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A Nearly Uniformly Convex Space which is not a (β)-Space

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An example is given of a nearly uniformly convex Banach space which is not a (β)-space. This answers a question of Rolewicz.

The Kuratowski measure of noncompactness of a set A in a Banach space X is the infimum $\alpha(A)$ of those $\varepsilon > 0$ for which there is a covering of A by a finite number of sets A_i with diam $(A_i) < \varepsilon$.

A norm $\|\cdot\|$ in a Banach space X is said to be Δ-uniformly convex (see [3] and [8]) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each convex set E contained in the closed unit ball with $\alpha(E) > \varepsilon$, we have $\inf\{\|x\|: x \in E\} < 1 - \delta$. This is equivalent to the notion of nearly uniform convexity of the norm (NUC), introduced by Huff [4].

Let $(X, \|\cdot\|)$ be a Banach space with closed unit ball B. By the drop $D(x, B)$ defined by an element $x \in X$, $x \notin B$, we mean the convex hull of the set $\{x\} \cup B$. Denote $R(x, B) = D(x, B) \setminus B$. The norm is called to satisfy condition (β) (cf. [8]) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\alpha(R(x, B)) < \varepsilon$. The space X is a (β)-space if it admits an equivalent norm which satisfies the condition (β).

Rolewicz [8] has proved that uniform convexity $\Rightarrow$ condition (β) $\Rightarrow$ (NUC) and he has posed the question about the converse implications up to renorming. In [6] we have proved that the class of (β)-spaces does not coincide with that of super-reflexive spaces (independently shown by Montesinos and Torregrosa [7]). In this paper we shall show that it does not coincide with the class of nearly uniformly convexifiable spaces, either.

Consider the example from [5] of a reflexive Banach space which does not admit an equivalent norm, uniformly differentiable in every direction. Let $\Gamma = \prod_{i=2}^{\infty} \{1, 2, \ldots, i\}$. That is, Γ is the family of all sequences $\gamma = \{\gamma^i\}_{i=1}^{\infty}$ of positive integers such that $1 \leqq \gamma^i \leqq i + 1$. Denote by Φ_Γ the family of all finite subsets of Γ which have the

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property that, if $A \in \Phi_\Gamma$, then there is a positive integer m such that, if $\gamma_k = \{\gamma_k^i\}_{i=1}^\infty$ and $\gamma_j = \{\gamma_j^i\}_{i=1}^\infty$ are different members of A, then $\gamma_k^m \neq \gamma_j^m$ and $\gamma_k^i = \gamma_j^i$ for $1 \leqq i \leqq$ $\leqq m - 1$. We denote by X the space of all real – valued functions x on Γ such that

$$\|x\| = \sup \{[\sum_{n \in N} (\sum_{\gamma \in A_n} |x(\gamma)|)^2]^{1/2}\} < \infty ,$$

where the supremum is taken over all finite systems $\{A_n\}_{n \in N}$ with each $A_n \in \Phi_\Gamma$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

It is shown in [6] that the space X is Δ-uniformly convex. The proof of the following statement is inspired by ideas of Day [2].

Theorem. The space X is not a (β) – space.

Proof. Suppose the contrary, i.e. there exists an equivalent norm $|\cdot|$ in X which satisfies the condition (β). We may assume without loss of generality that

$$(1) \qquad \|x\| \leqq |x| \leqq M\|x\| \quad \text{for every} \quad x \in X ,$$

where $1 \leqq M < \infty$.

Put $\varepsilon = 1/M$.

Denote by B_1 the closed unit ball with respect to the norm $|\cdot|$. By the assumption, $|\cdot|$ satisfies (β), thus we may choose and fix a $\delta > 0$ so that

$$(2) \qquad x \in X , \quad 1 < |x| < 1 + 2\delta \quad \text{imply} \quad \alpha(R(x, B_1)) < \varepsilon/2 .$$

Fix n large enough so that

$$(3) \qquad \varepsilon(1 + \delta/2)^n > 1 .$$

For $\gamma = \{\gamma^i\}_{i=1}^\infty \in \Gamma$ let $\pi_j(\gamma) = \gamma^j$, $j = 1, 2, \ldots$.

First step. Define the following sets in X

$T_i^{(0)} = \{\varepsilon\chi_{\{\gamma\}}: \gamma \in \Gamma,\ \pi_j(\gamma) = 1 \text{ for } 1 \leqq j < 2^n - 1 \text{ and } \pi_j(\gamma) = i, \text{ for } j = 2^n - 1\}$,
where $1 \leqq i \leqq 2^n$.

Consider a couple $(T_{2k-1}^{(0)}, T_{2k}^{(0)})$, $1 \leqq k \leqq 2^{n-1}$. Choose and fix an arbitrary $t' \in T_{2k-1}^{(0)}$. Put $x = (1 + \delta)\, t'$. We have that $\|x\| = \varepsilon(1 + \delta)$, whence $|x| \leqq 1 + \delta$.

First case. $|x| > 1$.

Let $y \in T_{2k}^{(0)}$ be arbitrary. Obviously, $\|y\| = \varepsilon$ and thus $|y| \leqq 1$. Then, $(x + y)/2 \in$ $\in D(x, B_1)$. For different elements we have

$$|(x + y_1)/2 - (x + y_2)/2| = |y_1 - y_2|/2 \geqq \|y_1 - y_2\|/2 > \varepsilon/2 .$$

On the other hand, it follows from (2) that $\alpha(x, B_1) < \varepsilon/2$. Therefore, the relation $(x + y)/2 \in R(x, B_1)$ is fulfilled for at most finite subset of $T_{2k}^{(0)}$. Since the set $T_{2k}^{(0)}$ is infinite, we obtain that for infinitely many $y \in T_{2k}^{(0)}$ the following holds

$$|(x + y)/2| \leqq 1 .$$

Moreover, $\|(x + y)/2\| = \varepsilon(1 + \delta/2)$.

Second case. $|x| \leqq 1$. Then the element x itself will do but for the sake of uniformity we shall consider again $(x + y)/2$ for arbitrary $y \in T_{2k}^{(0)}$. Clearly, $|(x + y)/2| \leqq 1$ and

$$\|(x + y)/2\| = \varepsilon(1 + \delta/2).$$

Now in both cases, let us vary the element $t' \in T_{2k-1}^{(0)}$. It is easy to observe that we may find an infinite set $T_k^{(1)}$ of elements $t = (x + y)/2$ of the form above with $|t| \leqq 1$ so that the conditions $t_1, t_2 \in T_k^{(1)}$, $t_1 \neq t_2$ imply $\operatorname{supp} t_1 \cap \operatorname{supp} t_2 = \emptyset$ and hence $\|t_1 - t_2\| > \varepsilon(1 + \delta/2) > \varepsilon$.

Second step. Similarly, consider $(T_{2k-1}^{(1)}, T_{2k}^{(1)})$, $1 \leqq k \leqq 2^{n-2}$. Fix an arbitrary $t' \in T_{2k-1}^{(1)}$ and put $x = (1 + \delta)\, t'$. We have that

$$\|x\| = \varepsilon(1 + \delta/2)(1 + \delta) \quad \text{and} \quad |x| \leqq 1 + \delta.$$

Consider the case $|x| > 1$. Let $y \in T_{2k}^{(1)}$ be arbitrary. We have $\|y\| = \varepsilon(1 + \delta/2)$ and $|y| \leqq 1$. Since $\operatorname{supp} x \cup \operatorname{supp} y \in \Phi_\Gamma$, then $\|x + y\| = \varepsilon(1 + \delta/2)(2 + \delta)$, i.e.

$$\|(x + y)|2\| = \varepsilon(1 + \delta/2)^2.$$

According to the choice of $T_i^{(1)}$, $1 \leqq i \leqq 2^{n-1}$, we have that for different elements $y_1, y_2 \in T_{2k}^{(1)}$ the inequality $\|y_1 - y_2\| > \varepsilon$ holds, which gives

$$|(x + y_1)/2 - (x + y_2)/2| > \varepsilon/2.$$

As in the first step, it follows from (2) that for infinitely many $y \in T_{2k}^{(1)}$ it is true that

$$|(x + y)/2| \leqq 1.$$

If $|x| \leqq 1$, we have again for each $y \in T_{2k}^{(1)}$,

$$|(x + y)/2| \leqq 1 \quad \text{and} \quad \|(x + y)/2\| = \varepsilon(1 + \delta/2)^2.$$

Then, we vary $t' \in T_{2k-1}^{(1)}$. Find an infinite set $T_k^{(2)}$ of elements $t = (x + y)/2$ with $|t| \leqq 1$ so that different members $t_1, t_2 \in T_k^{(2)}$ have disjoint supports and hence $\|t_1 - t_2\| > \varepsilon(1 + \delta/2)^2 > \varepsilon$.

We can repeat $n - 2$ more times and thus we shall obtain a non-void set $T_1^{(n)}$ such that for $t \in T_1^{(n)}$,

$$|t| \leqq 1 \quad \text{and} \quad \|t\| = \varepsilon(1 + \delta/2)^n.$$

By (1) and (3), this is a contradiction, which completes the proof.

Remark. We may also give a separable example of a (NUC)-space which is not a (β)-space. Consider the example of A. Baernstein [1] of a reflexive Banach space which does not possess the Banach-Saks property. In order to show that it is not a (β)-space, we can use the same argument. More precisely, represent the set of all integers, greater than 2^n, as a union of 2^n disjoint infinite subsets and procede as above. In [6] we have proved that this space is (NUC).

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