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Inverse systems and pretopological spaces

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Given a pretopological space $S=(X,P)$, we associate to any interior covering $\mathcal{X}$ of S a symmetrical pf-space $S_{\mathcal{X}}$ on the set X. Precisely, to obtain the pretopology of $S_{\mathcal{X}}$, we take for each point x of X the principal filter of base the star of x with respect to $\mathcal{X}$. Taking the pf-spaces $S_{\mathcal{X}}$ as terms, we obtain the inverse system $\hat{S}$ of the pretopological space S. Generally the inverse limit $S^*$ of $\hat{S}$ is different from S; yet $S^*=S$ when S is a Tychonoff topological space.

For each dimension n, we associate to $\hat{S}$ an inverse system of prehomotopy groups $\Pi_n(S_{\mathcal{X}},a)$ and an inverse system of singular homology groups $H_n(S_{\mathcal{X}})$. Taking the inverse limits $\varprojlim \Pi_n(S_{\mathcal{X}},a)$ and $\varprojlim H_n(S_{\mathcal{X}})$, we obtain the shape groups $\check{\Pi}_n(S,a)$ and the Čech homology groups $\check{H}_n(S)$ of the pretopological space S.

Our shape groups have the characteristical properties of the classical shape groups. Similarly we can say for our Čech groups. All proofs, except those for the homotopy conditions, are similar to the classical ones.

The relations between our groups and the classical shape groups or Čech homology groups of a compact topological space will be expounded in another paper.

## 1. The inverse system of a pretopological space.

Let X be a nonempty set and $P=\{F_x\}(x\in X)$ a family of filters of X such that $\overline{x}\leq F_x$ for each $x\in X$. Such a family $P$ is called a pretopology in X, and the pair $(X,P)$ is called a pretopological space (see [2]). Here we will denote by S the pretopological space $(X,P)$, since we need to consider different pretopologies on the set X.

We recall that S is a pf-space, if each filter $F_x$ is principal, i.e. $F_x=\overline{A_x}$ with $x\in A_x$. Moreover we say that the pf-space S is symmetrical, if $y\in A_x$ implies $x\in A_y$ for any $x,y\in X$.

We also recall that (see [1]) a covering $\mathcal{X}$ of X is an interior covering of S, if for any $x\in X$ there is at least one element A of $\mathcal{X}$ such that $A\in F_x$.

Now we consider the collection Cov(S) of all interior coverings of S and we preorder it by the following:

1.1 *Definition* Let $\mathcal{X}$, $\mathcal{X}'\in$Cov(S). We write $\mathcal{X}\leq\mathcal{X}'$ iff $\mathcal{X}'$ is a refinement of $\mathcal{X}$.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

1.2 *Remark.* Clearly $(\mathrm{Cov}(S), \leq)$ is a directed set, since $\mathcal{X}, \mathcal{X}'\in\mathrm{Cov}(S)$ implies $\mathcal{X}\wedge\mathcal{X}'\in\mathrm{Cov}(S)$.

1.3 *Definition* Given $\mathcal{X}\in\mathrm{Cov}(S)$, we denote by $P(\mathcal{X})$ the pretopology in X, that we obtain taking for each $x\in X$ the principal filter of base the star $St(x,\mathcal{X})$ of x with respect to $\mathcal{X}$. Then we put $S_{\mathcal{X}}=(X,P(\mathcal{X}))$.

1.4 *Remark.* $S_{\mathcal{X}}$ is a symmetrical pf-space, and the identity $p_{\mathcal{X}}:S\to S_{\mathcal{X}}$ is a precontinuous map. Moreover, if $\mathcal{X}, \mathcal{X}'\in\mathrm{Cov}(S)$ and $\mathcal{X}\leq\mathcal{X}'$, the identity $p_{\mathcal{X}\mathcal{X}'}:S_{\mathcal{X}'}\to S_{\mathcal{X}}$ is a precontinuous map, and $p_{\mathcal{X}} = p_{\mathcal{X}\mathcal{X}'}p_{\mathcal{X}'}$.

1.5 *Definition* We will denote by $\hat{S}$ the inverse system $(S_{\mathcal{X}}, p_{\mathcal{X}\mathcal{X}'}, \mathrm{Cov}(S))$, and we will call it the inverse system of the pretopological space S. The projection $(p_{\mathcal{X}}):S\to\hat{S}$ will be denoted by $\hat{p}$.

1.6 *Remark.* The inverse limit $\varprojlim S_{\mathcal{X}}$ is the pretopological space $S^*=(X,P^*)$, where $P^*$ is obtained taking for each $x\in X$ the filter on X of base $\{St(x,\mathcal{X})\}(\mathcal{X}\in\mathrm{Cov}(S))$. Generally the pretopology $P^*$ is coarser than $P$; yet $S^*=S$, if S is a completely regular topological space.

## 2. The morphism induced by a precontinuous map.

Let us consider two pretopological spaces S and T, their inverse systems $\hat{S} = (S_{\mathcal{X}}, p_{\mathcal{X}\mathcal{X}'}, \mathrm{Cov}(S))$ and $\hat{T} = (T_{\mathcal{Y}}, q_{\mathcal{Y}\mathcal{Y}'}, \mathrm{Cov}(T))$, and the projections $\hat{p}:S\to\hat{S}$ and $\hat{q}:T\to\hat{T}$.

2.1 *Proposition* Any precontinuous map $f:S\to T$ induces a morphism from $\hat{S}$ to $\hat{T}$.

*Proof:*

a) For any $\mathcal{Y}\in\mathrm{Cov}(T)$, the family $\{f^{-1}(Y)\}(Y\in\mathcal{Y})$ is an interior covering of S. So $f^{-1}$ induces a function from Cov(T) to Cov(S), which preserves the preorder. We will denote also this function by $f^{-1}$.

b) For each $\mathcal{Y}\in\mathrm{Cov}(T)$ we obtain a precontinuous map $f_{\mathcal{Y}}:S_{f^{-1}(\mathcal{Y})}\to T_{\mathcal{Y}}$ putting $f_{\mathcal{Y}}(x)=f(x)$ for any $x\in X$.

c) $(f_{\mathcal{Y}}, f^{-1})$ is a morphism from $\hat{S}$ to $\hat{T}$. In fact, given $\mathcal{Y}, \mathcal{Y}'\in\mathrm{Cov}(T)$ such that $\mathcal{Y}\leq\mathcal{Y}'$, clearly $f^{-1}(\mathcal{Y})\leq f^{-1}(\mathcal{Y}')$ and the following diagram commutes:

$$\begin{tikzcd}
S_{f^{-1}(\mathcal{Y})} \arrow[d, "f_{\mathcal{Y}}"'] & S_{f^{-1}(\mathcal{Y}')} \arrow[l, "p_{f^{-1}(\mathcal{Y})f^{-1}(\mathcal{Y}')}"'] \arrow[d, "f_{\mathcal{Y}'}"] \\
T_{\mathcal{Y}} & T_{\mathcal{Y}'} \arrow[l, "q_{\mathcal{Y}\mathcal{Y}'}"]
\end{tikzcd}$$

2.2 *Definition* The morphism $(f_{\mathcal{Y}}, f^{-1})$ will be denoted by $\hat{f}:\hat{S}\to\hat{T}$, and we will call it the morphism induced by f.

2.3 *Remark.* Let us define another function $\phi:\mathrm{Cov}(T)\to\mathrm{Cov}(S)$, taking for each $\mathcal{Y}\in\mathrm{Cov}(T)$ an interior covering $\phi(\mathcal{Y})$ of S, such that $f^{-1}(\mathcal{Y})\leq\phi(\mathcal{Y})$. Then for each $\mathcal{Y}\in\mathrm{Cov}(T)$, consider the precontinuous map $f'_{\mathcal{Y}}:S_{\phi(\mathcal{Y})}\to T_{\mathcal{Y}}$ given by $f'_{\mathcal{Y}}(x)=f(x)$ for any $x\in X$. It is easy to see that $(f'_{\mathcal{Y}}, \phi)$ is a morphism from $\hat{S}$ to $\hat{T}$, which is equivalent to $\hat{f}$.

2.4 *Remark*. The morphism $\hat{f}$ induced by f makes commutative the following diagram:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\hat{p}"'] & T \arrow[d, "\hat{q}"] \\ \hat{S} \arrow[r, "\hat{f}"] & \hat{T} \end{tikzcd}$$

Moreover, any morphism $g = (g_Y, \psi)$ from $\hat{S}$ to $\hat{T}$ such that $g\hat{p} = \hat{f}\hat{p}$ is equivalent to $\hat{f}$.

3. The morphism associated to a prehomotopy.

Let us consider two pretopological spaces S and T, the closed interval $I=[0,1]$ of the real line with the pretopology $\{U_t\}(t\in I)$ (where $U_t$ is the neighbourhood filter of the point t), the pretopological space $Z=S\times I$, and the inverse systems $\hat{S} = (S_X, p_{XX'}, Cov(S))$, $\hat{T} = (T_Y, q_{YY'}, Cov(T))$ and $\hat{Z} = (Z_R, \pi_{RR'}, Cov(Z))$. Then let $f:S\to T$ and $g:S\to T$ be homotopic precontinuous maps, and $H: S\times I\to T$ a prehomotopy of f to g.

3.1 *Theorem* We can associate to the map $H:Z\to T$ a morphism $K:\hat{Z}\to\hat{T}$, which is equivalent to $\hat{H}$ and has properties analogous with those of homotopies.

*Proof:*

a) Define a function $\Phi:Cov(T)\to Cov(Z)$ as follows.

Given $Y\in Cov(T)$, consider $H^{-1}(Y)\in Cov(Z)$. For each point $(x,t)\in Z$, take $C_{x,t}\in H^{-1}(Y)$, and then $A_x^{x,t}\in F_x$ and an open interval $V_t^{x,t}\in U_t$ such that $A_x^{x,t}\times V_t^{x,t}\subseteq C_{x,t}$. $\{U_{x,t}\}((x,t)\in Z)$, where $U_{x,t}=A_x^{x,t}\times V_t^{x,t}$, is an interior covering of Z which refines $H^{-1}(Y)$.

For any $x\in S$, the family $\{U_{x,t}\}(t\in I)$ is an interior covering of the subspace $\{x\}\times I$ of Z. Since $\{x\}\times I$ is compact, there is a finite number $n(x)$ of points $t_h$ of I such that $\bigcup_{1\leq h\leq n(x)} U_{x,t_h}\supseteq\{x\}\times I$. Now observe that $A_x=\bigcap_{1\leq h\leq n(x)} A_x^{x,t_h}$ belongs to the filter $F_x$, and put $R_x=\{W_{x,t_h}\}(1\leq h\leq n(x))$, where $W_{x,t_h}$ is the set $A_x\times V_{t_h}^{x,t_h}$.

Then consider the family $R = \bigvee_{x\in S} R_x$.

Clearly R is a covering of Z; moreover R refines $H^{-1}(Y)$, since $W_{x,t_h}\subseteq C_{x,t_h}$.

Given any $(x,t)\in Z$, we have $(x,t)\in W_{x,t_h}$ for some positive integer $h\leq n(x)$. Since $V_{t_h}^{x,t_h}\in U_t$, we have $W_{x,t_h}\in F_{(x,t)}$.

So $R\in Cov(Z)$, and we put $\Phi(Y)=R$.

b) For each $Y\in Cov(T)$, we consider the map $K_Y:Z_{\Phi(Y)}\to T_Y$, given by $K_Y(x,t)=H(x,t)$. $K_Y$ is a precontinuous map, since for each $(x,t)\in Z$ we obtain $H(St((x,t),\Phi(Y))\subseteq$ $\subseteq St(H(x,t),Y)$.

c) $K = (K_Y, \Phi)$ is a morphism from $\hat{Z}$ to $\hat{T}$.

In fact, given $Y$, $Y'\in Cov(T)$ such that $Y\leq Y'$, consider $\Phi(Y)=\bigvee_{x\in S} R_x$ and $\Phi(Y')=\bigvee_{x\in S} R'_x$, where $R_x = \{A_x\times V_{t_h}^{x,t_h}\}(1\leq h\leq n(x))$ and $R'_x = \{A'_x\times V'^{x,t_k}_{t_k}\}(1\leq k\leq m(x))$.

Then we take the family $\mathcal{R}''_x$ of all subsets of Z of form $A''_x \times V_{h,k}$ $(1\leq h\leq n(x),\ 1\leq k\leq m(x))$ with $A''_x = A_x \cap A'_x$ and $V_{h,k} = V^{x,t_h}_{t_h} \cap V'^{x,t_k}_{t_k} \neq \emptyset$, and put $\mathcal{R}'' = \bigvee_{x\in S} \mathcal{R}''_x$.
Given a point $(x,t)\in Z$, we have $A''_x \times V_{h,k} \in \mathcal{F}_{(x,t)}$ iff $t\in V_{h,k}$, since $A''_x \in \mathcal{F}_x$ and $V_{h,k}$ is an open subset of I. But we find two positive integers $h\leq n(x)$ and $k\leq m(x)$ such that $t\in V^{x,t_h}_{t_h}$ and $t\in V'^{x,t_k}_{t_k}$. Hence $\mathcal{R}''\in \mathrm{Cov}(Z)$.
Clearly $\mathcal{R}''$ refines both $\Phi(\mathcal{Y})$ and $\Phi(\mathcal{Y}')$. Moreover the following diagram commutes:

$$\begin{tikzcd}
 & Z_{\mathcal{R}''} \arrow[dl, "\pi_{\Phi(\mathcal{Y})\mathcal{R}''}"'] \arrow[dr, "\pi_{\Phi(\mathcal{Y}')\mathcal{R}''}"] & \\
Z_{\Phi(\mathcal{Y})} \arrow[d, "K_{\mathcal{Y}}"'] & & Z_{\Phi(\mathcal{Y}')} \arrow[d, "K_{\mathcal{Y}'}"] \\
T_{\mathcal{Y}} & & T_{\mathcal{Y}'} \arrow[ll, "q_{\mathcal{Y}\mathcal{Y}'}"']
\end{tikzcd}$$

d) The morphism K is equivalent to $\hat{H}$, since, for each $\mathcal{Y}\in \mathrm{Cov}(T)$, $\Phi(\mathcal{Y})$ is an interior covering of Z which refines $H^{-1}(\mathcal{Y})$.

e) Observe that, for each $t\in I$, the map $h_t : S\to T$ given by $h_t(x) = H(x,t)$ is precontinuous.

Then define a function $\phi : \mathrm{Cov}(T) \to \mathrm{Cov}(S)$ as it follows.
Given $\mathcal{Y}\in\mathrm{Cov}(T)$, consider $\Phi(\mathcal{Y}) = \bigvee_{x\in S} \mathcal{R}_x$ where $\mathcal{R}_x = \{A_x \times V^{x,t_h}_{t_h}\}(1\leq h\leq n(x))$, and take $\mathcal{A} = \{A_x\}(x\in S)$. Clearly $\mathcal{A}\in \mathrm{Cov}(S)$, since $A_x \in \mathcal{F}_x$ for any $x\in S$. Hence we put $\phi(\mathcal{Y}) = \mathcal{A}$.
Now consider the function $h^t_{\mathcal{Y}} : S_{\phi(\mathcal{Y})} \to T_{\mathcal{Y}}$, given by $h^t_{\mathcal{Y}}(x) = h_t(x)$ for any $x\in S$.
To prove that $h^t_{\mathcal{Y}}$ is precontinuous, we have to show that $h_t(\mathrm{St}(x,\mathcal{A}))\subseteq \mathrm{St}(H(x,t),\mathcal{Y})$.
To this purpose take a point $x_0\in S$ such that $x\in A_{x_0}$. For any positive integer $h\leq n(x_0)$ such that $t\in V^{x_0,t_h}_{t_h}$, we have $A_{x_0} \times V^{x_0,t_h}_{t_h} \subseteq \mathrm{St}((x,t),\Phi(\mathcal{Y}))$; therefore (see b)) $h_t(A_{x_0}) \subseteq \mathrm{St}(H(x,t),\mathcal{Y})$.
Hence $(h^t_{\mathcal{Y}}, \phi)$ is a morphism from $\hat{S}$ to $\hat{T}$.
Moreover $(h^t_{\mathcal{Y}}, \phi)$ is equivalent to the morphism $\hat{h}_t : \hat{S}\to\hat{T}$ induced by $h_t$, because the interior covering $\phi(\mathcal{Y})$ of S is a refinement of $h_t^{-1}(\mathcal{Y})$.
Observe that the covering $\phi(\mathcal{Y})$, and consequently the function $\phi : \mathrm{Cov}(T) \to \mathrm{Cov}(S)$ do not depend on the point t of I.
For $t=0$ we have $h_t = f$; therefore the morphism $(h^0_{\mathcal{Y}}, \phi)$ is equivalent to $\hat{f}$.
Then for $t=1$ we have $h_t = g$; thus the morphism $(h^1_{\mathcal{Y}}, \phi)$ is equivalent to $\hat{g}$.

3.2 *Remark.* We proved in b) that H is a precontinuous map from $(S\times I)_{\Phi(\mathcal{Y})}$ to $T_{\mathcal{Y}}$, for each $\mathcal{Y}\in \mathrm{Cov}(T)$. But generally we cannot say that H is a precontinuous map from $S_{\phi(\mathcal{Y})}\times I$ to $T_{\mathcal{Y}}$.

## 4. Inverse systems of pairs.

Let us consider a pretopological space $S=(X,\mathcal{P})$ and a subset A of X.

4.1 *Definition* Let J' be a subset of the index set J, and for each $i\in J$ let $A_i\subseteq X$. We say that $\mathcal{A} = \{A_i\}(i\in J,J')$ is an interior covering of the pair (S,A) with

$(J,J')$ as indexing pair, if:

(1) $\{A_i\}(i\in J)\in \mathrm{Cov}(S)$;

(2) for each $x\in A$ there is at least one index $j\in J'$ such that $A_j\in F_x$.

The collection of all interior coverings of the pair $(S,A)$ will be denoted by $\mathrm{Cov}(S,A)$.

4.2 *Remark.* Let $A=\{A_i\}(i\in J,J')\in \mathrm{Cov}(S,A)$. The families $\{A_i\}(i\in J)$ and $\{A_i\}(i\in J')$ will be denoted by $A_J$ and $A_{J'}$, respectively. $A$ induces in $X$ the pretopology $P(A_J)=\{\overline{\mathrm{St}(x,A_J)}\}(x\in X)$ and in $A$ the pretopology $P(A_{J'})=\{\overline{\mathrm{St}(x,A_{J'})}\}(x\in A)$. Clearly $P(A_J)$ induces in $A$ a pretopology $P(A_J)^*$ which is coarser then $P(A_{J'})$. The pair $((X,P(A_J)),\ (A,P(A_{J'})))$ will be denoted by $(S,A)_A$. Clearly the identity $p_A:(S,A)\to(S,A)_A$ is a precontinuous map.

4.3 *Definition* Let $A=\{A_i\}(i\in J,J')$ and $B=\{B_h\}(h\in H,H')$ be interior coverings of the pair $(S,A)$. We write $A\leq B$ iff:

(1) $B_H$ is a refinement of $A_J$;

(2) $B_{H'}$ is a refinement of $A_{J'}$.

4.4 *Remark.* $(\mathrm{Cov}(S,A),\leq)$ is a directed set.
If $A, A'\in\mathrm{Cov}(S,A)$ and $A\leq A'$, the identity $p_{AA'}:(S,A)_{A'}\to(S,A)_A$ is a precontinuous map, and $p_A=p_{AA'}p_{A'}$.

4.5 *Definition* The inverse system $((S,A)_A,\ p_{AA'},\ \mathrm{Cov}(S,A))$ will be called the inverse system of the pair $(S,A)$, and it will be denoted by $\widehat{S,A}$. $\hat{p}=(p_A)$ will be called the projection from $(S,A)$ to $\widehat{S,A}$.

4.6 *Proposition* Let $S$ and $T$ be pretopological spaces, $A$ a subset of $S$, $B$ a subset of $T$, $\widehat{S,A}=((S,A)_A,\ p_{AA'},\ \mathrm{Cov}(S,A))$ and $\widehat{T,B}=((T,B)_B,\ q_{BB'},\ \mathrm{Cov}(T,B))$. Any precontinuous map $f:(S,A)\to(T,B)$ induces a morphism $\hat{f}:\widehat{S,A}\to\widehat{T,B}$.

*Proof:* Given $B=\{B_i\}(i\in J,J')\in\mathrm{Cov}(T,B)$, the family $f^{-1}(B)=\{f^{-1}(B_i)\}(i\in J,J')$ belongs to $\mathrm{Cov}(S,A)$. Then, for each $B\in\mathrm{Cov}(T,B)$, we define a precontinuous map $f_B:(S,A)_{f^{-1}(B)}\to(T,B)_B$, putting $f_B(x)=f(x)$ for each $x\in S$. $(f_B,\ f^{-1})$ is a morphism from $\widehat{S,A}$ to $\widehat{T,B}$, and we will denote it by $\hat{f}$.

4.7 *Remark.* For each $B\in\mathrm{Cov}(T,B)$, let us take $\phi(B)\in\mathrm{Cov}(S,A)$ such that $f^{-1}(B)\leq\phi(B)$. We obtain a precontinuous map $f'_B:(S,A)_{\phi(B)}\to(T,B)_B$ putting $f'_B(x)=f(x)$ for any $x\in S$. $(f'_B,\ \phi)$ is a morphism from $\widehat{S,A}$ to $\widehat{T,B}$ which is equivalent to $\hat{f}$.

4.8 *Theorem* Let $S$ and $T$ be pretopological spaces, $A\subseteq S$ and $B\subseteq T$. Then let $f$ and $g$ be homotopic precontinuous maps from $(S,A)$ to $(T,B)$, and let $H:(S\times I,\ A\times I)\to(T,B)$ be a relative prehomotopy of $f$ to $g$. We can associate to the map $H$ a morphism $K:\widehat{S\times I,A\times I}\to\widehat{T,B}$, which is equivalent to $\hat{H}$ and has properties analogous with those of relative homotopies.

*Proof:* To simplify notations, we put $S\times I=Z$ and $A\times I=C$.

a) Given $B=\{B_i\}(i\in J,J')\in\mathrm{Cov}(T,B)$, observe that $H^{-1}(B)=\{H^{-1}(B_i)\}(i\in J,J')$ belongs to $\mathrm{Cov}(Z,C)$. For each $(x,t)\in L$, take $C_{x,t}\in H^{-1}(B)$ such that:

(i) $C_{x,t}\in F_{(x,t)}$;

(ii) if $x\in A$, then $C_{x,t}\in H^{-1}(\mathcal{B}_{J'})$.

Afterwards, with the same process of Theorem 3.1, for each $x\in S$ we construct a finite refinement $\mathcal{R}_x=\{W_{x,t_h}\}(1\leq h\leq n(x))$ of the family $\{C_{x,t}\}(t\in I)$, such that each $W_{x,t_h}$ is of form $A_x\times V_{t_h}^{x,t_h}$ where $A_x\in F_x$ and $V_{t_h}^{x,t_h}$ is an open interval of I containing $t_h$. The family $\bigvee_{x\in S,A}\mathcal{R}_x$ belongs to Cov(Z,C), and we put $\Phi(\mathcal{B})=\bigvee_{x\in S,A}\mathcal{R}_x$.

b) For each $\mathcal{B}\in$Cov(T,B), we obtain a precontinuous map $K_{\mathcal{B}}:(Z,C)_{\Phi(\mathcal{B})}\to(T,B)_{\mathcal{B}}$ putting $K_{\mathcal{B}}(x,t)=H(x,t)$ for any $(x,t)\in Z$.

c) $K=(K_{\mathcal{B}},\ \Phi)$ is a morphism from $\widehat{Z,C}$ to $\widehat{T,B}$, which is equivalent to $\hat{H}$.

d) For each $\mathcal{B}\in$Cov(T,B) consider $\Phi(\mathcal{B})=\bigvee_{x\in S,A}\mathcal{R}_x$ and put $\phi(\mathcal{B})=\{A_x\}(x\in S,A)$. Clearly $\phi(\mathcal{B})\in$Cov(S,A). Afterwards, given $t\in I$, we obtain a precontinuous map $h^t_{\mathcal{B}}:(S,A)_{\phi(\mathcal{B})}\to(T,B)_{\mathcal{B}}$ putting $h^t_{\mathcal{B}}(x)=H(x,t)$ for any $x\in S$. Then $(h^t_{\mathcal{B}},\ \phi)$ is a morphism from $\widehat{S,A}$ to $\widehat{T,B}$, which is equivalent to the morphism $\hat{h}_t:\widehat{S,A}\to\widehat{T,B}$, where $h_t:(S,A)\to(T,B)$ is the precontinuous map given by $h_t(x)=H(x,t)$ $\forall x\in S$. The function $\phi$:Cov(T,B) $\to$Cov(S,A) does not depend on t. Finally $(h^0_{\mathcal{B}},\ \phi)$ is equivalent to $\hat{f}$ and $(h^1_{\mathcal{B}},\ \phi)$ is equivalent to $\hat{g}$.

The proofs of b), c), d) are analogous to the corresponding ones from Theorem 3.1.

## 5. Shape groups and relative shape groups.

Let us consider a pretopological space S and its inverse system $\hat{S}=(S_X,\ p_{XX'},\ \text{Cov}(S))$.

Let x be a point of S. For any $X\in$Cov(S) and each dimension n, we can calculate (see [2]) the prehomotopy group $\Pi_n(S_X,x)$ of $S_X$ based at x. Moreover, given $X\leq X'$ in Cov(S), the precontinuous map $p_{XX'}:S_{X'}\to S_X$ induces a homomorphism $p^*_{XX'}$ from $\Pi_n(S_{X'},x)$ to $\Pi_n(S_X,x)$. So, for each positive integer n, we obtain the inverse system $(\Pi_n(S_X,x),\ p^*_{XX'},\ \text{Cov}(S))$.

5.1 *Definition* We put $\check{\Pi}_n(S,x)=\varprojlim\Pi_n(S_X,x)$. The group $\check{\Pi}_n(S,x)$ will be called the n-dimensional shape group of the pretopological space S based at the point x. We will write $\check{\Pi}_n(S)$ instead of $\check{\Pi}_n(S,x)$, when $\check{\Pi}_n(S,x)$ does not depend on the point x of S.

5.2 *Remark.* If S={x}, clearly $\check{\Pi}_n(S)=0$ for each dimension n.

Now take a subset A of the pretopological space S, and consider the inverse system $\widehat{S,A}=((S,A)_A,\ p_{AA'},\ \text{Cov}(S,A))$ of the pair (S,A).

Let x be a point of A. For each dimension n, we can consider the inverse system $(\Pi_n(S,A,x)_A,\ p^*_{AA'},\ \text{Cov}(S,A))$ of relative prehomotopy groups. (Observe that $\Pi_n(S,A,x)_A$ denotes the n-dimensional relative prehomotopy group of the pair $(S,A)_A$ based at x).

5.3 *Definition* We put $\check{\Pi}_n(S,A,x)=\varprojlim\Pi_n(S,A,x)_A$. The group $\check{\Pi}_n(S,A,x)$ will be called the n-dimensional relative shape group of the pair (S,A) based at x. We will write $\check{\Pi}_n(S,A)$ instead of $\check{\Pi}_n(S,A,x)$, when $\check{\Pi}_n(S,A,x)$ does not depend on the point x of A.

## 6. Homomorphisms between shape groups.

Let S and T be pretopological spaces, $\hat{S}=(S_X, p_{XX'}, \mathrm{Cov}(S))$ and $\hat{T}=(T_Y, q_{YY'}, \mathrm{Cov}(T))$ their inverse systems, $f:S\to T$ a precontinuous map. Then consider the morphism $\hat{f}=(f_Y, f^{-1})$ from $\hat{S}$ to $\hat{T}$ induced by f.
For each dimension n, the precontinuous map $f_Y:S_{f^{-1}(Y)}\to T_Y$ induces a homomorphism $f^*_{Y,n}$ from $\Pi_n(S_{f^{-1}(Y)},x)$ to $\Pi_n(T_Y,f(x))$.

6.1 *Definition* We denote by $\check{f}_n:\check{\Pi}_n(S,x)\to\check{\Pi}_n(T,f(x))$ the homomorphism $\varprojlim f^*_{Y,n}$, and we say that it is induced by the precontinuous map $f:S\to T$.

6.2 *Remark*. Similarly, given two subsets A of S and B of T, and given a point x of A, for each dimension n we obtain the homomorphism $\check{f}_n:\check{\Pi}_n(S,A,x)\to\check{\Pi}_n(T,B,f(x))$ induced by a precontinuous map $f:(S,A)\to(T,B)$.

6.3 *Proposition* If $f:(S,A)\to(S,A)$ is the identity, then $\check{f}_n:\check{\Pi}_n(S,A,x)\to\check{\Pi}_n(S,A,x)$ is the identical isomorphism.

6.4 *Proposition* Let $f:(S,A)\to(T,B)$ and $g:(T,B)\to(Z,C)$ be precontinuous maps and $h=gf$. Then $\check{h}_n=\check{g}_n\check{f}_n$.

## 7. The homomorphisms $\check{\delta}_n:\check{\Pi}_n(S,A,x)\to\check{\Pi}_{n-1}(A,x)$, $\check{i}_n:\check{\Pi}_n(A,x)\to\check{\Pi}_n(S,x)$ and $\check{j}_n:\check{\Pi}_n(S,x)\to\check{\Pi}_n(S,A,x)$.

Let us take a pretopological space $S=(X,P)$, a subset A of X carrying the pretopology $P^*$ induced by $P$, and a point x of A. Then consider the following three functions.

1) $\psi:\mathrm{Cov}(A)\to\mathrm{Cov}(S,A)$ associates to $\{X_i\}(i\in J')$ the family $\{A_i\}(i\in J,J')$, where $J=J'\cup\{j\}$ (with $j\notin J'$), $A_j=X$, and $A_i=X_i\cup(X-A)$ for $i\in J'$.
2) $\overline{\psi}:\mathrm{Cov}(S)\to\mathrm{Cov}(A)$ associates to $\{A_i\}(i\in J)$ the family $\{A_i\cap A\}(i\in J)$.
3) $\tilde{\psi}:\mathrm{Cov}(S,A)\to\mathrm{Cov}(S)$ associates to $\{A_i\}(i\in J,J')$ the family $\{A_i\}(i\in J)$.

For any $R\in\mathrm{Cov}(A)$ we can define a boundary homomorphism $\delta^*_{R,n}$ from $\Pi_n(S,A,x)_{\psi(R)}$ to $\Pi_{n-1}(A_R,x)$ in the usual way (see [2]). It is easy to prove that $(\delta^*_{R,n}, \psi)$ is a morphism from $(\Pi_n(S,A,x)_A, p^*_{AA'}, \mathrm{Cov}(S,A))$ to $(\Pi_{n-1}(A_R,x), p^*_{RR'}, \mathrm{Cov}(A))$.

Afterwards, considering the usual homomorphisms $i^*_{X,n}:\Pi_n(A_{\overline{\psi}(X)},x)\to\Pi_n(S_X,x)$ and $j^*_{A,n}:\Pi_n(S_{\tilde{\psi}(A)},x)\to\Pi_n(S,A,x)_A$, we obtain the morphisms $(i^*_{X,n}, \overline{\psi})$ from $(\Pi_n(A_R,x), p^*_{RR'}, \mathrm{Cov}(A))$ to $(\Pi_n(S_X,x), p^*_{XX'}, \mathrm{Cov}(S))$ and $(j^*_{A,n}, \tilde{\psi})$ from $(\Pi_n(S_X,x), p^*_{XX'}, \mathrm{Cov}(S))$ to $(\Pi_n(S,A,x)_A, p^*_{AA'}, \mathrm{Cov}(S,A))$.

7.1 *Definition* We put $\check{\delta}_n=\varprojlim \delta^*_{R,n}$, $\check{i}_n=\varprojlim i^*_{X,n}$, $\check{j}_n=\varprojlim j^*_{A,n}$.

With a standard proof we obtain:

7.2 *Proposition* Let $f:(S,A)\to(T,B)$ be a precontinuous map, $g:A\to B$ the restriction of f to A, $x\in A$, $y=f(x)$. For each dimension n, the following diagram commutes:

$$\begin{array}{ccc} \check{\Pi}_n(S,A,x) & \xrightarrow{\check{f}_n} & \check{\Pi}_n(T,B,y) \\ \downarrow \check{\delta}_n & & \downarrow \check{\delta}_n \\ \check{\Pi}_{n-1}(A,x) & \xrightarrow{\check{g}_n} & \check{\Pi}_{n-1}(B,y) \end{array}$$

7.3 *Proposition* Let S be a pretopological space, $A\subseteq S$, and $x\in A$. We obtain the following 0-sequence:

$$\ldots \xrightarrow{\check{\delta}_{n+1}} \Pi_n(A,x) \xrightarrow{\check{i}_n} \Pi_n(S,x) \xrightarrow{\check{j}_n} \Pi_n(S,A,x) \xrightarrow{\check{\delta}_n} \Pi_{n-1}(A,x) \xrightarrow{\check{i}_{n-1}} \ldots$$

8. The homotopy condition for shape groups.

To prove the homotopy condition (i.e. Theorem 8.3), we need a definition and a lemma.

8.1 *Definition* Let X and Y be sets, $\mathcal{P}$ a partition of X, $f:X\to Y$ a function. We say that f is quasiconstant with respect to $\mathcal{P}$, if f is constant in each element of $\mathcal{P}$.

8.2 *Lemma* Let $h:I^n\to X$ be a precontinuous map from the unit n-cube $I^n$ to a symmetrical pf-space X. It is possible to find a finite partition $\mathcal{P}$ of $I^n$ in open cells (of dimensions n, n-1,..., 0) and a precontinuous map $k:I^n\to X$, such that:

(i) k is quasiconstant with respect to $\mathcal{P}$ and homotopic to h;

(ii) moreover, if h is a n-preloop of X based at a, then also k is n-preloop of X based at a.

*Proof:* Let $\{F_x\}(x\in X)$ be the pretopology of X.

a) First we consider the case n=1.

Since $h:I\to X$ is precontinuous, for each $z\in I$ there is an open interval $V_z$ of I such that $h(V_z)\subseteq F_{h(z)}$.

Since I is compact, we find a finite number m of points $z_i$ of I, such that $\{V_{z_i}\}(1\leq i\leq m)$ is a minimal linked covering of I, where $z_1=0$, $z_i<z_j$ for $i<j$, $z_m=1$. Then we take $y_0=0$, $y_m=1$, and for each positive integer $i<m$ we choose a point $y_i\in V_{z_i}\cap V_{z_{i+1}}$. Afterwards we consider the partition

$$\mathcal{P}=\{[0,y_1[,\ \{y_1\},\ ]y_1,y_2[,\ \ldots,\ \{y_{m-1}\},\ ]y_{m-1},1]\}$$

of I, and we define a precontinuous map $k:I\to X$ putting:

$$k(y_i) = h(y_i) \qquad \text{for } i=0,\ 1,\ \ldots,\ m;$$
$$k(]y_i,y_{i+1}[) = \{h(z_{i+1})\} \qquad \text{for } 0\leq i<m.$$

Then we obtain a prehomotopy K of k to h, putting:

$$K(z,t) = \begin{cases} k(z) & \text{if } 0\leq t\leq \frac{1}{2} \\ h(z) & \text{if } \frac{1}{2}<t\leq 1 \end{cases}$$

Moreover, if h is a preloop based at a, also k is a preloop of X based at a, since $k(0)=h(0)=a$ and $k(1)=h(1)=a$.

b) Now we consider the case n>1, assuming that the lemma is true for n-1.

To use simple notations, given a point $w=(z_1,z_2,\ldots,z_n)$ of $I^n$, we put $(z_1,\ldots,z_{n-1})=z$, $z_n=u$, $(z,u)=w$.

For each $u\in I$, the function $h_u:I^{n-1}\to X$ given by $h_u(z)=h(z,u)$ is a precontinuous map. So we find a finite partition $\mathcal{P}_u$ of $I^{n-1}$ in open cells (of dimensions n-1, n-2, ..., 0) and precontinuous map $k_u:I^{n-1}\to X$ for which conditions (i) and (ii) hold.

Now take a point $u\in I$. For any cell Z of the partition $\mathcal{P}_u$, the image $\{k_u(Z)\}$ is a point of X. For each $Z\in\mathcal{P}_u$ there exists a point $z\in Z$, such that $\{k_u(Z)\}=\{h_u(z)\}$ and moreover z has an open neighbourhood $V_z$ which contains the closure $\overline{Z}$ of Z. Then, since $\overline{Z}$ is compact, we find an open interval $W_{u,Z}$ of I containing u and such that $h(V_z\times W_{u,Z})\subseteq F_{h(z,u)}$. Put $W_u=\bigcap_{Z\in\mathcal{P}_u} W_{u,Z}$.

Since I is compact, we find a finite number m of points $u_i\in I$ such that $\{W_{u_i}\}(1\leq i\leq m)$ is a minimal linked covering of I, where $u_1=0$, $u_i<u_j$ for $i<j$, $u_m=1$. Then we take $v_0=0$, $v_m=1$, and we choose $v_i\in W_{u_i}\cap W_{u_{i+1}}$ for each positive integer $i<m$. Afterwards we consider the following partition of I:

$$\mathcal{P}_n = \{[0,v_1[,\ \{v_1\},\ ]v_1,v_2[,\ \dots,\ \{v_{m-1}\},\ ]v_{m-1},1]\}.$$

By means of $\mathcal{P}_n$ and by means of the partitions $\mathcal{P}_{u_i}$ $(1\leq i\leq m)$ and $\mathcal{P}_{v_i}$ $(1\leq i<m)$ of $I^{n-1}$, we obtain a finite partition $\mathcal{P}$ of $I^n$ in open cells of dimensions n, n-1, ..., 0. We define a function $k:I^n\to X$, which is quasiconstant with respect to $\mathcal{P}$ and precontinuous, putting:

$$\begin{aligned} k(z,v_i) &= k_{v_i}(z) && \text{for } i=0,1,\dots,m;\\ k(z,u) &= k_{u_{i+1}}(z) && \text{for } u\in]v_i,v_{i+1}[ \text{ and } 0\leq i<m. \end{aligned}$$

We obtain a prehomotopy K of k to h, putting:

$$K(w,t) = \begin{cases} k(w) & \text{if } 0\leq t\leq \frac{1}{2}\\ h(w) & \text{if } \frac{1}{2}<t\leq 1 \end{cases}$$

Moreover, if h is a n-preloop of X based at a, clearly also k is a n-preloop of X based at a.

8.3 *Theorem* Let S and T be pretopological spaces, $a\in S$, $b\in T$, and let $f:(S,a)\to(T,b)$ and $g:(S,a)\to(T,b)$ be precontinuous maps. If f and g are homotopic, then $\check{f}_n=\check{g}_n$ for each dimension n.

*Proof:* Let $H:(S\times I,\{a\}\times I)\to(T,b)$ be a prehomotopy of f to g.

a) Given $Y\in\mathrm{Cov}(T)$, consider the elements $\Phi(Y)\in\mathrm{Cov}(S\times I)$ and $\phi(Y)\in\mathrm{Cov}(S)$ we mentioned in 3, and recall that both $f^{-1}(Y)\leq\phi(Y)$ and $g^{-1}(Y)\leq\phi(Y)$.

The theorem is proved if the following diagram commutes:

$$\begin{tikzcd}[column sep=small] & \Pi_n(S_{\phi(Y)},a) \arrow[dl, "p^*_{f^{-1}(Y)\phi(Y)}"'] \arrow[dr, "p^*_{g^{-1}(Y)\phi(Y)}"] & \\ \Pi_n(S_{f^{-1}(Y)},a) \arrow[dr, "f^*_{Y,n}"'] & & \Pi_n(S_{g^{-1}(Y)},a) \arrow[dl, "g^*_{Y,n}"] \\ & \Pi_n(T_Y,b) & \end{tikzcd}$$

b) Observe that $S_{\phi(Y)}$ is a symmetrical pf-space. Hence, by Lemma 8.2, in each class [h] of $\Pi_n(S_{\phi(Y)},a)$ we find a n-preloop k, which is quasiconstant with respect to a suitable finite partition $\mathcal{P}$ of $I^n$ in open cells.

c) The map $k\times 1_I:I^n\times I\to(S\times I)_{\Phi(Y)}$ is precontinuous.

In fact, let $(w,t)\in I^n\times I$. Since the map $k:I^n\to S_{\phi(Y)}$ is precontinuous, there is a neighbourhood $U_w$ of w such that $k(U_w) \subseteq St(k(w),\phi(Y))$. But $k(U_w)$ is a finite subset of $S_{\phi(Y)}$. Put $k(U_w)=\{x_1,\dots,x_m\}$, and take a positive integer $r\leq m$. The point $x_r$ belongs to the element $A_{x_r}$ of $\phi(Y)$. Moreover, for each $t\in I$ we find a point $t_r\in I$ and an open neighbourhood $V_{t_r}^{x_r,t_r}$ of $t_r$ such that $A_{x_r}\times V_{t_r}^{x_r,t_r}\ \Phi(Y)$. Then $V_t = \bigcap_{1\leq r\leq m} V_{t_r}^{x_r,t_r}$ is a neighbourhood of t, such that $k(U_w)\times V_t \subseteq St((k(w),t),\Phi(Y))$.

d) The prehomotopy H of f to g is a precontinuous map from $(S\times I)_{\Phi(Y)}$ to $T_Y$. By c), $K = H(k\times 1_I)$ is a precontinuous map from $I^n\times I$ to T. Moreover K is a prehomotopy of fk to gk.

Therefore the foregoing diagram commutes, because $f^*_{Y,n} p^*_{f^{-1}(Y)\phi(Y)}([h]) = [fk]$ and $g^*_{Y,n} p^*_{g^{-1}(Y)\phi(Y)}([h]) = [gk]$.

## 9. Čech homology groups.

Let us consider a pretopological space $S=(X,P)$ and its inverse system $\hat{S} = (S_X,\ p_{XX'},\ Cov(S))$.

For any $X\in Cov(S)$ and each dimension n, we can calculate (see [2]) the singular homology group $H_n(S_X)$ of $S_X$. Moreover, given $X\leq X'$, the precontinuous map $p_{XX'}:S_{X'}\to S_X$ induces a homomorphism $p_*^{XX'}:H_n(S_{X'})\to H_n(S_X)$ for each dimension n. So, for each integer $n\geq 0$, we obtain the inverse system $(H_n(S_X),\ p_*^{XX'},\ Cov(S))$.

9.1 *Definition* We put $\check{H}_n(S) = \varprojlim H_n(S_X)$. The group $\check{H}_n(S)$ will be called the n-dimensional Čech homology group of the pretopological space S.

9.2 *Remark.* Clearly, if $S=\{x\}$, $\check{H}_n(S)=0$ for $n>0$, and $\check{H}_0(S)=Z$.

Now let A be a subset of S and $\widehat{S,A} = ((S,A)_A,\ p_*^{AA'},\ Cov(S,A))$ the inverse system of the pair (S,A). For each dimension n, we can consider the inverse system $(H_n(S,A)_A,\ p_*^{AA'},\ Cov(S,A))$, where $H_n(S,A)_A$ is the n-dimensional relative singular homology group of the pair $(S,A)_A$.

9.3 *Definition* We put $\check{H}_n(S,A) = \varprojlim H_n(S,A)_A$. The group $\check{H}_n(S,A)$ will be called the n-dimensional relative Čech homology group of the pair (S,A).

## 10. Homomorphisms between Čech homology groups.

Let S and T be pretopological spaces, $\hat{S}=(S_X,\ p_{XX'},\ Cov(S))$ and $\hat{T} = (T_Y,\ q_{YY'},\ Cov(T))$ their inverse systems, $f:S\to T$ a precontinuous map, and $\hat{f} = (f_Y,f^{-1})$ the morphism from $\hat{S}$ to $\hat{T}$ induced by f.

For each dimension n, the precontinuous map $f_Y:S_{f^{-1}(Y)}\to T_Y$ induces a homomorphism $f_*^{Y,n}$ from $H_n(S_{f^{-1}(Y)})$ to $H_n(T_Y)$.

10.1 *Definition* We denote by $\check{f}_n:\check{H}_n(S)\to\check{H}_n(T)$ the homomorphism $\varprojlim f_*^{Y,n}$, and we say that it is induced by the precontinuous map $f:S\to T$.

10.2 *Remark.* Similarly, given two subsets A of S and B of T, for each dimension n we obtain the homomorphism $\check{f}_n:\check{H}_n(S,A)\to\check{H}_n(T,B)$.

10.3 *Proposition* If $f:(S,A)\to(S,A)$ is the identity, then $\check{f}_n:\check{H}_n(S,A)\to\check{H}_n(S,A)$ is the identical isomorphism.

10.4 *Proposition* Let $f:(S,A)\to(T,B)$ and $g:(T,B)\to(Z,C)$ be precontinuous maps and $h=gf$. Then $\check{h}_n=\check{g}_n\check{f}_n$.

10.5 *Proposition (Excision Theorem)* Let A and U be nonempty subsets of a pretopological space S, such that $cl(U)\subseteq int(A)$. Then the canonical injection $f:(S-U,\ A-U)\to(S,A)$ induces an isomorphism $\check{f}_n:\check{H}_n(S-U,\ A-U)\to\check{H}_n(S,A)$.

*Proof:* In fact we have:

(i) Let $\mathcal{A}=\{A_i\}(i\in J,J')$ be an element of $Cov(S,A)$ such that $P(\mathcal{A}_J)$ induces in A the pretopology $P(\mathcal{A}_{J'})$. Then (see [2]) the injection $f:(S-U,\ A-U)\to(S,A)$ induces an isomorphism $f_*^n:H_n(S-U,\ A-U)\to H_n(S,A)$.

(ii) Let $\mathcal{A}=\{A_i\}(i\in J,J')\in Cov(S,A)$. Then $\bar{\mathcal{A}}=\{A_i\}(i\ J,J^*)$ (where $J^*=\{i\in J/A_i\cap A\neq\emptyset\}$) is such that $P(\bar{\mathcal{A}}_J)$ induces in A the pretopology $P(\bar{\mathcal{A}}_{J^*})$. Moreover $\bar{\mathcal{A}}\leq\mathcal{A}$.

## 11. The homomorphisms $\check{\partial}_n:\check{H}_n(S,A)\to\check{H}_{n-1}(A)$, $\check{i}_n:\check{H}_n(A)\to\check{H}_n(S)$, $\check{j}_n:\check{H}_n(S)\to\check{H}_n(S,A)$.

Now consider a subspace A of a pretopological space S and the functions $\psi:Cov(A)\to Cov(S,A)$, $\bar{\psi}:Cov(S)\to Cov(A)$, $\tilde{\psi}:Cov(S,A)\to Cov(S)$ we mentioned in 7. For any $\mathcal{R}\in Cov(A)$ we can define a boundary homomorphism $\partial_*^{\mathcal{R},n}$ from $H_n(S,A)_{\psi(\mathcal{R})}$ to $H_{n-1}(A_{\mathcal{R}})$ in the usual way (see [2]), and $(\partial_*^{\mathcal{R},n},\ \psi)$ is a morphism from $(H_n(S,A)_{\mathcal{A}},\ p_*^{\mathcal{A}\mathcal{A}'},\ Cov(S,A))$ to $(H_{n-1}(A_{\mathcal{R}}),\ p_*^{\mathcal{R}\mathcal{R}'},\ Cov(A))$.

Afterwards, considering the usual homomorphisms $i_*^{\mathcal{X},n}:H_n(A_{\bar{\psi}(\mathcal{X})})\to H_n(S_{\mathcal{X}})$ and $j_*^{\mathcal{A},n}:H_n(S_{\tilde{\psi}(\mathcal{A})})\to H_n(S,A)_{\mathcal{A}}$, we obtain the morphisms $(i_*^{\mathcal{X},n},\ \bar{\psi})$ from $(H_n(A_{\mathcal{R}}),\ p_*^{\mathcal{R}\mathcal{R}'},\ Cov(A))$ to $(H_n(S_{\mathcal{X}}),\ p_*^{\mathcal{X}\mathcal{X}'},\ Cov(S))$ and $(j_*^{\mathcal{A},n},\ \tilde{\psi})$ from $(H_n(S_{\mathcal{X}}),\ p_*^{\mathcal{X}\mathcal{X}'},\ Cov(S))$ to $(H_n(S,A)_{\mathcal{A}},\ p_*^{\mathcal{A}\mathcal{A}'},\ Cov(S,A))$.

11.1 *Definition* We put $\check{\partial}_n=\varprojlim\partial_*^{\mathcal{R},n}$, $\check{i}_n=\varprojlim i_*^{\mathcal{X},n}$, $\check{j}_n=\varprojlim j_*^{\mathcal{A},n}$.

With a standard proof, we obtain:

11.2 *Proposition* Let $f:(S,A)\to(T,B)$ be a precontinuous map and $g:A\to B$ the restriction of f to A. For each dimension n the following diagram commutes:

$$\begin{array}{ccc}
\check{H}_n(S,A) & \xrightarrow{\check{f}_n} & \check{H}_n(T,B)\\
\downarrow{\scriptstyle\check{\partial}_n} & & \downarrow{\scriptstyle\check{\partial}_n}\\
\check{H}_{n-1}(A) & \xrightarrow{\check{g}_n} & \check{H}_{n-1}(A)
\end{array}$$

11.3 *Proposition* Let S be a pretopological space and $A\subseteq S$. We obtain the following 0-sequence:

$$\cdots\xrightarrow{\check{\partial}_{n+1}}\check{H}_n(A)\xrightarrow{\check{i}_n}\check{H}_n(S)\xrightarrow{\check{j}_n}\check{H}_n(S,A)\xrightarrow{\check{\partial}_n}\check{H}_{n-1}(A)\xrightarrow{\check{i}_{n-1}}\cdots$$

## 12. The homotopy condition for Čech homology groups.

To prove the homotopy condition (i.e. Theorem 12.5), we need some previous statements.

Let $\Delta_p=[a_0a_1\ldots a_p]$ be the standard p-simplex, and let $i_1,\ i_2,\ \ldots,\ i_n$ be integers such that $0\le i_1<i_2<\ldots<i_n\le p$ where $1\le n\le p$. Given a singular p-simplex $\sigma^\lambda$ on a pretopological space X, we denote by $\sigma^\lambda_{i_1\ldots i_n}$ the singular (n-1)-simplex $\sigma^\lambda(a_{i_1}\ldots a_{i_n})$ product of $\sigma^\lambda:\Delta_p\to X$ and of the singular (n-1)-simplex $(a_{i_1}\ldots a_{i_n})$ on $\Delta_p$. Moreover, given a function $F^\lambda:\Delta_p\times I\to X$, we will denote by $F^\lambda_{i_1\ldots i_n}$ the function $F^\lambda((a_{i_1}\ldots a_{i_n})\times 1_I):\Delta_p\times I\to X$. We will wright $\sigma^\lambda_{\hat{i}}$ and $F^\lambda_{\hat{i}}$ instead of $\sigma^\lambda_{0\ldots\hat{i}\ldots p}$ and $F^\lambda_{0\ldots\hat{i}\ldots p}$.

12.1 *Definition* Let $\alpha = \Sigma\alpha_\lambda\sigma^\lambda$ and $\beta = \Sigma\alpha_\lambda\tau^\lambda$ be singular p-chains on a pretopological space X. We say that $\alpha$ is homotopic to $\beta$, if:

(1) for each $\lambda$ with $\alpha_\lambda\neq 0$, there is a prehomotopy $F^\lambda$ of $\sigma^\lambda$ to $\tau^\lambda$;

(2) if $\sigma^\lambda_{\hat{i}} = \tau^\lambda_{\hat{j}}$, then $F^\lambda_{\hat{i}} = F^\mu_{\hat{j}}$.

With a process which is similar to the one of the classical case (see for example |5|), it is possible to prove the following:

12.2 *Proposition* Let X be a pretopological space, $\alpha = \Sigma\alpha_\lambda\sigma^\lambda$ a p-cycle on X and $\beta = \Sigma\alpha_\lambda\tau^\lambda$ a p-chain on X. If $\beta$ is homotopic to $\alpha$, then also $\beta$ is a p-cycle on X; moreover $\alpha$ and $\beta$ are homologous.

12.3 *Lemma* Let X be a symmetrical pf-space and $\alpha = \Sigma\alpha_\lambda\sigma^\lambda$ a singular p-chain on X. For each $\lambda$ such that $\alpha_\lambda\neq 0$, we find a finite partition $\mathcal{C}_\lambda$ of $\Delta_p$ in open cells and a singular p-simplex $\tau^\lambda$ on X such that:

(i) $\tau^\lambda$ is quasiconstant with respect to $\mathcal{C}_\lambda$;

(ii) $\Sigma\alpha_\lambda\tau^\lambda$ is homotopic to $\Sigma\alpha_\lambda\sigma^\lambda$.

*Proof:* Let us consider successively the faces of all simplices $\sigma^\lambda$ of dimensions 0, 1, ..., p and let us define the corresponding faces of the simplices $\tau^\lambda$. For any 0-dimensional face $\sigma^\lambda_{i_1}$ we take $\tau^\lambda_{i_1} = \sigma^\lambda_{i_1}$.

Now let $\sigma^\lambda_{i_1\ldots i_n}$ be a n-dimensional face of a simplex $\sigma^\lambda$. Assume that all simplices $\tau^\lambda_{i_1\ldots i_m}$ $(0<m<n)$ and the prehomotopies $F^\lambda_{i_1\ldots i_m}$ of $\tau^\lambda_{i_1\ldots i_m}$ to $\sigma^\lambda_{i_1\ldots i_m}$ has been defined, in a way such that:

if $\sigma^\lambda_{i_1\ldots i_m} = \sigma^\mu_{j_1\ldots j_m}$ for some $\lambda$, $\mu$, then $\tau^\lambda_{i_1\ldots i_m} = \tau^\mu_{j_1\ldots j_m}$ and $F^\lambda_{i_1\ldots i_m} = F^\mu_{j_1\ldots j_m}$.

We observe that we can consider $\Delta_n$ as the nth cone $C^n(a_0)$ on $\{a_0\}$, and we denote by $\pi^n$ the projection from $I^n$ to $\Delta_n=C^n(a_0)$. The product function $\sigma^\lambda_{i_1\ldots i_n}\pi^n$ is a precontinuous map $f^\lambda_{i_1\ldots i_n}:I^n\to X$.

So we have to construct a precontinuous map $g^\lambda_{i_1\ldots i_n}:I^n\to X$ which must be homotopic to $f^\lambda_{i_1\ldots i_n}$ and quasiconstant with respect to a suitable finite partition $\mathcal{C}^\lambda_{i_1\ldots i_n}$ of $I^n$ in open cells. To obtain the map $g^\lambda_{i_1\ldots i_n}$ and the prehomotopy $H^\lambda_{i_1\ldots i_n}$ of $g^\lambda_{i_1\ldots i_n}$ to $f^\lambda_{i_1\ldots i_n}$, we follow a process similar to the one of Lemma 8.2; but now we have to recall that $H^\lambda_{i_1\ldots i_n}$ is already determined in $\dot{I}^n\times I$ by the inductive hypothesis.

Finally we observe that the function $H^{\lambda}_{i_1\ldots i_n}$ and its restriction $g^{\lambda}_{i_1\ldots i_n}$ to $I^n\times\{0\}$ are relation preserving. So the functions $F^{\lambda}_{i_1\ldots i_n}$ and $\tau^{\lambda}_{i_1\ldots i_n}$ are given by the following commutative diagrams:

$$\begin{tikzcd}
I^n\times I \arrow[r, "H^{\lambda}_{i_1\ldots i_n}"] \arrow[d, "\pi^n\times 1_I"'] & X \\
\Delta_n\times I \arrow[ur, "F^{\lambda}_{i_1\ldots i_n}"']
\end{tikzcd}$$

$$\begin{tikzcd}
I^n\times\{0\} \arrow[r, "g^{\lambda}_{i_1\ldots i_n}"] \arrow[d, "\pi^n"'] & X \\
\Delta_n \arrow[ur, "\tau^{\lambda}_{i_1\ldots i_n}"']
\end{tikzcd}$$

12.4 *Remark.* Now let $(X,A)$ be a pair of pretopological spaces. Since generally $A$ carries a pretopology finer than the one induced by $X$, we have to add to Definition 12.1 the following condition:

(3) if $\sigma^{\lambda}_{i_1\ldots i_n}$ is a singular $(n-1)$-simplex on $A$, then $\tau^{\lambda}_{i_1\ldots i_n}$ and $F^{\lambda}_{i_1\ldots i_n}$ must be precontinuous maps into $A$.

12.5 *Theorem* Let $S$ and $T$ be pretopological spaces, $A\subseteq S$, $B\subseteq T$, and let $f:(S,A)\to(T,B)$ and $g:(S,A)\to(T,B)$ be precontinuous maps. If $f$ and $g$ are homotopic, then $\check{f}_n=\check{g}_n$ for each dimension $n$.

*Proof:* Let $H:(S\times I, A\times I)\to(T,B)$ be a prehomotopy of $f$ to $g$. Given $\mathcal{B}\in\mathrm{Cov}(T,B)$, consider the elements $\Phi(\mathcal{B})\in\mathrm{Cov}(S\times I, A\times I)$ and $\phi(\mathcal{B})\in\mathrm{Cov}(S,A)$ from Theorem 4.8, and recall that $\phi(\mathcal{B})$ refines both $f^{-1}(\mathcal{B})$ and $g^{-1}(\mathcal{B})$.

Then consider the following diagram:

$$\begin{tikzcd}
 & H_n(S,A)_{\phi(\mathcal{B})} \arrow[dl, "p_*^{f^{-1}(\mathcal{B})\phi(\mathcal{B})}"'] \arrow[dr, "p_*^{g^{-1}(\mathcal{B})\phi(\mathcal{B})}"] & \\
H_n(S,A)_{f^{-1}(\mathcal{B})} \arrow[dr, "f_*^{\mathcal{B}}"'] & & H_n(S,A)_{g^{-1}(\mathcal{B})} \arrow[dl, "g_*^{\mathcal{B}}"] \\
 & H_n(T,B)_{\mathcal{B}} &
\end{tikzcd}$$

Let $[\alpha]\in H_n(S,A)_{\phi(\mathcal{B})}$ and $\alpha=\Sigma\alpha_{\lambda}\sigma^{\lambda}$. By Lemma 12.3 and Remark 12.4, we construct a n-chain $\beta=\Sigma\alpha_{\lambda}\tau^{\lambda}$ such that:

i) $\beta$ is a linear combination of a finite number of n-simplices that are quasi-constant with respect to a suitable finite partition of $\Delta_n$;

ii) $\beta$ is homotopic to $\alpha$.

Therefore $\beta\in[\alpha]$ since, by Proposition 12.2 and Remark 12.4, $\beta$ is a relative cycle homologous to $\alpha$. Now consider the chains $f\beta=\Sigma\alpha_{\lambda}(f\tau^{\lambda})$ and $g\beta=\Sigma\alpha_{\lambda}(g\tau^{\lambda})$, and observe that $f_*^{\mathcal{B}}p_*^{f^{-1}(\mathcal{B})\phi(\mathcal{B})}([\alpha])=[f\beta]$ and $g_*^{\mathcal{B}}p_*^{g^{-1}(\mathcal{B})\phi(\mathcal{B})}([\alpha])=[g\beta]$.

With a proof analogous to the one of Theorem 8.3, we see that $K^{\lambda}=H(\tau^{\lambda}\times 1_I)$ is a prehomotopy of $f\tau^{\lambda}$ to $g\tau^{\lambda}$; moreover $f\beta$ and $g\beta$ are homotopic chains. Hence $[f\beta]=$ $=[g\beta]$.

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