is obvious. Now, let $d = |c| \geq n$ and the statement hold for $|c| < d$. Take an n-tiling A of P. Put $k_1 = \min(|\mathrm{succ}_P(c)|, n)$, $k_2 = \min(|\mathrm{succ}_P^{-1}(c)|, n)$. We can assume that c is in the normal position. Now, the K-convexity of I implies the existence of $i_1, \dots, i_{2\alpha+1} \in \mathbb{N}_0$ and of $j_1, \dots, j_{\alpha+1} \in \{1, \dots, n\}$ such that $i_1 + \dots + i_{2\alpha+1} = |c|$ and

$$A \supset Q = \frac{i_1}{n} N^n_{j_1} + \sum_{m=1}^{\alpha} \left(M^n_{i_{2m}} + \frac{i_{2m+1}}{n} N^n_{j_{m+1}}\right).$$

Moreover, we can assume $i_2, \dots, i_{2\alpha}$ nonzero. Put $P' = \bigcup (A \setminus Q)$. Then P' is a polyomino and for $2 \leq m \leq \alpha$ is either

$$c_m = \sigma^{i_1 + \dots + i_{2m-2}} \tau^{j_m} \overline{(0,0)(0,i_{2m-1})} \qquad (+)$$

an edge of P' and $|\mathrm{succ}_{P'}(c_m)|, |\mathrm{succ}_{P'}^{-1}(c_m)| \geq n - j_m$, or $j_{m+1} = n$. Similarly, for $m = 1$ (resp. $m = \alpha + 1$) (+) is either an edge of P' and $|\mathrm{succ}_{P'}(c_m)| \geq k_1 - j_m$ (resp. $|\mathrm{succ}_{P'}^{-1}(c_m)| \geq k_2 - j_m$), or $j_1 \geq k_1$ or $i_1 = 0$ (resp. $j_{\alpha+1} \geq k_2$ or $i_{2\alpha+1} = 0$).

In any case, if c_m is an edge of P', the sector I_m of P' over c_m is K-convex. Moreover, the sectors I_m are mutually disjoint. Consequently, by the induction hypothesis,

$$A \setminus Q \sim_{\bigcup I_m} B, \text{ where } P^n(c_m, P') \subset B$$

for some numbers α_m. Since, however, $n \mid |c_m| = i_{2m-1}$, we have

$$P^n_{\alpha_m}(c_m, P) \supset \sigma^{i_0+\dots+i_{2m-2}}\, \varsigma^{j_m} N^n_{\varepsilon - j_m},$$

where

$$\varepsilon = \begin{cases} k_1 & \text{for } m = 1 \\ k_2 & \text{for } m = \alpha + 1 \\ n & \text{for other } m. \end{cases}$$

Thus,

$$P \sim_I (B \cup Q) \supset \frac{i_1}{n} N^n_{k_1} + \sum_{m=1}^{\alpha-1} (M^n_{i_{2m}} + \frac{i_{2m+1}}{n} N^n_n) + M^n_{i_{2\alpha}} + \frac{i_{2\alpha+1}}{n} N^n_k \sim$$

$$\sim \frac{i_1}{n} N^n_{k_1} + M^n_{\sum_{p=2}^{2\alpha} i_p} + \frac{i_{2\alpha+1}}{n} N^n_{k_2} := C.$$

Let $\sum_{p=2}^{2\alpha} i_p = n\beta + b$ ($\beta \in \mathbb{N}_0$, $b = 0,\dots,n-1$). Then $C \sim \frac{i_1}{n} N^n_{k_1} + M^n_b + \beta N^n_n + \frac{i_{2\alpha+1}}{n} N^n_{k_2} \supset \frac{i_1}{n} N^n_{k_1} + M^n_b + (\beta + \frac{i_{2\alpha+1}}{n}) N^n_{k_2} = P^n_{i_1/n}(c,P)$. (see fig.7)

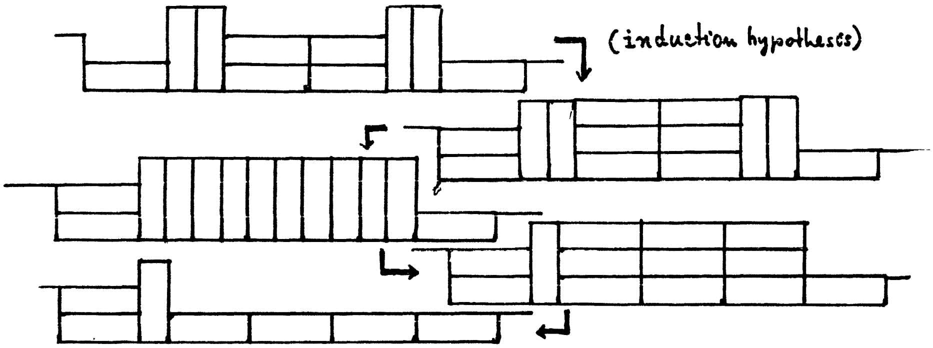


fig.7

2.2.2. Corollary: Let us have besides of the assumption of 2.2.1. moreover $|\mathrm{succ}_P(c)| \geq n$ (resp. $|\mathrm{succ}_P^{-1}(c)| \geq n$). Then each n-tiling of P is I-equivalent to an n-tiling containing $P^n_0(c,P)$ (resp. $P^{n\prime}_0(c,P)$). If we have both $|\mathrm{succ}_P(c)| \geq n$, $|\mathrm{succ}_P^{-1}(c)| \geq n$, then each n-tiling of P is I-equivalent with an n-tiling containing $\Phi^n(c,P)$.

Proof: follows from 2.2.1. and the formulas

$$P^n_i(\ell; n, k_2) = iN^n_n + M^n_\beta + (\alpha - i)N^n_{k_2} \sim M^n_{\beta+in} + (\alpha - i)N^n_{k_2} \sim$$

$$\sim M^n_\beta + iN^n_n + (\alpha - i)N^n_{k_2} \supset M^n_\beta + \alpha N^n_{k_2} = P^n_0(\ell; n, k_2).$$

$$P^n_i(\ell; k_1, n) \sim P^n_{[\ell/n]}(\ell; k_1, n) \text{ (analogously)}$$

$$P^n_i(\ell; n, n) = iN^n_n + M^n_\beta + (\alpha - i)N^n_n \sim M^n_\ell. \ \square$$

2.2.3. Theorem: Let c be an edge of a polyomino P. Assume that all the edges of P have length at least n. If the left (resp. right) semisector I of P over c is K-convex, then each n-tiling A of P is I-equivalent with an n-tiling containing $P_0^n(c,P)$ (resp. $P_0^{n'}(c,P)$).
Proof will be done for the left semisector. According to 1.4.2., the edge c is left regular. Let $h_i = \mathrm{succ}_P^i(c)$ and denote by ℓ the least natural number such that $h_{2\ell+1}$ is an edge of P. We will use the induction on ℓ. It will be of an advantage to consider the fact in a somewhat stronger formulation: we will restrict the assumption to $|h_{2\ell+1}| \geq n$ only.

For $\ell = 0$, $k_1 = |h_{2\ell+1}| \geq n$ and the theorem follows directly from 2.2.2. Let $\ell = \ell_0 > 0$ and the theorem hold for $\ell = \ell_0 - 1$. Put $k_1 = \min(|\mathrm{succ}_P(c)|, n)$, $k_2 = \min(|\mathrm{succ}_P^{-1}(c)|, n)$. If $k_1 = n$, the theorem follows from 2.2.2. Let $k_1 < n$. According to 2.2.1., we can assume $P_i^n(c,P) \subset A$ for some i. If either $i = 0$ or $n \mid |c|$ and $k_2 \leq k_1$, we have $P_0^n(c,P) \subset P_i^n(c,P)$ and the proof is finished. Assume the contrary. Put $P' = \bigcup(A \setminus P_i^n(c,P))$. We can assume that c is in the normal position. Then $c' = h_2 \cup \mathcal{T}^{k_1}(\overline{(0,0)(0,ni)})$ is an edge of P'. The left semisector I' of P' over c' is contained in I and hence K-convex. Putting $h_i' = \mathrm{succ}_P^i(c)$, we have $h_i' = h_{i+2}$ and hence $\ell - 1$ is the least natural ℓ' such that $h_{2\ell'+1}$ is an edge of P'. From the induction hypothesis it follows that $A \setminus P_i^n(c,P)$ is I'-equivalent with an n-tiling B, containing $P_0^n(c',P') \supset \mathcal{T}^{k_1}(iN_{\ell-k_1}^n)$, where

$$\ell = \begin{cases} n & \text{for } n \nmid |c| \\ k_2 & \text{for } n \mid |c| \end{cases}$$

Hence, setting $|c| = \alpha n + \beta$, $\alpha \in \mathbb{N}_0$, $\beta = 1,\dots,n-1$, we have $A \sim_I B \cup$ $\cup\, P_i^n(c,P) \supset iN_n^n + M_\beta^n + (\alpha - i)N_{k_2}^n \sim M_{ni+\beta}^n + (\alpha - i)N_{k_2}^n \sim M_\beta^n + iN_n^n +$ $+ (\alpha - i)N_{k_2}^n \supset M_\beta^n + \alpha N_{k_2}^n = P_0^n(c,P)$. □

Remark: The assumption of $|h_{2\ell+1}| \geq n$ is essential (see fig.8)

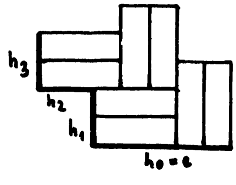


fig.8

2.3. The necessary and sufficient condition for n-tilability of acyclic polyomina

2.3.1. Consider a polyomino P with acyclic I-components. We have proved the following facts:

(1) P is n-tilable, iff each of its I-components is n-tilable.

(2) Assume P has an edge c of length $< n$. Then P is n-tilable, iff $\bigcup P^n_o(c,P) \subset P$ and $[P \setminus \bigcup P^n_o(c,P)]$ is n-tilable.

(3) Let P have no edges of length $< n$. Then there exists an edge c of P such that the left (resp. right) semisector of P over c is K-convex. Now P is n-tilable, iff $\bigcup P^n_o(c,P) \subset P$ (resp. $\bigcup P^{n\prime}_o(c,P) \subset P$) and $[P \setminus \bigcup P^n_o(c,P)]$ (resp. $[P \setminus \bigcup P^{n\prime}_o(c,P)]$) is n-tilable (see 1.4.3., 2.2.3.)

These statements yield an obvious "reduction algorithm" for testing the n-tilability, which consists of a construction of a certain n--tiling of P. The time complexity of this algorithm depends on the time needed for finding the convex semisectors. If we use the trial and error, we obtain the complexity of $O(|P|^3)$.

2.3.2. Theorem 2.2.3. gives a stronger result than the reduction algorithm. By the same method, we obtain quite analogously Theorem (The connectedness theorem): Let P have acyclic I-components. Then any two n-tilings of P are n-equivalent. □

2.4. Tilings of complements of subpolyomina

Let P be a polyomino. Assume that for a set $j \subset \partial P$ it holds

$$j = b \cup succ_P(a) \cup succ_P^2(a) \cup \ldots \cup succ_P^k(a) \cup c$$

for some $k \in \mathbb{N}_0$ and for some segments $b \subset a, c \subset succ_P^{k+1}(a)$, which are subcomplexes of K and satisfy $T(b) = T(a)$, $O(c) = O(succ_P^{k+1}(a))$ in the standard orientation. Then j is called an interval of ∂P. An interval is sais to be n-correct, if $|b|, |succ_P(a)|, \ldots, |succ_P^k(a)|, |c| < n$. A polyomino P is said to be an n-correct subpolyomino of Q, if $P \subset Q$ and $\partial P \setminus \partial Q \subset j$ for some n-correct interval $j \subset \partial P$.

2.4.1. Proposition: Let P be a 2-tilable polyomino and a 2-correct subpolyomino of Q. Then any 2-tiling of Q contains a 2-tiling of P.

Proof: Color a cell $\langle i,j \rangle$ black (resp. white) if $i+j$ is odd (resp. even). It is obviously necessary for a polyomino to be 2-tilable to have the same number of black and white cells.

Suppose now that a 2-tiling A of Q not containing a 2-tiling of P exists. In particular, the polyomino $Q' = \bigcup \{x \in A \mid]x \cap P[\neq \emptyset\} \supsetneq Q$ is 2-tilable. Since all the cells inciding with a 2-correct interval are obviously of the same color and since P is 2-tilable, the number of black and white cells in $[P \setminus Q']$ is not equal, which is a contradiction. □

Note that 2.4.1. obviously does not generally hold for $n > 2$. It holds, however, under the assumption P is acyclic. Our aim in the rest of this paragraph will be to prove this fact.

2.4.2. Lemma: Let P be an acyclic n-tilable polyomino and let $j \subset \partial P$ be an n-correct interval. Then there exists an edge c of P and an n-tiling A of P such that

$$c \cap j = \emptyset \quad (1)$$

$$\Phi^n(c,P) \subset A. \quad (2)$$

Proof will be done by induction on the volume $|P|$ of P. For $|P| \leq n$ the fact is obvious. Let now $|P| = m > n$ and the fact hold for $|P| < m$. We have three cases:

1) P contains an edge c, $|c| < n$, $c \cap j = \emptyset$. Then any n-tiling A of P satisfies $\Phi^n(c,P) \subset A$.

2) For all edges c of P, satisfying $c \cap j = \emptyset$, it holds $|c| \geq n$, but there exists an edge d of P with $d \cap j \neq \emptyset$ and $|d| < n$. Then any n-tiling of P contains $\Phi^n(d,P)$. Let C be an I-component of $[P \setminus \bigcup \Phi^n(d,P)]$. Put $j' = \partial C \cap (j \cup \bigcup \Phi^n(d,P))$. Then j' is an n-correct interval of ∂C. By the induction hypothesis there exists an n-tiling B of C and an edge $c \subset \partial C$ with $c \cap j' = \emptyset$ such that $B \supset \Phi^n(c,C)$. Obviously, c is also an edge of P with $c \cap j = \emptyset$ and any n-tiling A of P containing B (which necessarily exists) satisfies (1) and (2).

3) All the edges of P are of length $\geq n$. By Theorem 1.4.3., there exists an edge $c \subset \partial P$ such that one of the semisectors (assume it is the left one) of P over c is K-convex. Then there exists an n-tiling B of P with $P_0^n(c,P) \subset B$. Denote by Q an I-component of $[P \setminus \bigcup P_0^n(c,P)]$ inciding with $\mathrm{succ}_P^{-1}(c)$. There exists a $B' \subset B$ with $\bigcup B' = Q$. Let $\Phi = \bigcup (P_0^n(c,P) \cap \Phi^n(c,P))$. Put $\bar{j} = \partial Q \cap \Phi$. Evidently, $\bar{j}$ is an n-correct interval of ∂Q. By the induction hypothesis there exists an edge $\bar{d} \subset Q$ not inciding with $\bar{j}$ and an n-tiling D' of Q such that $\Phi^n(\bar{d},D) \subset D'$. It is easy to see that $\bar{d}$ is either an edge of P or $\bar{d} \subset \mathrm{succ}_P^{-1}(c)$. In any case we have an edge d of P containing $\bar{d}$ such that $\Phi^n(d,P) \subset (B \setminus B') \cup D' = D$. If $d \cap j = \emptyset$, we can put $D = A$, $c = d$, concluding the proof. Let now $d \cap j \neq \emptyset$. Assume, for instance, $T(d) \subset j$. Then D is obviously equivalent to an n-tiling E with $E \supset P_0^n(d,P)$. Denote by Q' an I-component of $[P \setminus \bigcup P_0^n(d,P)]$, inciding with $\mathrm{succ}_P^{-1}(d)$. There exists an $E' \subset E$ with $\bigcup E' = Q'$. Put $j' = \partial Q' \cap \cap (\bigcup (P_0^n(c,P) \cap \Phi^n(c,P)) \cup j)$. Obviously, j' is an n-correct interval of $\partial Q'$. By the induction hypothesis, we have an edge c' of Q' not inciding with j' and an n-tiling F of Q' with $\Phi^n(c',Q') \subset F$. As above, c' is either an edge of P or c' $\mathrm{succ}_P^{-1}(d)$. In any case, we have an edge c_1 of P with $c' \subset c_1$ and $\Phi^n(c_1,P) \subset A = F \cup (E \setminus E')$. We see easily that $c_1 \cap j = \emptyset$. □

2.4.3. Theorem: (The separation theorem) Let P be an n-correct acyclic subpolyomino of Q. (Q needn't be acyclic). If P is n-tilable then any n-tiling of Q contains an n-tiling of P. In particular, if P,Q are n-tilable, then $[Q \setminus P]$ is n-tilable.

Proof will be done by induction on $|P|$. For $|P| \leqslant n$ the statement is obvious. Let now $|P| = m > n$, and the theorem hold whenever $|P| < m$. Let j be the n-correct interval of ∂P such that $\partial P \setminus \partial Q \subset j$. If P is n-tilable, then, by lemma 2.4.2., there exists an edge $c \subset \partial P$, $c \cap j = \emptyset$ and an n-tiling A of P such that $\Phi^n(c,P) \subset A$. Now suppose that c is in the normal position. Let B be an n-tiling of Q. As c is obviously an edge of Q, there exists a $C \in \{C_{1,n}, C_{n,1}\}$ with $C \in B$. In any case, A is equivalent with an n-tiling A'' containing C. Then $[P \setminus C]$ is n-tilable. As all the I-components of $[P \setminus C]$ are obviously n-correct subpolyomina of $[Q \setminus C]$, $B \setminus \{C\}$ must contain an n-tiling of $[P \setminus C]$. Putting $A' = A'' \cup \{C\}$, we have $B \supset A'$. □

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