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In our paper we propose and study the notion of pathological infinity of measures.

If $(X,\mathcal{S},\mu)$ is a measure space then $A\in\mathcal{S}$ is called a pathological infinity if $\mu(A)=\infty$, A does not contain a set of positive finite measure and A contains no infinite atom.

In Section 1 there are introduced theorems connected with the role of the pathological infinity in classification of properties of infinite measures (see [5]) which are: the characterization of semifiniteness (Theorem 1) and the characterization of countable chain condition of nonnegative measures without pathological infinity. In Section 2 a measurable space admitting the pathological infinity is studied.

Definitions and Notations

Throughout the paper $(X,\mathcal{S})$ will denote a measurable space with a σ-ring $\mathcal{S}$ of subsets of X. If $\mathcal{E}$ is a family of subsets of the set X then "$\mathcal{E}C$" denotes that every family of pairwise disjoint elements from $\mathcal{E}$ is at most countable (therefore $\emptyset\notin\mathcal{E}$) (see Ficker [6]). If $A\subset X$ then we use the symbol $A|\mathcal{E}$ in the Hahn's sense ([7]), i.e. $A|\mathcal{E}=\{E\in\mathcal{E}: E\subset A\}$. The contraction μ_E of a measure μ by the set E ($E\in\mathcal{S}$) is understood in the sense of [1,p.12]. If $\mathcal{M}\subset\mathcal{S}$, then by the "contraction of $\mathcal{M}$ by the set E" we mean $\mathcal{M}_E=\{A\in\mathcal{S}: A\cap E\in\mathcal{M}\}$.

We use Sikorski's monograph [9] as a standard reference for the following notions used in the present paper (the page given in parentheses refers the respective definition in [9]): Boolean algebra (p.4), field of sets (p.4), atom of a Boolean algebra (p.27), atomic Boolean algebra (p.28), quotient algebra (p.29), Boolean $\mathfrak{m}$-algebra (p.65), the $\mathfrak{m}$-chain condition (p.72). But in the case when $\mathfrak{m}$ is a cardinal number of all integers $\aleph_0$ we write "a Boolean σ-algebra" or "the σ-chain condition" e.g. a Boolean σ-algebra $\mathcal{B}$ is said to satisfy the σ-chain condition provided

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every set of disjoint elements in $\mathcal{B}$ has cardinal number less than or equal to $\aleph_0$.

Definition 1. Let $(X,\mathcal{S},\mu)$ be a measure space and let $\mathcal{M}=\{E\in\mathcal{S}:\mu(E)=0\}$ and $\mathcal{F}=\{E\in\mathcal{S}:\mu(E)<\infty\}$. Then we say that $A\in\mathcal{S}$

a) is semifinite if for every $E\in A|(\mathcal{S}-\mathcal{F})$ is $E|(\mathcal{F}-\mathcal{M})\neq\emptyset$

b) is an infinite atom if $A\in\bigcap_{E\in\mathcal{S}}\mathcal{M}_E\cup\mathcal{M}_{E^\perp}-\mathcal{F}$

c) is a pathological infinity if $A\in(\mathcal{S}-\mathcal{F})$, $A|(\bigcap_{E\in\mathcal{S}}\mathcal{M}_E\cup\mathcal{M}_{E^\perp})\subset\mathcal{F}$ and $A|\mathcal{F}\subset\mathcal{M}$.

Further, we say that μ

d) is semifinite if every $A\in\mathcal{S}$ is semifinite

e) is pathological-infinite-less (shortly p.i.l.) if $\mathcal{S}$ contains no pathological infinity

f) satisfies CCC if $(\mathcal{S}-\mathcal{M})C$ holds.

We remark that according [5, Remark 1] the family $\mathcal{M}=\{E\in\mathcal{S}:\mu(E)=0\}$ is called a null-system of the measure μ and $\mathcal{F}=\{E\in\mathcal{S}:\mu(E)<\infty\}$ is called the finite-valued system of the measure μ. For an abstract aproach some notions in measure theory, such as the notion of an atom, see e.g. [3], [4], [5].

1. The Meaning of Pathological Infinity in the Classification of Infinite Measures

Example 1. The measure introduced here is an example of a measure satisfying CCC which is not an countable sum of finite measures. The example has been given and explained in more detail by K.P.S.Bhaskara Rao and M.Bhaskara Rao [2, Section 2].

Let $\mathcal{B}$ be the Boolean σ-algebra of all Borel subsets of the real line modulo first category Borel sets. Let X be the Stone space of $\mathcal{B}$, $\mathcal{A}$ the Baire σ-algebra on X, and $\mathcal{Y}$ the collection of all first category Baire subsets of X. By Loomis' Theorem (see, e.g. [8, p.102]) the quotient Boolean σ-algebra $\mathcal{A}/\mathcal{Y}$ and $\mathcal{B}$ are isomorphic. Then we define the measure μ as follows: $\mu(A)=0$, if $A\in\mathcal{Y}$; $\mu(A)=\infty$ if $A\in\mathcal{A}-\mathcal{Y}$.

The notion of a pathological infinity was inspired by the above example. Two theorems introduced in this section support the usefulness of this notion. The first one analyses the notion of semifiniteness. It refers to the fact that semifiniteness may be violated by the existence of an infinite atom or a pathological infinity. The second theorem characterizes the notion CCC in the case of p.i.l. measure spaces.

Theorem 1. Let $(X, \mathcal{S}, \mu)$ be a measure space. Then μ is semifinite if and only if it contains neither a pathological infinity nor an infinite atom.

Proof is given in [5] in a more general version for a quasi-measure space.

Definition 2. Let $(X, \mathcal{S}, \mu)$ be a measure space. We say that μ is a countable sum of finite measures (shortly c.s.f.m.) if $\mu = \sum_{n=1}^{\infty} \mu_n$ where μ_n are finite measures.

V.Ficker in [6] introduces a very elegant but non-valid assertion that if μ is a measure on σ-algebra then μ satisfies CCC iff μ is c.s.f.m. The above example by K.P.S.Bhaskara Rao and M.Bhaskara Rao from [2] shows that Ficker's assertion is false. Both authors introduce a characterization of measures satisfying CCC assuming semifiniteness of measure.

In the following theorem we introduce this characterization under slightly weaker assumptions which together with Theorem 1, Theorem 2 and Lemma 5 from [5] give a generalization of Theorem from [2]. In the proof of Theorem 2 we use some ideas from the proof of the Ficker's assertion.

Lemma 1. Let a measure μ be a countable sum of finite measures. Then measure μ satisfies CCC.

Proof. Denote by $\mathcal{M}_n$ the null-system of measures μ_n and by $\mathcal{M}$ the null-system of measure μ. Then $\mathcal{S} - \mathcal{M} = \mathcal{S} - \bigcap_{n=1}^{\infty} \mathcal{M}_n = \bigcup_{n=1}^{\infty} (\mathcal{S} - \mathcal{M}_n)$.

As μ_n are finite measures, the condition $(\mathcal{S} - \mathcal{M}_n)C$ is true for all $n \in N$. (See e.g. Berberian [1], Section 44). Hence we get that $\bigcup_{n=1}^{\infty} (\mathcal{S} - \mathcal{M}_n)C$ holds which proves that μ satisfies CCC.

Theorem 2. Let $(X, \mathcal{S}, \mu)$ be a pathological-infinite-less measure space. Then μ satisfies CCC if and only if μ is a countable sum of finite measures.

Proof. The sufficient condition follows from Lemma 1.

Necessary condition. Let $\mathcal{F}(\mathcal{M})$ be a finite-valued (null) system of measure μ. Let $(A_k)_{k \in K}$ be a maximal family of pairwise disjoint infinite atoms. As μ satisfies CCC, K is countable and thus the set $A = \bigcup_{k \in K} A_k \in \mathcal{S}$. Let $(B_i)_{i \in I}$ be a maximal family of pairwise disjoint sets from $(X-A) | (\mathcal{F} - \mathcal{M})$. As μ satisfies CCC, I is countable and thus $B = \bigcup_{i \in I} B_i \in \mathcal{S}$.

From the maximility of the families $(A_k)_{k\in K}$ and $(B_i)_{i\in I}$ we obtain that if $F\in(X-A\cup B)|\mathcal{S}$ then $(F|\bigcap_{E\in\mathcal{S}}\mathcal{M}_E\cup\mathcal{M}_{E^\perp})\subset\mathcal{F}$ and $F|\mathcal{F}\subset\mathcal{M}$.

If there existed $F\in(X-A\cup B)|(\mathcal{S}-\mathcal{F})$, then F would be a pathological infinity. Hence $(X-A\cup B)|(\mathcal{S}-\mathcal{F})=\emptyset$ and as $(X-A\cup B)|\mathcal{F}\subset\mathcal{M}$ we obtain $(X-A\cup B)|\mathcal{S}\subset\mathcal{M}$. So $\mu=\mu_{A\cup B}=\mu_A+\mu_B=\sum_{k\in K}\mu_{A_k}+\sum_{i\in I}\mu_{B_i}$. As $B_i\in\mathcal{F}$, μ_{B_i} are finite measures and to complete the proof it sufficies to show that, for all $k\in K$, μ_{A_k} can be written as a countable union of finite measures. Let us define the set function ν on $\mathcal{S}$ as follows: for $E\in\mathcal{S}$, $\nu(E)=1$ if $\mu(A_k\cap E)=\infty$ and $\nu(E)=0$ if $\mu(A_k\cap E)=0$. As A_k is an infinite atom of the measure μ, we have that ν is a measure. Put $\nu_n=\nu$ for all $n\in N$. From the definition of ν it is evident that $\mu_{A_k}=\sum_{n=1}^{\infty}\nu_n$.

Remark 1. Theorem 2 reads:"Let μ be a p.i.l. measure. Then μ satisfies CCC iff μ is c.s.f.m.". There arises a natural question whether Theorem 2 can be strengthened in the following way: "The measure μ is c.s.f.m. iff it is both p.i.l. and satisfies CCC".

The following example gives a negative answer. Let μ_n be a Lebesgue measure on $\langle 0,1)$ for all $n\in N$ and let $\mu=\sum_{n=1}^{\infty}\mu_n$. Then μ is c.s.f.m. while $\langle 0,1)$ is a pathological infinity of μ.

2.The Structure of Measurable Spaces Admitting Pathological Infinity

Theorem 3. Let $(X,\mathcal{S},\mu)$ be a measure space, $\mathcal{S}$ being a countably generated σ-algebra and let $Y\in\mathcal{S}$ be a pathological infinity of the measure μ. Then a field of sets $Y|\mathcal{S}$ is an atomic σ-algebra on Y containing an uncountable family of atoms. Moreover, for every atom A of the σ-algebra $Y|\mathcal{S}$ we have $\mu(A)=0$.

Proof. As $\mathcal{S}$ is a countably generated σ-field of sets, it is an atomic field of sets according to 24.5 in [9]. Then $Y=\bigcup_{A\in\mathcal{R}}A$, where $\mathcal{R}$ is the set of all atoms A of Boolean σ-algebra $\mathcal{S}$ such that $A\subset Y$.

As the pathological infinity Y contains neither an infinite atom nor a set of positive finite measure, it can easily be seen that $\mu(A)=0$ for all $A\in\mathcal{R}$. If the family $\mathcal{R}$ were countable, then we would have $\mu(Y)=\sum_{A\in\mathcal{A}}\mu(A)=0$, contradicting the assumption $\mu(Y)=\infty$. Thus $\mathcal{A}$ is uncountable.

Theorem 4. Let $(X,\mathcal{S},\mu)$ be a measure space and let $Y\in\mathcal{S}$ be a pathological infinity of the measure μ. Then X is an uncountable set.

Proof. Let the set X be countable. Then $Y\subset X$ is also countable and thus the field of sets $Y|\mathcal{S}$ satisfies the σ-chain condition. As, moreover, $Y|\mathcal{S}$ is a σ-algebra according to 20.5 from [9], $Y|\mathcal{S}$ is a complete algebra. Hence, according to 25.1 from [9], $Y|\mathcal{S}$ is atomic and it is isomorfic with the algebra 2^B, where B is a countable set of all its atoms. As every σ-algebra of this type is countably generated, by Theorem 3 it contains no pathological infinity of the measure μ, which contradicts the assumption.

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