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An Application of Set-Pair Systems for Multitransversals

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Let $\mathbf{H}$ be a hypergraph (= finite set system) on an underlying set X, and let k be a natural number. Using the definition of [4], a set $Y \subseteq X$ is called a *k-transversal set* of $\mathbf{H}$ if $|Y \cap H| \geqq k$ for all $H \in \mathbf{H}$, $|H| \geqq k$, and $H \subseteq Y$ for $H \in \mathbf{H}$, $|H| \leqq k$. (Hence, a 1-transversal set is a transversal in the sense of Berge's [1].) Define the *k-transversal number* $\tau_k(\mathbf{H})$ of $\mathbf{H}$ as the minimum cardinality of a k-transversal set in $\mathbf{H}$.

It is well-known that from an algorithmic point of view, finding $\tau_1(\mathbf{H})$ belongs to the 'hard' problems even on the class of graphs (i.e. when the $\mathbf{H}$ are supposed to be 2-uniform); that is, a polynomial algorithm exists if and only if P = NP. Let us choose now a $(k - 1)$-element set Z, $Z \cap X = \emptyset$. For every graph G we can define a $(k + 1)$-uniform hypergraph $G + Z$ whose edges are of the form $e \cup Z$, where e is an edge of G. Then $\tau_k(G + Z) = \tau_1(G) + k - 1$. Thus, the following result holds.

Theorem 1. For every natural number k, it is NP-complete to determine the k-transversal number of $(k + 1)$-uniform hypergraphs. □

We note that the same statement is valid for the class of r-uniform hypergraphs whenever $r \geqq k + 1$. (For larger r, the edges should be completed by adding distinct vertices.) For $r \leqq k$, however, the k-transversal number is equal to the number of non-isolated vertices, so that it is trivial to compute $\tau_k(\mathbf{H})$ in this case.

Similarly to other 'hard' parameters (like stability number, chromatic number, matching number etc.), let us introduce the notion of critical structures. Call $\mathbf{H}$ *k-transversal critical* if $\tau_k(\mathbf{H} \setminus \{H\}) < \tau_k(\mathbf{H})$ for each $H \in \mathbf{H}$.

We say that $\mathbf{H}$ has *rank* r if $|H| \leqq r$ for all $H \in \mathbf{H}$. The number of edges in $\mathbf{H}$ is denoted by $|\mathbf{H}|$.

The following result generalizes the classical theorem of Jaeger and Payan [3] who considered the case $k = 1$.

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Theorem 2. If $\boldsymbol{H}$ is a k-transversal critical hypergraph of rank r with $\tau_k(\boldsymbol{H}) = t$ $(r \geqq k,\ t \geqq k)$, then $|\boldsymbol{H}| \leqq \binom{r+t+1-2k}{r+1-k}$. This bound is sharp for every r, k and t.

Proof. Let $|X| = r + t - k$, $|W| = k - 1$, $W \subset X$. Define $\boldsymbol{H}$ as the collection of all r-element subsets of X that contain W. Hence, $|\boldsymbol{H}| = \binom{r+t+1-2k}{r+1-k}$. It is easily seen that $\tau_k(\boldsymbol{H}) = t$ and $\boldsymbol{H}$ is k-transversal critical.

To prove the upper bound, let $\boldsymbol{H}$ be a k-transversal critical hypergraph of rank r with $\tau_k(\boldsymbol{H}) = t$. Say, $\boldsymbol{H} = \{H_1, H_2, \ldots, H_m\}$. For every i, $1 \leqq i \leqq m$, we have a k-transversal set Y_i of $\boldsymbol{H} \setminus \{H_i\}$ with $|Y_i| \leqq t - 1$, since $\boldsymbol{H}$ is critical. Then the pairs (H_i, Y_i) satisfy the following two requirements:

$$|H_i \cap Y_i| \leqq k - 1 \quad \text{for} \quad 1 \leqq i \leqq m\,,$$

$$|H_i \cap Y_j| \geqq k \quad \text{for} \quad i \neq j\,, \quad 1 \leqq i, j \leqq m\,.$$

(The first property follows by $\tau_k(\boldsymbol{H}) > |Y_i|$.) Since $|H_i| \leqq r$ and $|Y_i| \leqq t - 1$, a theorem of Füredi [2] implies that the number $m = |\boldsymbol{H}|$ of those pairs cannot exceed $\binom{r+(t-1)-2(k-1)}{r-(k-1)}$. □

The following (equivalent) formulation of Theorem 2 provides a more convenient sufficient condition for set systems having a small k-transversal number.

Theorem 3. Let $\boldsymbol{H}$ be a hypergraph of rank r. If for every $\boldsymbol{H}' \subseteq \boldsymbol{H}$ with $|\boldsymbol{H}'| \leqq \binom{r+t+2-2k}{r+1-k}$ we have $\tau_k(\boldsymbol{H}') \leqq t$, then $\tau_k(\boldsymbol{H}) \leqq t$.

Proof. Suppose that the assumptions hold for $\boldsymbol{H}$, and choose a *minimal* $\boldsymbol{H}' \subseteq \boldsymbol{H}$ with $\tau_k(\boldsymbol{H}') > t$. Then $\boldsymbol{H}'$ is k-transversal critical with $\tau_k(\boldsymbol{H}') = t + 1$. By Theorem 2, $|\boldsymbol{H}'| \leqq \binom{r+t+2-2k}{r+1-k}$, so that $\tau_k(\boldsymbol{H}') \leqq t$ should hold — a contradiction. □

We note that Theorem 3 does not provide a fast algorithm for finding $\tau_k(\boldsymbol{H})$. Although we can list all subhypergraphs $\boldsymbol{H}'$ having $\binom{r+t+2-2k}{r+1-k}$ edges, it remains NP-complete to decide whether or not $\tau_k(\boldsymbol{H}') \leqq t$.

We mention that 2-transversal critical *graphs* have a very simple structure; namely, all of their connected components are stars. More generally, if a hypergraph of rank r is r-transversal critical, then none of its edges is contained in the union of the others. (This property is not only necessary but also sufficient.)

References

[1] Berge C., "Graphs and Hypergraphs", North-Holland, 1973.

[2] Füredi Z., Geometrical solution of an intersection problem for two hypergraphs, Europ. J. Combinatorics **5** (1984) 133—136.

[3] Jaeger F. and Payan C., Nombre maximal d'arêtes d'un hypergraphe τ-critique de rang h, C. R. Acad. Sci. Paris **273** (1971) 221—223.

[4] Tuza Zs, Critical hypergraphs and intersecting set-pair systems, J. Combinatorial Theory Ser. B **39** (1985) 134—145.