Richard Delanghe

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DECOMPOSABLE SYSTEMS OF DIFFERENTIAL OPERATORS AND GENERALIZED INVERSES

by R. Delanghe

0. Introduction

In his paper [4], M.R. Hestenes showed that each closed linear operator $L:H\to H'$, H and H' being Hilbert spaces, admits a generalized inverse $L^{-1}:H'\to H$ and he developed a "spectral theory" for such operators. As an example he considered the gradient operator which satisfies the relation $-\Delta=(-\mathrm{div})\mathrm{grad}$. In [3] H.G. Garnir built up a framework for studying abstract Dirichlet-Neumann problems for decomposable systems of differential operators with constant coefficients i.e. operators $L(\partial/\partial x)$ of the form $L(\partial/\partial x)=L^+(-\partial/\partial x)L(\partial/\partial x)$ where $L(\partial/\partial x)$ is a matrix differential operator. In this paper we combine the results of the cited authors in the case where the (D-N)-problem for the operators under consideration is well-posed. In particular, a spectral decomposition is obtained for the operator L which factorizes L and for its generalized inverse L^{-1}.

1. Generalized inverses

Let H,H' be Hilbert spaces and let $L:H\to H'$ be a closed densely defined linear operator with domain $\mathrm{dom}(L)$, kernel $\eta(L)$ and range $R(L)$. Then the generalized inverse L^{-1} of L is defined as follows. Call $C(L)=\mathrm{dom}(L)\cap\eta(L)^{\perp}$; then $\mathrm{dom}(L)=C(L)\oplus\eta(L)$ whence for each $v\in\mathrm{dom}L$, $v=\hat{v}+v_0$ with $\hat{v}\in C(L)$, $v_0\in\eta(L)$. As $L|C(L)$ is injective and $R(L|C(L))=R(L)$, the inverse $\tilde{L}$ of L with $\mathrm{dom}(\tilde{L})=R(L)$ and $R(\tilde{L})=C(L)$, may be extended to the linear operator $L^{-1}:H'\to H$ defined by

(i) $\mathrm{dom}(L^{-1})=R(L)\oplus R(L)^{\perp}$

(ii) If $w\in\mathrm{dom}(L^{-1})$ with $w=\hat{w}+w_0$, $\hat{w}\in R(L)$, $w_0\in R(L)^{\perp}$, then $L^{-1}w=\tilde{\hat{w}}=\hat{v}$ if and only if $(L|C(L))\hat{v}=\hat{w}$.

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From (i) and (ii) it follows that $R(L^{-1})=C(L)$. L^{-1} is called the generalized inverse of L (also called pseudo-inverse or generalized reciprocal of L).

Among other properties we mention (see [4], [5] and [6])

(i) $L^{-1}:H'\to H$ is a closed densely defined linear operator

(ii) $(L^{-1})^{-1}=L$

(iii) $(L^{-1})^{*}=(L^{*})^{-1}$

2. Decomposable differential operators

In this section we first recall the abstract setting for studying the Dirichlet-Neumann problem posed for a decomposable system of differential operators $L(\partial/\partial x=L^{+}(-\partial/\partial x)L(\partial/\partial x)$ as it was worked out in [3]. As an example we give the case of the negative Laplacian which is decomposed by its "square root" the Dirac operator.

In the second subsection we derive spectral decompositions of the operators L and L^{-1} in the case where the (D-N)-problem is well-posed for L.

2.1. The (D-N)-problem for decomposable operators

Let Ω be an open subset of R^m, let $N\in N$ ($N\geq 1$) and let $L_{2,N}(\Omega)$ be the Hilbert space of $C^{N\times 1}$-valued L_2-functions in Ω, i.e. $\vec{u}\in L_{2,N}(\Omega)$ if

$$\vec{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_N\end{bmatrix}\qquad\text{with } u_j\in L_2(\Omega),\ j=1,\dots,N.$$

The inner product and norm on $L_{2,N}(\Omega)$ are defined by

$$\langle\vec{u},\vec{v}\rangle_N=\int_\Omega \vec{u}\times\bar{\vec{v}}dx=\sum_{j=1}^{N}\int_\Omega u_j(x)\bar{v}_j(x)dx,$$

$$\|\vec{u}\|_N^2=\sum_{j=1}^{N}\int_\Omega|u_j(x)|^2dx.$$

Furthermore, let $L=L(\partial/\partial x)$ be an MxN matrix such that its elements L_{ij} are linear partial differential operators with constant coefficients and put

$$L=L(\partial/\partial x)=L^{+}(-\partial/\partial x)L(\partial/\partial x)$$

where $L^{+}=L^{+}(-\partial/\partial x)$ is obtained by taking the adjoint of $L(\partial/\partial x)$ and replacing $\partial/\partial x_j$ by $-\partial/\partial x_j$, $j=1,\dots,m$.

In general, if $L^{loc}_{1,N}(\Omega)$ and $\mathcal{D}(\Omega;\mathbb{C}^{N\times 1})$ denote respectively the space of $\mathbb{C}^{N\times 1}$-valued locally integrable functions in Ω and the space of $\mathbb{C}^{N\times 1}$-valued testfunctions in Ω, then the action of an MxN matrix differential operator $P(\partial/\partial x)$ having constant coefficients on $\vec{u}\in L^{loc}_{1,N}(\Omega)$ is defined to be element $P\vec{u}\in L^{loc}_{1,M}(\Omega)$, provided that it exists, such that for all $\vec{\varphi}\in\mathcal{D}(\Omega;\mathbb{C}^{M\times 1})$

$$\int_\Omega P(\partial/\partial x)\vec{u}\times\vec{\varphi}\,dx=\int_\Omega \vec{u}\times P^{+}(-\partial\ \partial x)\vec{\varphi}\,dx.$$

Returning to the decomposable differential operator $L=L^{+}L$, put

$$Z_{1,L}=\{\vec{u}\in L_{2,N}(\Omega):L\vec{u}\in L_{2,M}(\Omega)\}$$

and equip this space with the inner product

$$\langle\vec{u},\vec{v}\rangle=\langle\vec{u},\vec{v}\rangle_N+\langle L\vec{u},L\vec{v}\rangle_M.$$

Then $Z_{1,L}$ is a Hilbert space containing $\mathcal{D}(\Omega;\mathbb{C}^{N\times 1})$.
Furthermore let $\dot{\Omega}$ be the boundary of Ω and let $\dot{\Omega}_D$ and $\dot{\Omega}_N$ be two subsets of $\dot{\Omega}$ such that $\dot{\Omega}=\dot{\Omega}_D\cup\dot{\Omega}_N$ and $\dot{\Omega}_D\cap\dot{\Omega}_N=\phi$. Then $V_{\dot{\Omega}_D}$ stands for the closure in $Z_{1,L}$ of the set of functions $\vec{u}\in Z_{1,L}$ such that $\vec{u}$ is identically zero in a neighbourhood of $\dot{\Omega}_D$, this neighbourhood defending upon $\vec{u}$.
Finally define the subspace N of $V_{\dot{\Omega}_D}$ as follows :
$\vec{u}\in N$ if and only if

(N_1) $\vec{u}\in L_{2,N}(\Omega)$, $L\vec{u}\in L_{2,M}(\Omega)$, $\mathrm{L}\vec{u}\in L_{2,N}(\Omega)$

(N_2) (Dirichlet condition on $\dot{\Omega}_D$) $\vec{u}\in V_{\dot{\Omega}_D}$

(N_3) (Neumann condition on $\dot{\Omega}_N$)

$$\langle \mathrm{L}\vec{u},\vec{v}\rangle_N=\langle L\vec{u},L\vec{v}\rangle_M \text{ for all } \vec{v}\in V_{\dot{\Omega}_D} .$$

Taking N=dom(L), then clearly $\mathcal{D}(\Omega;\mathbb{C}^{N\times 1})$ is contained in N. Moreover L is a non-negative self-adjoint operator and its domain N is dense in $V_{\dot{\Omega}_D}$ for the $Z_{1,L}$-norm (see [3]).

Taking $V_{\dot{\Omega}_D}$ =dom(L) we have

2.1.1. <u>Theorem</u> (i) $\mathcal{L}$ is a closed densely defined linear operator
(ii) $L=\mathcal{L}^*\mathcal{L}$
(iii) $\mathcal{L}^*$ is a closed extension of $\mathcal{L}^+$.

<u>Proof</u>. (i) As $\mathcal{D}(\Omega;\mathcal{C}^{N\times 1})\subset V_{\dot{\Omega}_D}$, $\mathcal{L}$ is densely defined.

Now let $(\vec{u}_k)_{k\in\mathbb{N}}$ be a sequence in $V_{\dot{\Omega}_D}$ such that $\vec{u}_k\to\vec{u}$ in $L_{2,N}(\Omega)$ and $\mathcal{L}\vec{u}_k\to\vec{w}$ in $L_{2,M}(\Omega)$. Then $(\vec{u}_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $Z_{1,\mathcal{L}}$ and as $V_{\dot{\Omega}_D}$ is closed in $Z_{1,\mathcal{L}}$, $\vec{u}\in V_{\dot{\Omega}_D}$ and $\mathcal{L}\vec{u}=\vec{w}$, whence $\mathcal{L}$ is closed.

(ii) Put $T=\mathcal{L}^*\mathcal{L}$. Then T is a self-adjoint linear operator in $L_{2,N}(\Omega)$ with $\mathcal{N}\subset\mathrm{dom}(T)$. Moreover $T|\mathcal{N}=L$. Indeed, take $\vec{n}\in\mathcal{N}$ and $\vec{\varphi}\in\mathcal{D}(\Omega;\mathcal{C}^{N\times 1})$. Then by virtue of condition $(\mathcal{N}_3)$

$$\langle L\vec{n},\vec{\varphi}\rangle_N=\langle\mathcal{L}\vec{n},\mathcal{L}\vec{\varphi}\rangle_M$$

while from $\mathcal{D}(\Omega;\mathcal{C}^{N\times 1})\subset\mathcal{N}\subset\mathrm{dom}(\mathcal{L}^*\mathcal{L})$ it follows that

$$\langle T\vec{n},\vec{\varphi}\rangle_N=\langle\mathcal{L}^*\mathcal{L}\vec{n},\vec{\varphi}\rangle_M=\langle\mathcal{L}\vec{n},\mathcal{L}\vec{\varphi}\rangle$$

whence, by the density of $\mathcal{D}(\Omega;\mathcal{C}^{N\times 1})$ in $L_{2,N}(\Omega)$, $L\vec{n}=T\vec{n}$ and so $T|\mathcal{N}=L$.
Consequently T is a self-adjoint extension of L so that, L being itself self-adjoint, T=L.
(iii) Obvious. ∎
For examples of decomposable differential operators occurring in mathematical physics, we refer to [3].
Note that since $L=\mathcal{L}^*\mathcal{L}$ is a non negative self-adjoint operator, L coincides with its Friedrichs extension. Moreover $V_{\dot{\Omega}_D}$ is the energy space of L and hence its square root $\sqrt{L}$ has $V_{\dot{\Omega}_D}$ as its domain (see [7] Satz 20.5).

2.1.2. <u>The generalized Cauchy-Riemann operator D</u>
As a further example of such operators L and $\mathcal{L}$ we consider the case of the negative Laplacian and the generalized Cauchy-Riemann operator (also called Dirac operator) acting on $L_2(\Omega;\mathcal{A}_m(\mathcal{C}))$.
Let $\mathcal{A}$ be the Clifford algebra constructed over an orthonormal basis $\{e_1,\dots,e_m\}$ of $\mathbb{R}^m$ with multiplication rules

$$e_ie_j+e_je_i=-2\delta_{ij},\ i,j=1,\dots,m.$$

Consider its basis elements $e_A=e_{h_1}e_{h_2}\dots e_{h_r}$ where $A=\{h_1,\dots,h_r\}$ $\subset\{1,\dots,n\}$ is ordered in such a way that $1\leq h_1<h_2<\dots<h_r\leq m$, $e_\phi=e_0$ being the identity of A. Furthermore put for each $A\in PN$,

$$\overline{e}_A=(-1)^{n(A)(n(A)+1)/2}e_A,$$

$n(A)$ being the cardinality of A, call

$$A_m(C)=A\otimes_R C.$$

and define for each $\lambda=\sum_A\lambda_A e_A\in A_m(C)$,

$$\overline{\lambda}=\sum_A\overline{\lambda}_A\overline{e}_A.$$

Order the basis elements e_A in a certain way, say $B=\{e_{(K)}:K=1,2,\dots,2^m\}$ whereby $e_{(1)}$ is taken to be e_0, associate to each $\lambda\in A_m(C)$ the linear operator $\Pi_\lambda:A_m(C)\to A_m(C)$ given by $\Pi_\lambda(u)=\lambda u$ for all $u\in A_m(C)$ and call $\theta(\lambda)$ the matrix representation of Π_λ with respect to B, i.e. $\theta(\lambda)_{K,L}=[\lambda e_{(K)}]_{(L)}$, $K,L=1,\dots,2^m$.

Then a faithful matrix representation is obtained of $A_m(C)$ into $C^{2^m\times 2^m}$ and it may be easily checked that for each $\lambda\in A_m(C)$ $\theta(\overline{\lambda})=(\theta(\lambda))^+$ (see also [1]). Moreover if for each $u\in A_m(C)$, we put $\vec{u}=[u]_B$, the coordinate vector of u with respect to B, then $\Pi_\lambda(u)=\lambda u=\theta(\lambda)\vec{u}$.

Now consider the generalized Cauchy-Riemann operator

$D=\sum_{j=1}^{m}e_j\frac{\partial}{\partial x_j}$. Then $D^2=DD=-\Delta_m e_0$, Δ_m being the Laplacian in R^m.

Call $L(\partial/\partial x)=\theta(D)$ and $L(\partial/\partial x)=\theta(-\Delta_m e_0)$. Then we have that $L(\partial/\partial x)=L^+(-\partial/\partial x)L(\partial/\partial x)$. Indeed, $\theta(\overline{D})=\theta(-D)$ and $\theta(\overline{D})=\theta(D)^T$ so that $\theta(D)=\theta(-D)^T$. But, as $\theta(D)$ is a homogeneous first order differential operator with real coefficients , $L^+(-\partial/\partial x)=\theta(-D)^T$, whence $L(\partial/\partial x)=L^+(-\partial/\partial x)$ and

$$L(\partial/\partial x)=\theta(-\Delta_m e_0)=\theta(D^2)=\theta(D)\theta(D)=L^+(-\partial/\partial x)L(\partial/\partial x).$$

We may thus define for $u\in L_2(\Omega;A_m(C))$, $w=Du\in L_2(\Omega;A_m(C))$ as being the unique element in $L_2(\Omega;A_m(C))$, provided that it exists, such that for all $\varphi\in D(\Omega;A_m(C))$,

$$<L(D)\vec{u},\vec{\varphi}>=<\vec{u},L^+(-D)\vec{\varphi}>$$
$$=<\vec{u},L(D)\vec{\varphi}>.$$

Call $Z_{1,L}=\{u\in L_2(\Omega;A_m(C)):Du\in L_2(\Omega;A_m(C))\}$ and equip this space with the inner product

$$[u,v]=<\vec{u},\vec{v}>+<L\vec{u},L\vec{v}>$$

Then $Z_{1,L}$ is a Hilbert space and as $L^+(-\partial/\partial x)=L(\partial/\partial x)$, $Z_{1,L}=Z_{1,L^+}$.

Now consider the pure Dirichlet problem for the operator $-\Delta_m e_0$ acting on $L_2(\Omega;A_m(C))$, i.e. take $\dot{\Omega}_D=\dot{\Omega}$. Then, as the set of functions $u\in V_{\dot{\Omega}}$ having bounded support is dense in $V_{\dot{\Omega}}$, $u\in V_{\dot{\Omega}}$ if and only if $u\in Z_{1,L}$ and

$$<Du,v>=<u,Dv> \text{ for all } v\in Z_{1,L}=Z_{1,L^+}$$

(see [3], pp.70-71).

Hence D is symmetric in $V_{\dot{\Omega}}$ and as D is closed (see also Theorem 2.1.1(i)), we have

Theorem. D is a self-adjoint linear operator in $L_2(\Omega;A_m(C))$.
Corollary. D^{-1} is self-adjoint.

2.2. Well-posed (D-N)-problems for decomposable operators

In this subsection we again consider differential operators of the form $L(\partial/\partial x)=L^+(-\partial/\partial x)L(\partial/\partial x)$ and the associated spaces $V_{\dot{\Omega}_D}$ and N.

The (D-N)-problem for L in N is said to be well posed if for each $\vec{f}\in L_{2,N}(\Omega)$ there exists a unique $\vec{n}\in N$ such that

(i) $L\vec{n}=\vec{f}$

(ii) $\vec{f}_k\to\vec{f}$ in $L_{2,N}(\Omega)$ implies that $\vec{n}_k\to\vec{n}$ in $L_{2,N}(\Omega)$.

As has been shown in [3], a necessary and sufficient condition for the (D-N)-problem to be well-posed in N for L is that there exists $C>0$ such that for all $\vec{u}\in V_{\dot{\Omega}_D}$,

$$\|\vec{u}\|_N^2 \leq \frac{1}{C}\|L\vec{u}\|_M^2 \qquad (2.2)$$

Assume hence forth that the (D-N)-problem is well-posed for L in N.

Condition (2.2) implies that $\eta(L)=\{0\}$ whence $C(L)=\mathrm{dom}(L)=V_{\mathring{\Omega}_D}$.

Moreover it means that L is reciprocally bounded in $V_{\mathring{\Omega}_D}$ or $R(L)$ is closed in $L_{2,M}(\Omega)$ (see [4], Theorem 3.3) and so $\mathrm{dom}(L^{-1})=L_{2,M}(\Omega)$.

Condition (2.2) together with the self-adjointness of L in N also implies the existence of a spectral measure M in C carried by $[C,+\infty[$ and of a bounded self-adjoint operator G(Z) in $L_{2,N}(\Omega)$ such that

$$L=\int_0^{+\infty}\lambda\, dM \text{ and } G(z)=\int_0^{+\infty}\frac{dM}{\lambda-z}$$

for all $z\in\rho(L)$, $\rho(L)\subset C$ being the resolvent set of L and G(z) being the Green's operator corresponding to L-z (see [3]). As $0\in\rho(L)$, we thus have for the operator

$G_0=G(0)=\int_0^{+\infty}\frac{dM}{\lambda}$ that $LG_0=1_{L_{2,N}(\Omega)}$ and $G_0L=1_N$ whence clearly $G_0=L^{-1}$.

Moreover, as both L and G_0 are positive-definite, their square roots are represented by

$$\sqrt{L}=\int_0^{+\infty}\sqrt{\lambda}\,dM \text{ and } \sqrt{G_0}=\int_0^{+\infty}\frac{1}{\sqrt{\lambda}}dM. \tag{2.3}$$

We so obtain

2.2.1 <u>Theorem</u>. Suppose that the (D-N)-problem is well-posed for the operator $L(\partial/\partial x)=L^+(-\partial/\partial x)L(\partial/\partial x)$ in N. Then there exists a partial isometry $R: L_{2,N}(\Omega)\to L_{2,M}(\Omega)$ such that

(i) $G_0=L^{-1}$ and $\sqrt{G_0}=(\sqrt{L})^{-1}$

(ii) $L_0=R\sqrt{L}$, $L^{-1}=\sqrt{G_0}R^*$ and $L^*=\sqrt{L}R^*$

(iii) (Spectral decomposition of L and L^*

$$L=\int_0^{+\infty}\sqrt{\lambda}\,d(RM),\quad L^*=\int_0^{+\infty}\sqrt{\lambda}\,d(MR^*) \tag{2.4}$$

(iv) (Spectral decomposition of L^{-1}) :

$$L^{-1}=\int_0^{+\infty} \frac{1}{\sqrt{\lambda}}d(MR^*) \qquad (2.5)$$

Proof. (i) As we have already remarked, $G_0=L^{-1}$ and as L is a non-negative self-adjoint operator , $\sqrt{L^{-1}}=(\sqrt{L})^{-1}$ (see [4], Theorem 5.2) whence $\sqrt{G_0}=(\sqrt{L})^{-1}$.
(ii) The polar decomposition of L yields that $L=R\sqrt{L^*L}$ or, taking account of Theorem 2.1.1.(ii), that $L=R\sqrt{L}$. Hereby $R:L_{2,N}(\Omega)\to L_{2,M}(\Omega)$ is a partial isometry with dom(R)= $\overline{R(L)}=L_{2,N}(\Omega)$, im(R)$=\overline{R(L)}=R(L)$ and satisfying $R^{-1}=R^*$ (see [8] Satz 7.20 and [4], Theorem 6.2).
Call $D=\sqrt{G_0}R^*=(\sqrt{L})^{-1}R^{-1}$. Then $D=L^{-1}$.
Indeed, R^* and $R^{*-1}=R$ are bounded while $\eta(R^{**})=\eta(R)=\eta(L)=\eta(\sqrt{G_0})$.
Hence , using the Corollary to [4] Theorem 3.5, the desired result is obtained. As $L=R\sqrt{L}$ with R bounded, we have that $L^*=(\sqrt{L})^*R^*$ $=\sqrt{L}R^*$ (see also [8] Satz 4.19).
(iii) and (iv). As $\sqrt{\lambda}$ and $\frac{1}{\sqrt{\lambda}}$ are M-integrable and R,R^* are partial isometries, $\sqrt{\lambda}$ and $\frac{1}{\sqrt{\lambda}}$ are respectively RM- and MR^*-integrable so that, using (2.3) and the results from [2], p. 43, the relations (2.4) and (2.5) are obtained.■

2.2.2. Remark. By means of (2.4) we have that for all $\vec{v}\in V_{\dot{\Omega}_D}$,

$$L\vec{v}=\int_0^{+\infty} \sqrt{\lambda}d(RM\vec{v}) .$$

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Rijksuniversiteit Gent

Seminarie voor Algebra en Functionaalanalyse

Galglaan 2

B-9000 Gent

Belgium