Jan Paseka

A note on the prime ideal theorem

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 131--136

Persistent URL: `http://dml.cz/dmlcz/701805`

A Note on the Prime Ideal Theorem

JAN PASEKA

Brno*)

Received 15 March 1989

The Prime Ideal Theorem is shown to be equivalent with the following two statements (1) Any compact nontrivial quantale has a prime element, and (2) Any compact normal complete nontrivial distributive lattice has a maximal element. Some another equivalents of the Prime Ideal Theorem are given.

The lattice-theoretical investigations of complete lattices equipped with an additional binary operation $\cdot$ which distributes over arbitrary joins in each variable can be traced back to Ward and Dilworth [14]. Such a gadget is called quantale (following C. J. Mulvey). Some topological properties of quantales were obtained by Borceux [4].

The original motivation for this paper was the question whether the Prime Ideal Theorem (every nontrivial distributive lattice has a prime ideal) is strong enough to ensure the existence of a prime element in arbitrary compact quantale, rather than just in compact frames (see [3], [7]). We present some new relationships between this principle and quantale-theoretic conditions, e.g. the existence of a maximal element in arbitrary compact normal frame.

All unexplained facts concerning frames and quantales can be found in [7], [9] or [10]. Detailed accounts of choice principles appear in [6].

§ 1. m-prime ideals yields prime elements

1.1. Definition. (i) A (weak) m-semilattice is a $\vee$-semilattice S with the top element 1 and the bottom element 0 equipped with an associative operation $\cdot$ so that $\cdot$ distributes over finite (finite nonempty) joins in each variable and $1 . x = x$ for all $x \in S$. A morphism of weak m-semilattices is a mapping preserving finite joins, $\cdot$ and 1.

---

*) Department of Mathematics, University J. E. Purkyně, 662 95 Brno, Janáčkovo nám. 2a, Czechoslovakia

(ii) An element a of a weak m-semilattice S is called

(1) 2-sided if $a \,.\, 1 \leqq a$,

(2) idempotent if $a \,.\, a = a$.

The set of all 2-sided (idempotent) elements will be denoted by $\hat{S}$ ($E(S)$). It is easy to check that $\hat{S}$ is a weak m-semilattice.

A weak m-semilattice S is said to be nontrivial if it has at least two 2-sided elements.

(iii) A weak m-semilattice S is called 2-sided (idempotent) if any its element is 2-sided (idempotent).

The following is well known (see [4], [5]).

1.2. Lemma. For any elements a, b, c in a weak m-semilattice S we have the following

(i) $b \leqq c$ implies $a \,.\, b \leqq a \,.\, c$,

(ii) $a \leqq b$ implies $a \,.\, c \leqq b \,.\, c$,

(iii) $a \,.\, 0 = 0$,

(iv) a is 2-sided implies $a \,.\, b \leqq a$,

(v) $a \in \hat{S} \cap E(S)$ implies $a \vee (b \,.\, c) = (a \vee b) \,.\, (a \vee c)$,

(vi) $a \in \hat{S} \cap E(S)$ implies $a \,.\, 1 = a$.

1.3. Corollary. Let S be a weak m-semilattice. If $E(S) = S$ then $\hat{S} \cap E(S)$ constitutes a distributive lattice. Moreover, a weak m-semilattice is a distributive lattice iff $S = E(S) = \hat{S}$.

1.4. Definition. Let S be a weak m-semilattice. (i) An ideal of S will be just an ideal of the $\vee$-semilattice S. An ideal I is called m-prime if $x \,.\, y \in I$ implies $x \in I$ or $y \in I$ for all $x, y \in S$.

(ii) An element $p \neq 1$ of S is called prime if $x \,.\, y \leqq p$ implies $x \leqq p$ or $y \leqq p$ for all $x, y \in S$. The set of all prime (maximal) elements of S will be denoted by $P(S)$ ($D(S)$).

As for weak m-semilattices, we shall introduce the notion of a weak quantale.

1.5. Definition. A (weak) quantale K is a complete (weak) m-semilattice K in which $\cdot$ distributes over arbitrary (nonempty) joins in each variable. Congruences on (weak) quantales are congruences with respect to $\cdot$ and $\bigvee$. A frame K is a quantale K satisfying $K = E(K) = \hat{K}$.

Let us recall that a complete lattice L is said to be compact if $E \subseteq L$, $\bigvee E = 1$ implies there is $F \subseteq E$, F finite such that $\bigvee F = 1$.

The following is well known for m-semilattices (see [10]).

1.6. Proposition. Let $Id(S)$ be the compact weak quantale of all ideals of a weak m-semilattice S ($I \,.\, J$ being generated by $\{x \,.\, y;\ x \in I,\ y \in J\}$). Then the prime elements of $Id(S)$ are precisely the m-prime ideals of S.

1.7. Remark. Let K be a weak quantale. We put $a\ C\ b$ if and only if $(a\,.\,1 \vee c = 1 \Leftrightarrow b\,.\,1 \vee c = 1$ for any $c \in K)$ for all $a, b \in K$.

1.8. Proposition. Let K be a compact weak quantale. Then C is a congruence on K.

Proof. Let $x\ C\ y$, $u\ C\ v$, $x\,.\,u\,.\,1 \vee c = 1$. Then $x\,.\,1 \vee c = u\,.\,1 \vee c = 1$ i.e. $y\,.\,1 \vee c = 1 = v\,.\,1 \vee c$. Now, we have $y\,.\,v\,.\,1 \vee c = 1$.

We have to show that $x_i\ C\ y_i$, $i \in I$, $I \neq \emptyset$ implies $x\ C\ y$; here $x = \bigvee\{x_i;\ i \in I\}$, $y = \bigvee\{y_i;\ i \in I\}$. Let $x\,.\,1 \vee c = 1$. Then by compactness of K there is $F \subseteq I$, F finite such that $1 = c \vee \bigvee\{x_i;\ i \in F\}\,.\,1$ i.e. $1 = c \vee \bigvee\{y_i;\ i \in F\}\,.\,1 = c \vee y\,.\,1$. The symmetry argument concludes the proof.

For a compact weak quantale K, we denote $K_{/C} = K_c$. It is easy to check that K_c is a conjunctive frame (see [11], [13]).

1.9. Lemma. Let K be a weak compact quantale. Then K_c is a compact frame.

Proof. Let $a_i \in K_c$, $i \in I$, ${}_{K_c}\bigvee a_i = 1$. Clearly, $\{a_i;\ i \in I\} \subseteq \hat{K}$. Since K is compact we have $\bigvee a_i\ C\ 1$ i.e. $1 = (\bigvee a_i)\,.\,1 \leqq \bigvee a_i$. Consequently, we have $\bigvee a_i = 1$ and the rest follows from the compactness of K.

1.10. Theorem. The following statements are equivalent.

(i) Any nontrivial compact frame has a prime element.

(ii) Any nontrivial compact (weak) quantale has a prime element.

(iii) Any nontrivial (weak) m-semilattice has an m-prime ideal.

(iv) The Prime Ideal Theorem.

Proof. (i) $\Rightarrow$ (ii) Let K be a compact (weak) quantale. From 1.9 and (i) we have that K_c has a prime element p which is clearly prime and 2-sided in K.

(ii) $\Rightarrow$ (iii) It follows from 1.6.

(iii) $\Rightarrow$ (iv) It is evident.

(iv) $\Rightarrow$ (i) It results from [1].

1.11. Lemma. Let S be a weak m-semilattice (weak quantale), $a \in \hat{S}$. Then the $\vee$-semilattice (complete lattice) $\uparrow(a) = \{x \in S;\ a \leqq x\}$ is a weak m-semilattice (weak quantale) with respect to a multiplication $\cdot$ defined by

$$x \cdot y = x\,.\,y \vee a \quad \text{for all} \quad x, y \in \uparrow(a)\,.$$

Moreover, if $a \in \hat{S} \cap E(S)$ then $\uparrow(a)$ is an m-semilattice (quantale).

Proof. It is immediate.

The following proposition is a generalization of the result of Banaschewski [2, 3].

1.12. Proposition. The following are equivalent:

(i) The Prime Ideal Theorem.

(ii) Any nontrivial compact complete m-semilattice such that its 2-sided elements constitute a complete sublattice has a prime element.

(iii) Any nontrivial compact complete weak m-semilattice such that its 2-sided elements constitute a complete sublattice has a prime element.

Proof. (i) $\Rightarrow$ (ii) Let K be a nontrivial compact complete m-semilattice, $\hat{K}$ a complete sublattice of K. Clearly, $\hat{K}$ is a compact complete m-semilattice. We can suppose that $\hat{K} \neq \{1\}$ i.e. the coproduct M of 2-sided m-semilattices $\uparrow(a) = \{x \in \hat{K}; a \leqq x\}$, $a \in \hat{K}$-$\{1\}$ is a nontrival m-semilattice M with coproduct maps h_a: $\uparrow(a) \to M$. For any $a \in \hat{K}$-$\{1\}$ there is an m-prime ideal $P_a = h_a^{-1}(P)$ of $\uparrow(a)$, P is an m-prime ideal of M. We may define a map σ: $\hat{K}$-$\{1\} \to \hat{K}$-$\{1\}$ putting $\sigma(a) = \bigvee P_a$. It is easy to check that the definition of σ is correct and the assumptions of Bourbaki's fix point lemma are satisfied. Now, we have that σ has a fix point, say c. Without any difficulties we see that c is prime in K.

(ii) $\Rightarrow$ (iii) Let K be a nontrivial compact complete weak m-semilattice, $\hat{K}$ complete sublattice of K. Clearly. $Q = \uparrow(a)$ is a nontrivial compact m-semilattice satisfying assumptions of (ii); here $a = 0 . 1 \in \hat{K} \cap E(K)$, $a \neq 1$. Since $\emptyset \neq P(Q) \subseteq P(K)$, we are ready.

(iii) $\Rightarrow$ (i) It is evident.

§ 2. Normality yields the Maximal Ideal Theorem

2.1. Definition. A weak m-semilattice S is said to be normal if, given $a, b \in S$ with $a \vee b = 1$, we can find $d, c \in S$ with $d . c = 0$, $d \vee a = 1 = b \vee c$.

The following result is well known for m-semilattices (see [10], Theorem 4.5).

2.2. Proposition. Let S be a weak m-semilattice. Then the following conditions are equivalent:

(i) S is normal.

(ii) $Id(S)$ is normal.

Note that 2-sided maximal means maximal with respect to the weak m-semilattice of all 2-sided elements.

2.3. Theorem. The following statements are equivalent:

(i) Any compact normal nontrivial (weak) quantale has a 2-sided maximal element.

(ii) Any compact normal nontrivial frame has a maximal element.

(iii) Any normal nontrivial (weak) m-semilattice has a 2-sided maximal ideal.

(iv) The Maximal Ideal Theorem for normal distributive lattices.

(v) Any compact normal nontrivial complete distributive lattice has a maximal element.

(vi) Any compact regular nontrivial frame has a maximal element i.e. it is spatial.

(vii) The Prime Ideal Theorem.

Proof. (i) $\Rightarrow$ (ii) It is evident.

(i) $\Rightarrow$ (iii) It follows immediately from 2.2.

(ii) $\Rightarrow$ (iv) Using 2.2 for distributive lattices.

(iii) $\Rightarrow$ (iv) It is evident.

(iv) $\Rightarrow$ (v) Let K be a compact normal nontrivial complete distributive lattice. Then there is a maximal ideal M of K. Clearly, by compactness of K we have that M is principal i.e. $M = \downarrow(m) = \{x \in K;\ x \leqq m\}$ for some $m \in M$. It is easy to check that m is a maximal element of K.

(v) $\Rightarrow$ (vi) Using the fact that any compact regular frame is normal (see [7]).

(vi) $\Rightarrow$ (vii) It is well known (see [7], [12]).

(vii) $\Rightarrow$ (i) Let K be a compact normal nontrivial (weak) quantale. Then K_c is exactly the compact regular coreflection of K i.e. there is a maximal element m of $\hat{K}$. because any prime element of a regular frame is maximal. The rest follows from the fact that there is not any 2-sided element $\neq 1$ which is greater than m.

Next we shall consider a particular class of weak m-semilattices (for distributive lattices see [8]).

2.4. Definition. We shall say that a weak m-semilattice S is semi-normal if, whenever $a \vee b = 1$, we can find elements $c, d \in S$ with $a \vee d = 1 = c \vee b$ and $\downarrow(d \,.\, c)\ C\ \{0\}$ in $Id(S)$. Clearly, any normal weak m-semilattice is semi-normal.

2.5. Proposition. Let K be a quantale so that C is a congruence on K. Then the following are equivalent:

(i) K is semi-normal.

(ii) $a \vee b = 1$ implies there are $c,\ d \in K$ such that $a \vee d = 1 = c \vee b$ and $d \,.\, c\ C\ 0$ in K.

Proof. It is enough to verify that $\downarrow(x)\ C\ \{0\}$ in $I\,d(K)$ if and only if $x\ C\ 0$ in K. But this is an immediate reformulation of 1.7.

2.6. Corollary. Let S be a weak m-semilattice. Then the following conditions are equivalent:

(i) S is semi-normal.

(ii) $Id(S)$ is semi-normal.

Proof. The proof is analogous to [10], Theorem 4.5.

2.7. Theorem. The following statements are equivalent:

(i) Any compact semi-normal nontrivial (weak) quantale has a 2-sided maximal element.

(ii) Any compact semi-normal nontrivial frame has a maximal element.

(iii) Any semi-normal nontrivial (weak) m-semilattice has a 2-sided maximal ideal.

(iv) The Maximal Ideal Theorem for semi-normal distributive lattices.

(v) Any compact semi-normal nontrivial complete distributive lattice has a maximal element.

(vi) Any compact normal nontrivial complete distributive lattice has a maximal element.

Proof. (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (vi) It is transparent.

(ii) $\Rightarrow$ (iv) Using 2.6 for distributive lattices.

(iv) $\Rightarrow$ (v) Let K be a compact semi-normal nontrivial complete distributive lattice. Then there is a maximal ideal M of K. Clearly, by compactness of K we have that M is principal i.e. $M = \downarrow(m) = \{x \in K;\ x \leqq m\}$ for some $m \in M$. It is easy to check that m is a maximal element of K.

(vi) $\Rightarrow$ (i) Let K be a nontrivial compact semi-normal (weak) quantale. Clearly, $Q = \uparrow(a)$ is a nontrivial compact normal weak quantale satisfying assumptions of 2.3(i); here $a = \bigvee\{x;\ x\ C\ 0\}$, $a \in \hat{K}$, $a \neq 1$. Since $\emptyset \neq D(\hat{Q}) \subseteq D(\hat{K})$ we are ready.

References

[1] Banaschewski B. and Harting R., Lattice aspects of radical ideals and choice principles, Proc. London Math. Soc. (3) 50 (1985), 385—404.

[2] Banaschewski B., Prime elements from prime ideals, Order 2 (1985), 211—213.

[3] Banaschewski B., More choice principles from the Prime Ideal Theorem, preprint.

[4] Borceux F., About quantales and quantic spaces, Sydney Category Seminar Reports (1984).

[5] Borceux F., Van den Bossche G., Quantales and their sheaves, Rapp. Sém. Math. Pure No. 63, Université Louvain (1985).

[6] Jech T., The Axiom of Choice, North Holland (1973).

[7] Johnstone P. T., Stone spaces, Cambridge University Press, Cambridge (1982).

[8] Johnstone P. T., Almost maximal ideals, Fund. Math. 123 (1984), 197—209.

[9] Johnstone P. T., Wallman compactification of locales, Houston J. Math., 10 (1984), 201—206.

[10] Paseka J., Regular and normal quantales, Arch. Math. (Brno), (4) 22 (1986), 203—210.

[11] Paseka J., Conjunctivity in quantales, Arch. Math. (Brno), (4) 24 (1988), 173—180.

[12] Paseka J., The Tychonoff product theorem for finite spaces implies the Boolean Ultrafilter Theorem. Journal of the University of Kuwait, (to appear).

[13] Simmons H., The lattice theoretical part of topological separation properties, Proc. Edinburgh Math. Soc. (2) 21 (1978), 41—48.

[14] Ward M., Dilworth R. P., Residuated lattices, Trans. Math. Soc. 45 (1939), 335—354.