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A family $\mathcal{D}$ of sets is an ω*-cover* of a set X if, for every finite set $F \subseteq X$, there is a set $D \in \mathcal{D}$ such that $F \subseteq D$. A topological space X has the *Gerlits-Nagy property* (abbreviated *GNP*) if, for every open ω-cover $\mathcal{D}$ of X, there is a sequence $\langle D_n : n \in \omega \rangle \in {}^{\omega}\mathcal{D}$ such that $X \subseteq \bigcup_{m<\omega} \bigcap_{m \leq n < \omega} D_n$.

Gerlits and Nagy [1] introduced the GNP (which they called "property γ") and proved that the following are equivalent for every completely regular space X:

(1) X has the GNP ;

(2) $C(X)$ is a Frèchet-Urysohn space;

(3) $C(X)$ is countably bisequential;

(4) $C(X)$ is a w-space.

Here $C(X)$ is the space of all continuous real-valued functions defined on X, with the topology of pointwise convergence. The notion of a w-space is due to Gruenhage [2]; a simple characterization of w-spaces was found by Sharma [5].

Let R be the real line. It was shown in [1] that every subset of R, which has the GNP, also has Rothberger's [4] property C'' and is always of the first category; hence in Laver's [3] model, the only subsets of R having the GNP are the countable sets. On the other hand, it was shown in [1] that, assuming $MA_\varkappa$, every subset of R of cardinality $\varkappa$ has the GNP; moreover, A.Hajnal has shown that the existence of an uncountable subset of R with the GNP is consistent with $ZFC+GCH$. The following theorem improves Hajnal's result:

Theorem. Assuming CH, there is a set $X \subseteq R$ such that $|X| = 2^{\aleph_0}$ and X has the GNP.

As the GNP is clearly preserved by continuous mappings, we have the following corollary, which answers a question of Sierpiński [6, p.86]:

Corollary. Assuming CH, there is a set $X \subseteq R$ of cardinality $2^{\aleph_0}$ such that every continuous image of X has property C'' and is always of the first category.

Problem. Can CH be replaced by MA in the theorem above, or in the corollary?

REFERENCES

1. J. Gerlits and Zs. Nagy, Some properties of $C(X)$, to appear.

2. Gary Gruenhage, Infinite games and generalizations of first-countable spaces, General Topology and Appl. 6(1976), 339-352.

3. Richard Laver, On the consistency of Borel's conjecture, Acta Math. 137 (1976), 151-169.

4. Fritz Rothberger, Sur les familles indènombrables de suites de nombres naturels et les problèmes concernant la propriètè C, Proc.Cambridge Philos. Soc. 37(1941), 109-126.

5. P.L.Sharma, Some characterizations of W-spaces and w-spaces, General Topology and Appl. 9(1978), 289-293.

6. Wacław Sierpiński, Hypothèse du continu, Monografie Matematyczne, Vol.4, Warszawa-Lwów, 1934.