Cosimo Guido

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CONNECTED SPACES WHICH ARE NOT STRONGLY CONNECTED (*)

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Sunto. *In questa nota vengono individuate due classi di spazi topologici, rispettivamente T_0 non-T_1 e T_1 non-T_2, che sono connessi ma non fortemente connessi.*

INTRODUCTION. We recall that a topological space (S,τ) is *maximal connected* if it is connected and no connected topology τ' on S exists which is strictly finer that τ, (S,τ) is *strongly connected* if τ is coarser than a maximal connected topology τ' on S.

The existence of maximal connected spaces verifying some separation axioms has been often investigated; until now it is an open question whether a regular maximal connected space exists or not.

Any-way every maximal connected space is a T_0 space.

We remark that each connected topology on a finite set is strongly connected.

Now if we consider only T_0 topological spaces with infinitely many points, the following questions can be asked.

Do there exist T_0 non-T_1 strongly connected spaces which have, respectively, maximal connected T_2 or maximal connected T_1 or only maximal connected T_0 expansions?

Do there exist T_1 non-T_2 strongly connected spaces which have, respectively, maximal connected T_2 or only maximal connected T_1 expansions?

Do there exist connected non-strongly connected spaces which are respectively T_2, T_1 non-T_2, T_0 non-T_1?

The answers to the first two questions are affirmative by the well-known existence of T_0 non-T_1, T_1 non-T_2, and T_2 maximal connected spaces.

In some papers, listed below, sufficiently large classes of spaces were pointed out in order to give examples of such spaces.

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Guthrie and Stone [5] and Baggs [1] found T_2 connected spaces which are not strongly connected.

In this note we show how to construct T_1 non-T_2 or T_0 non-T_1 connected topological spaces wich are not strongly connected.

If (S,τ) is a topological space, we shall denote by $\tau(x)$ the family of open neighbourhoods of x while $\mathcal{F}_\tau(x)$ will denote the neighbourhood filter of x in τ. $\tau_{|X}$ will be the induced topology on the subset $X \subseteq S$. If X, Y are subsets of S, $X \setminus Y$ will denote their difference. If, finally, X is a disconnected subspace of (S,τ), we shall say that A, B divide X in τ if A, B are non-empty open sets of $\tau_{|X}$ and $A \cup B = X$, $A \cap B = \emptyset$

See [3] for further notations not mentioned here.

1. Let (N,τ_0) be a T_2 connected space with a dispersion point $x_0 \in N$. Choose a point $x \neq x_0$ in N and a non-principal open ultrafilter υ on N containing the family $\{V \setminus \{x\} \ / \ V \in \tau_0(x)\}$.

Consider the topology τ on N defined by

$$A \subseteq N \ , \ A \in \tau \iff \begin{cases} x \notin A \Rightarrow A \in \tau_0 \\ x \in A \Rightarrow A \setminus \{x\} \in \upsilon \cap \tau_0 \end{cases}$$

Trivially (N,τ) is Hausdorff and x_0 is a dispersion point of (N,τ) too. Furthermore we have the following.

LEMMA 1. $U \subseteq N$, $U \cup \{x\} \Rightarrow (N \setminus U) \cup \{x\} \in \mathcal{F}_\tau(x)$.

Proof. Let U be a subset of N. If $U \in \upsilon$, then $A \in \upsilon \cap \tau_0$, $A \subseteq U$ exists. $A \cup \{x\}$ is open in τ and $U \cup \{x\} \in \mathcal{F}_\tau(x)$.

If $U \notin \upsilon$, then $B \in \upsilon \cap \tau_0$, $B \subseteq (N \setminus U)$ exists hence $B \cup \{x\} \in \tau$ and $(N \setminus U) \cup \{x\} \in \mathcal{F}_\tau(x)$.

LEMMA 2. (N,τ) *is connected.*

Proof. If (N,τ) were disconnected and A,B divided S in τ and $x \in A$, then we should have

$$B \in \tau_0 \quad \text{and} \quad A \setminus \{x\} \in \upsilon \cap \tau_0$$

hence B and $A \setminus \{x\}$ would divide $N \setminus \{x\}$ in τ_0 and consequently x would be a cut point of (N, τ_0).

So we have the assertion since the dispersion point x_0 is the only cut point of (N,τ_0)

Now consider a point $y \notin N$ and put $S = N \cup \{y\}$. Let σ be the topology on S defined by

$$A \subseteq S, \ a \in \sigma \iff \begin{cases} y \notin A \Rightarrow A \in \tau \\ y \in A \Rightarrow A \setminus \{y\} \in \tau_0 \cap \upsilon \end{cases}$$

Of course $\sigma|_N = \tau$ and (S,σ) is a T_1 non-T_2 space.

Furthermore (S,σ) is connected; indeed if A,B divided S in σ and $y \in A$, then $A\setminus\{y\}$ and B would divide N in τ_0.

Eventually it can be proved the following lemma.

LEMMA 3. N *remains connected in every connected expansion of* (S,σ)

Proof. Let (S,τ'), $\tau' \supseteq \sigma$, be a connected expansion of (S,σ).

If N were disconnected and X, Y divided N in τ', where $X = A \cap N$, $Y = B \cap N$ with $A,B \in \tau'$, we should have $y \in A \cup B$ whence $A,B\setminus\{y\}$ or A,B divide S in τ' which contradicts the assumptions.

Then let us consider the case $y \notin A \cup B$ and consequently assume $X = A \in \tau'$, $Y = B \in \tau'$; by lemma 1 we could suppose $X \cup \{x\} \in \mathcal{T}_\tau(x)$ and then $X \cup \{y\}$ would be open in $\sigma \subseteq \tau'$. $X \cup \{y\}$ and Y would divide S in τ' and, again, (S,τ') would be disconnected.

The following theorem can now be easily proved.

THEOREM 1. (S,σ) *is not a strongly connected space.*

Proof. Otherwise, the connected subspace N of some maximal connected expansion (S,τ') would be maximal connected and (N,τ_0) would be strongly connected which is absurd (see [5] theorem 15).

2. Let (M,τ_1) be a T_1 maximal connected topological space such that the non-empty open sets form an ultrafilter $\upsilon = \tau_1\setminus\{\emptyset\}$.

Take two points $x \in M$ and $y \notin M$, put $X = M \cup \{y\}$ and consider the topology τ on X defined by

$$A \subseteq X,\ A \in \tau \iff \begin{cases} A\setminus\{y\} \in \tau_1 & \text{and} \\ y \in A \Rightarrow x \in A. \end{cases}$$

(X,τ) is a T_0 non-T_1 connected space and it is not maximal connected (in fact its proper expansion $A \in \tau' \iff A\setminus\{y\} \in \upsilon$ or $A=\emptyset$ is connected).

Furthermore the following results hold for such a topology.

LEMMA 4. *If* $\tau' \supseteq \tau$ *is a maximal connected non-*T_1 *topology on* X, *then* M *is connected in* τ'.

Proof. Let M be disconnected and $A \cap M$, $B \cap M$ divide M; $A,B \in \tau'$; suppose $x \in A$, then $x \notin B$ and $y \notin B$ since $\tau'(y) \subseteq \tau'(x)$.

If now $y \in A$, then A,B divide X in τ'.

If $y \notin A$ and $A \in \tau_1$ we must have $A \cup \{y\} \in \tau \subseteq \tau'$ whence $A \cup \{y\}$, B divide X in τ'; in the same way A, $B \cup \{y\}$ divide X in τ' if $y \notin A$ and $B \in \tau_1$.

Anyway (X,τ') must be disconnected, a contradiction.

LEMMA 5. *If* $\tau' \supset \tau$ *is a maximal connected topology on* X *and* M *is connected in* τ', *then* τ' *is a* T_1 *topology.*

Proof. It follows from the assumption that $(M, \tau'|_M)$ is maximal connected since it is a connected subspace of a maximal connected space; on the other hand $\tau_1 = \tau|_M \subseteq \tau'|_M$ is a maximal connected topology too, so we have $\tau'|_M = \tau_1$.

Consider now $U \in \tau' \setminus \tau$; trivially $U \cap M = U \setminus \{y\} \in \tau_1$ and $y \in U$: furthermore it follows from $U \setminus \{y\} \in \tau_1$, $y \in U$ and $U \notin \tau$ that $x \notin U$. U is actually a neighbourhood of y, in τ', which does not contain x and consequently τ' is a T_1 topology.

Let now $\sigma \supseteq \tau$ be a T_0 non-T_1 connected topology on X which is T_1-disconnected, i.e. the least T_1 topology containing σ is disconnected (see [3]).

We are now able to prove the concluding result.

THEOREM 2. (X, σ) *is not strongly connected.*

Proof. If $\tau' \supseteq \sigma$ were a maximal connected expansion of σ one would have $\tau' \supseteq \sigma \supseteq \tau$; on the other hand τ' would not be T_1, hence $\tau'|_M$ would be connected, by lemma 4, which contradicts the assumption that τ' is maximal connected, by lemma 5.

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DIPARTIMENTO DI MATEMATICA
UNIVERSITA'
VIA ARNESANO

73100 LECCE (ITALIA)