Acta Universitatis Carolinae. Mathematica et Physica

Marián J. Fabián

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 51--56

Persistent URL: `http://dml.cz/dmlcz/701793`

Subdifferentiability and Trustworthiness in the Light of a New Variational Principle of Borwein and Preiss

MARIAN FABIAN

Prague

Received 15 March 1989

It was shown by N. V. Zhivkov and the author that every lower semicontinuous function on an Asplund space is ε-subdifferentiable at the points of a dense set of its domain, where for ε it can be taken an arbitrarily small positive number. Here we show that the same holds also with $\varepsilon = 0$. Previous results of the author on trustworthiness can be improved in a similar way. In proofs a new variational principle of Borwein and Preiss is used.

Let $(X, \|\cdot\|)$ be a Banach space with dual X^*, $f: X \to (-\infty, +\infty]$ a function, $x \in X$, with $f(x) < +\infty$, and $\varepsilon \geqq 0$. We define [7]

$$\Phi_\varepsilon^- f(x) = \{\zeta \in X^* : \liminf_{\|h\| \to 0} [f(x + u) - f(x) - \langle \zeta, h \rangle]/\|h\| \geqq -\varepsilon\},$$

$$\partial_\varepsilon^- f(x) = \{\zeta \in X^* : \liminf_{\substack{u \to h \\ t \downarrow 0}} [f(x + h) - f(x)]/t \geqq \langle \zeta, h \rangle - \varepsilon \|h\|$$

$$\text{for all} \quad h \in X\},$$

where $\langle \zeta, h \rangle$ means the value of ζ at h. If $\Phi_\varepsilon^- f(x)$ $(\partial_\varepsilon^- f(x))$ is nonempty we say that f is *Fréchet* (*Dini*) *ε-subdifferentiable* at x. If $\varepsilon = 0$ we simple speak about *Fréchet* (*Dini*) *subdifferentiability*.

The papers [6], [4], [5] deal with ε-subdifferentiability for $\varepsilon > 0$. The proofs are based, among other things, on the Ekeland's variational principle [3, Theorem 1], which roughly and incompletely sounds as: *Every lower semicontinuous function bounded from below is supported by a shift of the function $h \mapsto -\varepsilon\|h\|$.*

In the meantime there has appeared a very interesting smooth variational principle due to Borwein and Preiss [1], [8, Theorem 4.3]:

Theorem 0. (Borwein, Preiss). *Let $g: X \to (-\infty, +\infty]$ be a lower semicontinuous function bounded from below, let $\varepsilon > 0$, $\lambda > 0$ be given, and take $x_0 \in X$ such that*

$$g(x_0) < \inf g + \varepsilon .$$

*) Sibeliova 49, 162 00 Praha 6, Czechoslovakia

Then there exist sequences $\{\mu_n\}$, *with* $\mu_n \geqq 0$, $\mu_1 + \mu_2 + \ldots = 1$, *and* $\{v_n\} \subset X$, *with* $v_n \to v \in X$. *such that*

$$g(x) + \frac{\varepsilon}{\lambda^2} \sum_{n=1}^{\infty} \mu_n \|x - v_n\|^2 \geqq g(v) + \frac{\varepsilon}{\lambda^2} \sum_{n=1}^{\infty} \mu_n \|v - v_n\|^2 \quad \text{for all} \quad x \in X$$

and

$$\|x_0 - v\| < \lambda\,, \quad g(v) < \inf g + \varepsilon\,.$$

Moreover, if X admits an equivalent Fréchet (Gateaux) differentiable norm, then $\Phi_0^- g(v)$ $(\partial_0^- g(v))$ *is nonepty and contains a* ζ *with* $\|\zeta\| \leqq 2\varepsilon/\lambda$.

In [1] there are mentioned some easy consequences of this result. Let us recall one, perhaps the most important, of them:

Corollary 0. *If X has a Fréchet (Gateaux) differentiable norm, then it is an* S_0 – *space (a* WS – *space), that is, for every lower semicontinuous function* $f: X \to (-\infty, +\infty]$ *the set of points* $(v, f(v))$ *where* $\Phi_0^-(v)$ $(\partial_0^-(f(v))$ *is nonempty is dense in the graph* $\{(v, f(v)) : v \in X, f(v) < +\infty\}$ *of f.*

Having such a nice variational principle, theorems from [6] [4] and [5] call up immediately to an improvement. In fact, ε-subdifferentiability can be replaced by subdifferentiability. In this note we will formulate strenghtened versions of these results and provide sketchs of proofs.

Theorem 1. *A Banach space is Asplund (if and) only if it is an* S_0 – *space.*

If X is separable Asplund, then it admits an equivalent Frechet differentiable norm [2, p. 118] and so Corollary 0 applies. For the proof in the case of a non-separable Asplund space we need a separable reduction formulated in the next

Lemma 1. *Let* Y_0 *be a separable subspace of* X, *let* $f: X \to (-\infty, +\infty]$ *be a function locally bounded from below and let* $\varepsilon \geqq 0$.

Then there exists a separable subspace $Y_0 \subset Y \subset X$ *such that* $\Phi_\varepsilon^- f(x) \neq \emptyset$ *whenever* $x \in Y$ *and* $\Phi_\varepsilon^-(f_{|Y})(x) \neq \emptyset$, *where* $f_{|Y}$ *denotes the restriction of f to the subspace Y.*

The proof proceeds similarly like in [6] (where $\varepsilon > 0$) when replacing [6, Lemma] by a more general:

Lemma 2. *Let* $f: X \to (-\infty, +\infty]$ *be a function,* $x \in \operatorname{dom} f$, *and* $\varepsilon \geqq 0$.

Then $\Phi_\varepsilon^- f(x) \neq \emptyset$ *if and only if there are* $c \geqq 0$ *and a sequence* $\{\delta_j\}$ *of positive numbers such that*

$$\sum_{j=1}^{m} \beta_j \sum_{l=1}^{k_j} \alpha_{jl} \left[f(x + h_{jl}) + \left(\varepsilon + \frac{1}{j}\right) \|h_{jl}\| \right] \geqq f(x) - c \left\| \sum_{j=1}^{m} \beta_{jl} \sum_{l=1}^{k_j} \alpha_{jl} h_{jl} \right\|$$

whenever $h_{jl} \in \delta_j B_X$, $\alpha_{jl} \geqq 0$, $l = 1, \ldots, k_j$, $\alpha_{j1} + \ldots + \alpha_{jk_j} = 1$, $k_j = 1, 2, \ldots$, $\beta_j \geqq 0$, $j = 1, \ldots, m$, $\beta_1 + \ldots + \beta_m = 1$, $m = 1, 2, \ldots$.

Here B_X means the unit ball in X. The proof is omitted because Lemma 2 is in fact a consequence of Lemma 6. It should be noted that the most important case is when $\varepsilon = 0$. And since [6, Lemma] does not cover this case we needed the new lemma.

Next we approach a *trustworthiness*. This concept means that a very rough (fuzzy) analogy of a Moreau-Rockafellar theorem [8, Theorem 3.23] holds. It proves that the trustworthiness depends on the properties of the space in question; more precisely:

Theorem 2. *A Banach space X is an S_0 – space (a WS_0 – space) [if and] only if it is trustworthy (weak trustworthy) in the sense that for any two lower semicontinuous functions f_1, f_2: $X \to (-\infty, +\infty]$, for any $z \in X$, any $\varepsilon \geqq 0$, $\delta > 0$, and any weak* neighbourhood V of the origin in X^* the following inclusion holds*

$$\Phi_\varepsilon^-(f_1 + f_2)(z) \subset \bigcup\{\Phi_0^- f_1(z_1) + \Phi_0^- f_2(z_2):\ z_i \in X\,,\ \ \|z_i - z\| < \delta\,,$$
$$|f_i(z_i) - f_i(z)| < \delta\,,\quad i = 1, 2\} + \varepsilon B_X + V$$

(the same inclusion with Φ_ε^-, Φ_0^- replaced by ∂_ε^-, ∂_0^- respectively).

The proof proceedes like that of [5, Theorem 1]; the only difference is that [5, Lemma 2] should now be replaced by

Lemma 3. *Let X be an S_0 – space (a WS_0 – space), let f_1, f_2: $X \to (-\infty, \infty]$ be two functions such that their sum $f_1 + f_2$ attains sharp local minimum at some $z \in X$ and let $\delta > 0$ be given. Suppose moreover that the function f_2 is compact near z in the sense that the sets $\{x \in z + \delta B_X: f_2(x) \leqq t\}$ are norm compact for all real t.*

Then there exist $z_1, z_2 \in z + \delta B_X$, with $|f_j(z_j) - f_j(z)| < \delta$, $j = 1, 2$, such that

$$0 \in \Phi_0^- f_1(z_1) + \Phi_0^- f_2(z_2) \quad (0 \in \partial_0^- f_1(z_1) + \partial_0^- f_2(z_2))\,.$$

The proof is almost identical with that of [5, Lemma 2]. The only substantial change is that the convolution f of f_1 and f_2 is now defined by

$$f(x) = \begin{cases} \inf\{f_1(x + y) + f_2(z + y):\ y \in \delta B_X\} & \text{if}\quad x \in z \in \delta B_X \\ +\infty \quad \text{otherwise}\,. \end{cases}$$

An analogy of [4, Theorem 4], see also [5, Theorem 2], exists too:

Theorem 3. *X is an Asplund space (if and) only if it is trustworthy in the following sense: for any $\varepsilon \geqq 0$, $\delta > 0$, $\gamma > 0$, for any functions $f_1, \ldots, f_n$: $X \to$ $\to (-\infty, +\infty]$, $n \geqq 2$, and for any $z \in X$ such that f_1 is lower semicontinuous and $f_2, \ldots, f_n$ are Lipschitz in a neighbourhood of z the following inclusion holds*

$$\Phi_\varepsilon^-(f_1 + \ldots + f_n)(z) \subset \bigcup\{\Phi_0^- f_1(z_1) + \ldots + \Phi_0^- f_n(z_n):\ z_j \in z + \delta B_X\,,$$
$$|f_j(z_j) - f_j(z)| < \delta\,,\quad j = 1, \ldots, n\} + (\varepsilon + \gamma) B_X\,.$$

Corollary 1. *If X is Asplund, f: $X \to (-\infty, +\infty]$ lower semicontinuous, $z \in X$, and $\varepsilon \geqq 0$, $\delta > 0$, $\gamma > 0$, then*

$$\Phi_\varepsilon^- f(z) \subset \bigcup\{\Phi_0^- f(x): x \in z + \delta B_X,\ |f(x) - f(z)| < \delta\} + (\varepsilon + \gamma) B_X\,.$$

Theorem 3 can be easily obtained, see the proof of [5, Theorem 2], from the next

Lemma 4. *Let X be an Asplund space, let $\delta, \gamma > 0$ and let $f_1, \dots, f_n: X \to (-\infty, +\infty]$, $n \geqq 2$, be functions such that f_1 is lower semicontinuous and $f_2, \dots, f_n$ are Lipschitz in a neighbourhood of some $z \in X$. Finally assume that the sum $f_1 + \dots + f_n$ attains local minimum at z.*

Then there are $z_j + z + \delta B_X$, with $|f_j(z_j) - f_j(z)| < \delta$, $j = 1, \dots, n$, such that

$$0 \in \Phi_0^- f_1(z_1) + \dots + \Phi_0^- f_n(z_n) + \gamma B_X .$$

However we have not succeeded in proving this lemma by imitating the way used in the proof of [5, Lemma 3]. So we proceed like in [7] and [4]. First we consider X with Fréchet differentiable norm. Then a method from [7, Lemma 2] can be adapted when replacing the Ekeland's principle by the principle of Borwein-Preiss. Second, remarking that a separable Asplund space admits an equivalent Fréchet differentiable norm, we can obtain the statement of Lemma 4 with help of a separable reduction formulated in the following.

Lemma 5. *Let Y_0 be a separable subspace of X, let $f_1, \dots, f_n: X \to (-\infty, +\infty]$ be functions locally bounded from below and let $\varepsilon_1, \dots, \varepsilon_n \geqq 0$ be given.*

Then there exists a separable subspace $Y_0 \subset Y \subset X$ such that

$$0 \in \Phi_{\varepsilon_1}^- f_1(x_1) + \dots + \Phi_{\varepsilon_n}^- f_n(x_n)$$

whenever $x_1, \dots, x_n \in Y$ and

$$0 \in \Phi_{\varepsilon_1}^-(f_{1|Y})(x_1) + \dots + \Phi_{\varepsilon_n}^-(f_{n|Y})(x_n) .$$

The proof is an elaboration of that of [4, Theorem 2] (where $\varepsilon_i > 0$) in the sense that [4, Lemma 4] should now be replaced by

Lemma 6. *Let $f_i: X \to (-\infty, +\infty]$ be functions, $x_i \in \operatorname{dom} f_i$, and $\varepsilon_i \geqq 0$, $i = 1, \dots, n$. Then*

$$0 \in \Phi_{\varepsilon_1}^- f_1(x_1) + \dots + \Phi_{\varepsilon_n}^- f_n(x_n) \tag{1}$$

if and only if there are $c \geqq 0$ and sequences $\{\delta_{1j}\}, \dots, \{\delta_{nj}\}$ of positive numbers such that

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m_i} \beta_{ij} \sum_{l=1}^{k_{ij}} \alpha_{ijl} \left[f_i(x_i + h_{ijl}) + \left(\varepsilon_i + \frac{1}{j}\right) \|h_{ijl}\| \right] \geqq \\ \displaystyle\geqq \sum_{i=1}^{n} f_i(x_i) - c \sum_{i=1}^{n-1} \left\| \sum_{j=1}^{m_i} \beta_{ij} \sum_{l=1}^{k_{ij}} \alpha_{ijl} h_{ijl} - \sum_{j=1}^{m_n} \beta_{nj} \sum_{l=1}^{k_{nj}} \alpha_{njl} h_{njl} \right\| \end{cases} \tag{2}$$

whenever $h_{ijl} \in \delta_{ij} B_X$, $\alpha_{ijl} \geqq 0$, $l = 1, \dots, k_{ij}$, $\alpha_{ij1} + \dots + \alpha_{ijk_{ij}} = 1$, $k_{ij} = 1, 2, \dots$, $\beta_{ij} \geqq 0$, $j = 1, \dots, m_i$, $\beta_{i1} + \dots + \beta_{im_i} = 1$, $m_i = 1, 2, \dots$, $i = 1, \dots, n$.

Proof. *Necessity*. Let (1) hold. Let $\zeta_i \in \Phi_{\varepsilon_i}^- f_i(x_i)$ be such that $\zeta_1 + \ldots + \zeta_n = 0$ For $i = 1, \ldots, n$ find sequences $\{\delta_{ij}\}$ of positive numbers such that

$$f_i(x_i + h) - f_i(x_i) \geqq \langle \zeta_i, h \rangle - \left(\varepsilon_i + \frac{1}{j}\right) \|h\|$$

whenever $h \in \delta_{ij} B_X$. Then for any $h_{ijl}, \alpha_{ijl}, \beta_{ij}, k_{ij}, m_i$ as in Lemma the left hand side in (2) is

$$\geqq \sum_{i=1}^{n} \sum_{j=1}^{m_i} \beta_{ij} \sum_{l=1}^{k_{ij}} \alpha_{ijl} [f_i(x_i) + \langle \zeta_i, h_{ijl} \rangle] =$$

$$= \sum_{i=1}^{n} f_i(x_i) - \sum_{i=1}^{n-1} \langle \zeta_i, \sum_{j=1}^{m_i} \beta_{ij} \sum_{l=1}^{k_{ij}} \alpha_{ijl} h_{ijl} - \sum_{j=1}^{m_n} \beta_{nj} \sum_{l=1}^{k_{nj}} \alpha_{njl} h_{njl} \rangle \geqq$$

the right side in (2) where $c = \max \{\|\zeta_1\|, \ldots, \|\zeta_{n-1}\|\}$.

Sufficiency. For $i = 1, \ldots, n$ and $j = 1. 2, \ldots$ we define the functions $\phi_{ij}: X \to \to (-\infty, +\infty]$ by

$$\phi_{ij}(h) = \inf \left\{ \sum_{l=1}^{k} \alpha_l \left[f_i(x_i + h_l) + \left(\varepsilon_i + \frac{1}{j}\right) \|h_l\| \right] : h_l \in \delta_{ij} B_X \,, \right.$$

$$\left. \alpha_l \geqq 0, \; l = 1, \ldots, k, \; \alpha_1 + \ldots + \alpha_k = 1, \; \alpha_1 h_1 + \ldots + \alpha_k h_k = h, \; k = 1, 2, \ldots \right\}$$

$$\text{if} \quad h \in \delta_{ij} B_X \,,$$

$\phi_{ij}(h) = +\infty$ otherwise.

Clearly ϕ_{ij} are proper convex functions. It follows from (2) that

$$\phi_{ij}(0) \leqq f_i(x_i) \leqq \phi_{ij}(0)$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} \beta_{ij} \, \phi_{ij}(h_{ij}) \geqq \sum_{i=1}^{n} f_i(x_i) - c \sum_{i=1}^{n-1} \left\| \sum_{j=1}^{m_i} \beta_{ij} h_{ij} - \sum_{j=1}^{m_n} \beta_{nj} h_{nj} \right\|$$

for all $h_{ij} \in X$, $j = 1, \ldots, m_i$, $m_i = 1, 2, \ldots$, $i = 1, \ldots, n$. Further for $i = 1, \ldots, m$ define $\phi_i: X \to (-\infty, \infty]$ by

$$\phi_i(h) = \inf \{ \sum_{j=1}^{m} \beta_j \, \phi_{ij}(h_j) : h_j \in X, \; \beta_j \geqq 0, \; j = 1, \ldots, m \,,$$

$$\beta_1 + \ldots + \beta_m = 1, \; \beta_1 h_1 + \ldots + \beta_m h_m = h, \; m = 1, 2, \ldots \} \,, \quad h \in X \,.$$

Then the last inequality yields $\phi_i(0) = f_i(x_i)$ and

$$\sum_{i=1}^{n} \phi_i(h_i) \geqq \sum_{i=1}^{n} \phi_i(0)) - c \sum_{i=1}^{n-1} \|h_i - h_n\|$$

for all $h_1, \ldots, h_n \in X$. It follows by [4, Lemma 2] that

$$0 \in \partial \phi_1(0) + \ldots + \partial \phi_n(0) \,.$$

Here $\partial\phi_i$ means the usual subdifferential of ϕ_i known from convex analysis. Now in order to show (1) it suffices to remark that

$$\partial\phi_i(0) \subset \Phi_{\varepsilon_i}^{-} f_i(x_i) .$$

So let $\zeta_i \in \partial\phi_i(0)$. Then for $h \in \delta_{ij} B_X$ by the definition of ϕ_{ij} and ϕ_i we have

$$f_i(x_i + h) + \left(\varepsilon_i + \frac{1}{j}\right) \|h\| \geqq \phi_{ij}(h) \geqq \phi_i(h) \geqq$$

$$\geqq \phi_i(0) + \langle \zeta_i, h \rangle = f_i(x_i) + \langle \zeta_i, h \rangle$$

and hence

$$\liminf_{\|h\| \to 0} [f_i(x_i + h) - f_i(x_i) - \langle \zeta_i, h \rangle]/\|h\| \geqq -\left(\varepsilon_i + \frac{1}{j}\right)$$

for all $j = 1, 2, \dots$. Therefore ζ_i lies in $\Phi_{\varepsilon_i}^{-} f_i(x_i)$.

Remark. We confess that new ideas in this note can be found only in Lemma 2, eventually in Lemma 6.

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