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Inverse limits need not exist in the category of compact spaces and Feller kernels: a counterexample

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INVERSE LIMITS NEED NOT EXIST IN THE CATEGORY OF COMPACT SPACES AND FELLER KERNELS: A COUNTEREXAMPLE

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In this note, incorrect statements are typed in italics. For applications, it would be useful to know that *inverse limits exist in the category* $\mathfrak{D}$ *of compact spaces with Feller kernels as morphisms* (c.f. [5], Ch. IV). This assertion is the main part in [7] - it will be stated explicitely in a moment. J. Vestergaard pointed out that there must be an error in the proof; familiarity with the Poulsen simplex enforces this feeling. We do here the clerical work to disprove the result decisively and localize the error in [7]; all arguments used below are well-known.

Let X and Y be Hausdorff spaces, denote by $B(X)$ the Borel-σ-algebra, by $C(X)$ the space of bounded continuous functions and by $M(X)$ ($M_+(X)$, $M_+^1(X)$) the bounded (and positive,and normalized) Radon measures on X - "Radon" means "inner regular w.r.t. compact sets"; a mapping $P : X \to M_+^1(Y)$, $x \to P(x,\cdot)$, s.t. the functions $x \to P(x,B)$, $B \in B(Y)$, are Borel measurable is called a <u>Feller kernel</u> iff

$$\{P(\cdot,f) = \int f(y)\ P(\cdot,dy):\ f \in C(Y)\} \subset C(X),$$
$$\{\mu P \quad = \int P(x,\cdot)\ \mu(dx):\ \mu \in M(X)\} \subset M(Y);$$

the composition with a kernel Q from Y to Z is defined as usual: $PQ(x,B) = \int Q(y,B)\ P(x,dy)$, $x \in X$, $B \in B(Z)$. What we need from category theory is contained in Ch. III of [8].

Assume that $(I,\leq)$ is an increasing net; consider compact spaces X_i, $i \in I$, and kernels P_{ji}, $i \leq j$, such that

(*i) all P_{ji} are Feller kernels,

(*ii) $P_{ii}(x,\cdot)$, $i \in I$, $x \in X$, is the Dirac measure in x,
$P_{kj}P_{ji} = P_{ki}$ whenever $i \leq j \leq k$,

i.e. (X_i,P_{ji}) is an inverse system in $\mathfrak{D}$. $\mathbf{Ban}_1$ denotes the category of Banach spaces and linear contractions. The system (*) induces both:
an inverse system $(M(X_i),\Phi_{ji})$ in $\mathbf{Ban}_1$, where the $M(X_i)$ are Banach spaces in the norms v_i of total variation and morphisms Φ_{ji} defined by $\Phi_{ji}(\mu_j) = \mu_j P_{ji}$,

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a direct system $(C(X_i),\Phi^{ij})$ in $\mathfrak{Ban}_1$, where $C(X_i)$ has supremum norm and $\Phi^{ij}(f_i) = P_{ji}(\cdot,f_i)$.
The $\mathfrak{Ban}_1$ limits exist respectively (c.f. [8], 11.8.2(b) and (d)); denote them by M with norm v and C with norm $\|\cdot\|$ (concrete representations are given in the just cited reference). M is the dual Banach space of C, the duality being determined by the formulas

(1) $\quad \langle\Phi^j(f_j),(\mu_i)_{i\in I}\rangle = \mu_j(f_j),\ f_j \in C(X_j),\ (\mu_i)_{i\in I}\in M,$

where Φ^j is the canonical injection from $C(X_j)$ into C ([9]). M may be ordered by the cone

(2) $\quad M_+ := \{(\mu_i)_{i\in I}\in M\colon \mu_i \geq 0 \text{ for every } i\in I\}.$

The positive face of the unit ball is

$$M_+^1 := \{(\mu_i)_{i\in I}\in M_+\colon v((\mu_i)_{i\in I}) = 1\};$$

write ex M_+^1 for the set of extreme points.

The main part of the theorem in [7], p.1200 reads as follows:
Assertion: Assume that an inverse system (*) in $\mathfrak{D}$ *is given. Then:*
a. *the space* $X_o := M_+^1$ *is compact in the weak*-topology* $\sigma(M,C)$,
b. *there are Feller kernels* P_i *from* X_o *to the* X_i *such that* $P_i = P_jP_{ji}$ *whenever* $i \leq j$,
c. *if there are Feller kernels* Q_i *from some Hausdorff space* Y *to the* X_i, *then there is a unique Feller kernel* Q *from* Y *to* X_o, *such that* $QP_i = Q_i$ *if* $i\in I$.

In other words, *the system (*) has an inverse limit in* $\mathfrak{D}$.

We will show that the validity of this assertion implies that the inverse limit of Bauer simplices is a Bauer simplex whereas it may be even a Poulsen simplex. By a <u>simplex</u> we mean a compact Choquet simplex; inverse systems are considered in the category $\mathfrak{S}$ with simplices as objects and affine continuous maps $\varphi : S \to T$ between simplices as morphisms. In [3], thm. 13 it is shown that in $\mathfrak{S}$ every inverse system has an inverse limit (in [10] the corresponding results for noncompact simplices are obtained). A simplex with compact extreme boundary is called a <u>Bauer</u> simplex, a metrizable simplex with dense extreme boundary is a <u>Poulsen</u> simplex (which in fact exists).

The counterexample is based on the following observations I and II:

I. If T is a (compact) metrizable simplex then there is a locally convex space E such that i) E contains a simplex S affinely homeomorphic to T, ii) there is a decreasing sequence of Bauer simplices S_n in E whose intersection is S.

Proof: [4], thm. 9.

II. Consider an inverse system (S_i,φ_{ji}) in the category $\mathcal{S}$, where all S_i are Bauer simplices and denote by S the inverse limit in $\mathcal{S}$. Denote further by X_i the compact extreme boundaries ex S_i and define

$$P_{ji}(x,\cdot) := p_i(\varphi_{ji}(x),\cdot),\ i \le j,\ x \in X_j,$$

where $p_i(y,\cdot)$ is the unique element in $M_+^1(X_i)$ with barycenter y in S_i. Then: the mappings P_{ji} define an inverse system (*) in $\mathbf{D}$ and an inverse and a direct system in $\mathbf{Ban}_1$ according to the remarks above.
Finally, S and M_+^1 are affinely homeomorphic if M_+^1 is endowed with the weak*-topology $\sigma(M,C)$.

Proof: Because the S_i are Bauer simplices, the mappings

$$S_i \ni x \to p_i(x,f),\ f \in C(X_i),$$

are affine and continuous ([1], II.4.1), thus also the mappings

$$S_j \ni x \to p_i(\varphi_{ji}(x),f_i),\ f_i \in C(X_i). \tag{3}$$

A standard monotonicity argument shows that the mappings in (3) restricted to X_j = ex S_j define Feller kernels P_{ji} from X_j to X_i. As representing measures p satisfy the barycentrical formula $g(x) = \int g(y)\ p(dy)$ for affine continuous functions g, we have for $i \le j \le k$, $x \in X_k$ and $f \in C(X_i)$

$$P_{kj}P_{ji}(x,f) = \int p_i(\varphi_{ji}(y),f)\ p_j(\varphi_{kj}(x),dy) = p_i(\varphi_{ji}\circ\varphi_{kj}(x),f) =$$
$$= p_i(\varphi_{ki}(x),f) = P_{ki}(x,f).$$

Thus we have verified that the compact spaces X_i together with the Feller kernels P_{ji} are an inverse system (*) in $\mathbf{D}$. Let us now consider the induced $\mathbf{Ban}_1$ direct system of the spaces $C(X_i)$ with linear contractions Φ^{ij}.
For a simplex S, denote by $A(S)$ the space of affine continuous functions on S. If we take the functions f in (3) from $A(S_i)$ instead of $C(X_i)$ then we get a $\mathbf{Ban}_1$ direct system of the spaces $A(S_i)$ with supremum norm and linear contractions Ψ^{ij}: from [3], p.162, we learn that $A(S)$ is the $\mathbf{Ban}_1$ direct limit.
Again since the S_i are Bauer simplices, the spaces $A(S_i)$ and $C(X_i)$ are isometrically isomorphic ([2], 2.7.5) via

$$\begin{aligned} &A(S_i) \ni \overline{f} \to \overline{f}/\text{ex } S_i \in C(\text{ex } S_i) \\ &C(\text{ex}S_i) \ni f \to \overline{f} \qquad \in A(S_i) \end{aligned} \tag{4}$$

where $\overline{f}(x) = \int_{\text{ex}S_i} f(y)\ p(x,dy)$.
This shows that $A(S)$ is isometrically isomorphic to C, that M is the dual Banach space of $A(S)$, and that the duality is determined by the set of formulas

$$\langle \Psi^j(f_j),(\mu_i)_{i\in I}\rangle = \mu_j(f_j), \tag{5}$$

where Ψ^j is the canonical injection from $A(S_j)$ into $A(S)$.
By (4) and (5), we see that the positive cone of M determined by the evaluations on $A(S)$ is again M_+ as defined in (2). Recall that the convex compact set S is affinely homeomorphic to its "state space"

$$(A(S)')_+^1 := \{h \in A(S)' : h \geq 0, h \text{ has norm } 1\}$$

in the weak*-topology $\sigma(A(S)',C) = \sigma(M,C)$.
Since the state space is equal to M_+^1, the proof is complete.

Now we see easily that Scheffer's assertion from [7] reported above fails to be true. By I. there is a sequence S_n of Bauer simplices in some locally convex space which decreases to a Poulsen simplex S_P. By II. we get compact spaces X_n and Feller kernels P_{nm} satisfying the assumptions of Scheffer's assertion. Again by II. the set $X_o = \text{ex } M_+^1$ is affinely homeomorphic to the extreme boundary of the Poulsen simplex. Hence X_o is not compact as claimed - X_o is even dense in M_+^1.
In other words: Scheffer's assertion implies that the inverse limit in $\mathcal{S}$ of Bauer simplices is a Bauer simplex whereas we have seen that it can be a Poulsen simplex.

The error can be localized in a lemma which is the basis of Scheffer's proofs. We state from [7], p. 1199:
<u>Assertion</u>: Let B *be a Banach space such that its dual Banach space* B' *is an* AL-*space with* $\sigma(B',B)$-*closed positive cone* B'_+. *Then:*
B *is an* AM-*space in the order induced by the cone* $B_+ := \{x \in B : \langle x',x\rangle \geq 0 \text{ if } x' \in B'_+\}$ *and there are two alternatives:*
a. B *has a unit and* $X_o = \text{ex } B'^1_+$ *is compact,*
b. B *has no unit and* $X_o \cup \{0\} = \text{ex } B'^1_+$ *is compact.*

To see that this is wrong, we consider again the Poulsen simplex S_P. It is well-known that $A(S_P)'$ is an AL-space ([2], 2.7.1). Obviously, $A(S_P)'_+$ is closed w.r.t. $\sigma(A(S_P)',A(S_P))$, hence the assumptions are fulfilled. But $A(S_P)$ is no AM-space since it is no lattice - a space $A(S)$ is a lattice if and only if S is a Bauer simplex ([2], 2.7.5; in fact, $A(S_P)$ is even an anti-lattice). Moreover: neither $\text{ex } (A(S_P)')_+^1$ nor $\text{ex } (A(S_P)')_+^1 \cup \{0\}$ are weak*-compact since again $\text{ex } (A(S_P)')_+^1$ is dense in $(A(S_P)')_+^1$.

REFERENCES

[1] ALFSEN E. "Compact convex sets and boundary integrals", Springer Verlag: Berlin, Heidelberg, New York (1971)

[2] ASIMOV L. and ELLIS A.J. "Convexity theory and its applications in functional analysis", Academic Press: London, New York (1980)

[3] DAVIES E.B. and VINCENT-SMITH G.F. "Tensor products, infinite products and

projective limits of Choquet simplexes", Math.Scand. 22 (1968), 145-164

[4] EDWARDS D.A. "Système projectifs d'ensemble convexes compacts", Bull. Soc. Math. France 103 (1975), 225-240

[5] LAURITZEN S.L. "Statistical models as extremal families", Aalborg University Press (1982)

[6] SCHAEFER H.H. "Topological vector spaces", Springer Verlag: Berlin, Heidelberg, New York (1971)

[7] SCHEFFER C.L. "La limite projective des espaces topologiques de Hausdorff compacts dans la catégorie des diffusions continues sousmarkoviennes", C. R. Acad. Sci. Paris 272 (1971), 1198-1201

[8] SEMADENI Z. "Banach spaces of continuous functions", PWN - Polish Scientific Publishers: Warsawa (1971)

[9] SEMADENI Z. and ZIDENBERG H. "Inductive and inverse limits in the category of Banach spaces", Bull. Acad. Pol. Sc. 13 (1965), 579-583

[10] WINKLER G. "Inverse limits of simplices and applications in stochastics", Habilitationsschrift: München (1983)

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