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MULTIPARAMETRIC OSCILLATOR HAMILTONIANS WITH EXACT BOUND STATES IN INFINITE-DIMENSIONAL SPACE

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ABSTRACT. Bound states in quantum mechanics must almost always be constructed numerically. One of the best known exceptions concerns the central D−dimensional (often called "anharmonic") Hamiltonian $H = p^2 + a\,|\vec{r}|^2 + b\,|\vec{r}|^4 + \ldots + z\,|\vec{r}|^{4q+2}$ (where $z = 1$) with a complete and elementary solvability at $q = 0$ (central harmonic oscillator, no free parameters) and with an incomplete, N−level elementary analytic solvability at $q = 1$ (so called "quasi-exact" sextic oscillator containing one free parameter). In the limit $D \to \infty$, numerical experiments revealed recently a highly unexpected existence of a new broad class of the q−parametric quasi-exact solutions at the next integers $q = 2, 3, 4$ and $q = 5$. Here we show how a systematic construction of the latter, "privileged" $D \gg 1$ exact bound states may be extended to much higher qs (meaning an enhanced flexibility of the shape of the force) at a cost of narrowing the set of wavefunctions (with N restricted to the first few non-negative integers). At $q = 4K + 3$ we conjecture a closed formula for the $N = 3$ solution at all K.

1. INTRODUCTION

In quantum mechanics, bound states of a particle confined in a central potential well $V(|\vec{r}|)$ in D dimensions are constructed as normalizable solutions of the ordinary differential Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r)\right]\,\psi(r) = E\,\psi(r)\,. \tag{1}$$

Usually, this equation (or, more precisely, their infinite set numbered by integer argument $L = 0, 1, \ldots$ of the "angular momentum" $\ell = \ell(L) = L + (D-3)/2$) must be solved by purely numerical means. A notable exception concerns all the models where the spatial dimension D proves "sufficiently" large, $D \gg 1$. In what follows we shall pay attention to this case.

The technique we shall use is new. We shall see that its essence is fairly different from the standard semi-analytic constructions as reviewed and illustrated, e.g., in some technically more detailed papers [1]. Still, a certain overlap with these perturbative

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(often called large$-D$) expansion techniques will survive, so let us review them briefly in the introduction.

1.1. Usual large$-D$ methods and their shortcomings. The main purpose of the usual large$-D$ methods lies in an efficient use of our knowledge of the potential $V(r)$ in (1). Typically, an asymptotic growth of $V(r) \approx r^\alpha$ at $r \gg 1$ is required, the existence of which implies an appearance of a deep minimum at a point $r = R(\alpha)$ which lies somewhere not too close to the (by assumption, very strong) cental repulsion $\sim \ell(\ell+1)/r^2$ near the origin.

The first motivation of our present work derived from an elementary observation that an efficiency of virtually all the large$-D$ techniques *worsens* with the growth of α. In particular, one of the most popular classes of the (say, rational-power-law) multinomial potentials

$$V_{(q,k)}(r) = \frac{1}{r^2}\left[g_{-2} + g_{-1}\,r^{2/(k+1)} + g_0\,r^{4/(k+1)} + \ldots + g_{2q}\,r^{(4q+4)/(k+1)}\right] \tag{2}$$

with a fixed k (say, for the sake of simplicity, $k = 0$) and growing integers q leads to the routine large$-D$ approximations which do not seem promising at all. In fact, an explanation is almost trivial - once we have, due to the definition of a minimum,

$$R(\alpha) \approx \left[\frac{2\ell(\ell+1))}{\alpha}\right]^{1/(\alpha+2)} \gg 1$$

we see that our estimate of the "measure of the rate of convergence" $1/R(\alpha)$ (cf. [1] for details) does not look too small for virtually any positive integer q in eq. (2), indeed. Moreover, the smallness of $1/R$ will not notably improve with the growth of D, either.

In a search for an alternative semi-analytic approach to the problem (1) + (2) one may equally easily be dissatisfied by the various power-series ansatzs

$$\psi^{(PS)}_{(q,k)}(r) \approx r^{\ell+1} \times U(r) \times \exp W(r) \tag{3}$$

of the popular Hill-determinant method [2]. Here, approximants $U(r)$ and $W(r)$ are polynomials in $[r^{2/(k+1)}]$ of respective degrees m_U and m_W which may be re-constructed in a more or less algebraic manner after an insertion of the ansatz in the original differential equation. Still, many sophisticated implementations of this method prove numerically less efficient than the brute-force variational techniques and, moreover, the Hill-determinant approach seems only able to employ the advantage of the smallness of $1/R$ at a fairly high computational cost [2].

We come to the conclusion that the only remaining and eligible non-numerical approach to the desired large$-D$ constructions should be related to the method proposed by E. Magyari many years ago [3]. In essence, it modifies the above-mentioned power-series method by an additional (and apparently paradoxical) requirement that the *polynomial* solutions of the form (3) are *exact*. In the other words, as long as we all know that *all* solutions can *very rarely* have just an elementary polynomial form, we must bridge this obstacle by an *ad hoc* tuning of the potential (2) itself. This will also be our present key idea.

1.2. Quasi-exact approach and an outline of its relevance for the large$-D$ expansions. Basically, the above-mentioned Magyari's (also known as "quasi-exact") recipe of ref. [3] just extends the well known polynomial solvability of eq. (1) + (2) at $q = k = 0$ (harmonic oscillator) and $q = k - 1 = 0$ (Coulomb field). Its full details have been reviewed in ref. [4] but the patient reader of this text need not consult this reference when inserting two formulae (3) and (2) in the differential eq. (1). In the textbook spirit he/she arrives at the standard recurrence relations, and in the next step he/she simply requires that these recurrences will terminate.

Of course, it is well known that such a "quasi-exact" approach gives the exact solutions at $q = 0$ and a mere incomplete solution at any $q \geq 1$. Thus, people often conclude that the *practical* merits of the Magyari's approach are virtually non-existent at any larger q. Indeed, the termination requirements (called, sometimes, Magyari-Schrödinger equations) may be characterized as a coupled set of $q - 1$ determinantal equations which are extremely complicated to solve at *any* $q \geq 2$. Thus, just the first nontrivial $q = 1$ models seem to be tractable within the quasi-exact approach at the generic, small D [5].

The main reason why we decided to return today to the Magyari's approach in the new large$-D$ context is that many years ago we discovered an enormous simplification of the equations as well as solutions during an "experimental" limiting transition to the large dimensions $D \gg 1$. Unfortunately, we only mentioned in an unpublished preprint [6] that for one of the most popular, quartic polynomial interaction the complicated Magyari-Schrödinger equations exhibit a remarkable and fairly surprizing simplification.

Several years later, in another revival of interest in this project [7] we choose "the next" $q = 2$ oscillator and developed an (admittedly, rather clumsy) method of an explicit perturbative construction of the $1/D$ corrections. As long as we observed an improved convergence of the innovated solutions (= series in the powers of $1/D^2$) a return of interest has been completed. In the immediately following step we succeeded in showing that an innovative combination of the perturbative and quasi-exact approach might be extremely productive, indeed. Picking up the sextic example with $q = 1$ for illustration we proved that in a sharp contrast to the divergence of the series, say, in ref. [1], our innovated expansions in the powers of $1/R \approx 1/D^{1/4}$ were absolutely convergent [8].

1.3. An unexpected emergence of the new exact solutions in the large$-D$ limit. A real climax of our effort came with the papers [4] and [9] where we performed an explicit construction of the zero-order solutions and discovered that at the next few less trivial exponents $q = 3$, $q = 4$ and $q = 5$ we were still able to construct the $D \to \infty$ solutions in closed form. Although we were only able to work by the brute-force methods at the time, we proceeded via a direct solution of the Magyari-Schrödinger coupled algebraic nonlinear equations in the $D \to \infty$ limit. A formal key to the necessary new solutions has been found in the elimination method using the Groebner bases [10]. For this reason, we were never able to find any solution at $q \geq 6$.

This was also the main motivation for the forthcoming study and the source of the results which we are going to describe in what follows.

2. A concise formulation of the problem

In a way discussed thoroughly in [4], the use of our ansatzs in the limit $D \to \infty$ implies an immediate reduction of the *differential* radial Schrödinger equation (with potentials (2)) to its finite-dimensional algebraic equivalent $Q\vec{h} = 0$. Here, the dimension of the vector $\vec{h}$ of the Taylor coefficients is N while the matrix elements of the non-square, $N \times (N+q-1)$–dimensional matrix Q are merely linear functions of the large parameter D. This means that the asymptotic analysis of such an equation is straightforward.

2.1. Asymptotic Magyari-Schrödinger equations. It is easy to verify that in the limit $D \to \infty$ the reduced algebraic equivalent of the *differential* radial Schrödinger equation is [4]

$$\left(\begin{array}{cccccc} s_1 & 1 & & & & \\ s_2 & s_1 & 2 & & & \\ \vdots & & \ddots & \ddots & & \\ s_q & \vdots & & s_1 & N-2 & \\ N-1 & s_q & & & s_1 & N-1 \\ & N-2 & s_q & & \vdots & s_1 \\ & & \ddots & \ddots & & \vdots \\ & & & 2 & s_q & s_{q-1} \\ & & & & 1 & s_q \end{array} \right) \left(\begin{array}{c} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{array} \right) = 0 . \qquad (4)$$

Here we are going to study its solvability. Calling this deeply nonlinear non-square-matrix algebraic problem the Magyari-Schrödinger equation, we may just summarize that the N quantities p_j are in a one-to-one correspondence with the Taylor coefficients in the parts $U(r)$ of wavefunctions (3). Moreover, all q generalized eigenvalues s_a [cf. their definitions $s_a = s_a(g_{a-2})$ by eq. (30) in [4]] are just certain re-scaled forms of the energies and/or of coupling constants in our potential (2).

2.2. Upside-down symmetry. The practical use of the explicit quasi-exact (QE) solutions requires a purely numerical determination of their QE-compatible eigenvalues g_{a-2} with $a = 1, 2, \dots, q$. This means that the simplified Magyari-Schrödinger eq. (4) should be interpreted as an implicit definition of a QE solution *on a pre-determined level of a finite precision.* All the sufficiently small $\mathcal{O}(1/D)$ variations of the parameters should be ignored, within this convention, as negligible and irrelevant. Alternatively, they may all be treated, if necessary, by perturbation methods in a way exemplified in [11] at $q = 1$ and in [7] at $q = 2$.

In such a setting, one of our main introductory messages is that all the perturbation constructions of any type may remain exact and non-numerical, at the first few smallest q at least. This is suppoorted by the observation in [4] that all the exact QE parameters prove expressible through integers in the zero-order limit $D \to \infty$.

The necessary condition of this simplification lies in the elementary form of our algebraized QE Magyari-Schrödinger equation (4). Still, one has to overcome a few further obstacles. In particular, the brute-force origin of the results in [4] made it impossible to move beyond $q \leq 5$. In the other words, the key weakness of paper [4] lies in the too rapid growth of the difficulties with the increasing q. One needs all the capacity of the available computers to reveal the structure of solutions at the first few q. This means that whenever one needs an improvement of the insight in the structure of the solutions, one has to make a better use of their symmetries.

In our present continuation and completion of paper [4] we are going to exploit the most obvious symmetry of eq. (4) with respect to its upside-down transposition. For this purpose we may modify our notation slightly, replacing eq. (4) by its reincarnation

$$
(5)\qquad \begin{pmatrix}
\alpha_1 & 1 & & & & \\
\alpha_2 & \alpha_1 & 2 & & & \\
\vdots & & \ddots & \ddots & & \\
\tilde{\alpha}_1 & \vdots & & \alpha_1 & N-2 & \\
N-1 & \tilde{\alpha}_1 & & & \alpha_1 & N-1 \\
 & N-2 & \tilde{\alpha}_1 & & \vdots & \alpha_1 \\
 & & \ddots & \ddots & & \vdots \\
 & & & 2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 \\
 & & & & 1 & \tilde{\alpha}_1
\end{pmatrix}
\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ \tilde{p}_1 \\ \tilde{p}_0 \end{pmatrix} = 0 .
$$

However trivial, such a change of notation implies that all the separate lines have a tilded partner in this set. For example, the definition $p_0 = -p_1/\alpha$ (where $\alpha = \alpha_1$ is tacitly assumed non-zero) is *always* accompanied by its tilded counterpart $\tilde{p}_0 = -\tilde{p}_1/\tilde{\alpha}$ whenever $\tilde{\alpha} = \tilde{\alpha}_1 \neq 0$, etc.

3. Solutions of the symmetrized recurrences

3.1. **Trivial case at $N = 2$.** In our tilded notation, the first nontrivial version of our problem (5) with $N = 2$ is particularly transparent and instructive,

$$
(6)\qquad \begin{pmatrix}
\alpha_1 & 1 \\
\alpha_2 & \alpha_1 \\
\alpha_3 & \alpha_2 \\
\vdots & \vdots \\
\tilde{\alpha}_1 & \tilde{\alpha}_2 \\
1 & \tilde{\alpha}_1
\end{pmatrix}
\begin{pmatrix} p \\ \tilde{p} \end{pmatrix} = 0 .
$$

Its upside-down or "tilding" symmetry separates its first and last line as giving a constraint upon the doublet of unknowns $\alpha_1 = s_1 = \alpha$ and $\tilde{\alpha}_1 = s_q = \tilde{\alpha}$ in a tilding-symmetric manner,

$$
(7)\qquad \det \begin{pmatrix} \alpha & 1 \\ 1 & \tilde{\alpha} \end{pmatrix} = 0, \qquad \alpha\tilde{\alpha} = 1 .
$$

The fully tilding-symmetric way of dealing with the rest of eq. (6) consists now in the equivalence of its recurrent downward or upward treatment.

After we normalize $p_1 = \tilde{p} = 1$, the first line defines $p_0 = p = -1/\alpha$ and may be omitted. Step-by-step, the $k-$th line of eq. (6) may be multiplied by the factor $\tilde{\alpha}/\alpha^{k-1}$. This converts the old components into the known constants and we get the following sequence of definitions

$$\tilde{\alpha}\alpha = 1, \quad \tilde{\alpha}\alpha_2/\alpha = 1, \quad \tilde{\alpha}\alpha_3/\alpha^2 = 1, \dots , \tag{8}$$

with elementary consequence: $\alpha_k = \alpha^k$. We may imagine that the last definition prescribes that $\alpha_{q+1} = \alpha^{q+1} = 1$. This equation may be read as a boundary condition for our recurrences, fixing the physical value of our single free parameter α. It has many unphysical complex roots and just a single *real* one, viz.; the physical root $\alpha = 1$ at any even q. Similarly, *two* different real roots $\alpha = \pm 1$ become available at all the odd q's. This makes the final reconstruction of all the original QE-compatiblity "eigenvalues" $s_1, \dots , s_q$ trivial.

We may add a comment. Knowing that the last line of recurrences (6) defines the function of α ($\alpha_{q+1} = \alpha^{q+1}$) with a prescribed value ($\alpha^{q+1} = 1$), the latter constraint may be interpreted as an algebraic equation which fixes the eligible values of α. Such a type of the boundary condition is not unique. The same role may be played by any other line of eq. (6), once we re-direct these recurrences and demand that

$$\tilde{\alpha}_{1+j} = \alpha_{q-j} \tag{9}$$

at any shift $j \leq q - 1$. At $N = 2$ all this is trivial since after being multiplied by α^{j+1}, all relations (9) degenerate to the same rule $\alpha^{q+1} = 1$.

3.2. A transition to the single variable at $N = 3$. A separate treatment of the first nontrivial $N = 3$ version of eq. (4) is necessary in the degenerate case with $p_1 = 0$. We may infer that $s_1 = s_3 = s_5 = \ldots = s_q = 0$. This means that $q = 2Q + 1$ must be odd and that we in effect return to the $N = 2$ structure. We only have to replace the old unknowns s_k by the new ones, re-scaled to $s_{2k}/2$. Otherwise, the construction of the solutions remains strictly the same, giving the nontrivial roots $s_{2k} = 2\,\varrho^k$ where $\varrho^{Q+1} = 1$.

In what follows, similar detailed qualification will be omitted and, with the degenerate solutions ignored, we shall normalize $p_1 = 1$ at $N = 3$, etc.

From the two outer lines of the $N = 3$ version of eq. (4) we deduce that $p_0 = -1/\alpha$ while $p_2 = -1/\tilde{\alpha}$. The rest of equation (4) acquires the tilding-symmetric matrix form

$$\begin{pmatrix} \beta & \alpha & 2 \\ \gamma & \beta & \alpha \\ \delta & \gamma & \beta \\ \vdots & \vdots & \vdots \\ \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\ 2 & \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \begin{pmatrix} -1/\alpha \\ 1 \\ -1/\tilde{\alpha} \end{pmatrix} = 0\,. \tag{10}$$

It gets facilitated when pre-multiplied by an auxiliary row vector. This observation results from the step-by-step analysis of this system of equations re-written in the form

$$\tilde{\alpha}\,\beta/\alpha = \tilde{\alpha}\alpha - 2, \quad \tilde{\alpha}\,\gamma/\alpha = \tilde{\alpha}\beta - \alpha, \quad \tilde{\alpha}\,\delta/\alpha = \tilde{\alpha}\gamma - \beta, \quad \dots . \tag{11}$$

In the first item the right-hand-side part $\tilde{\alpha}\alpha - 2 = \alpha\tilde{\alpha} - 2 \equiv \xi - 2 = \tilde{\xi} - 2$ is tilding symmetric. This means that the same tilding-invariance must hold for the left-hand-side expression as well. The second item is not tilding-invariant but the invariance is restored after we divide all this equation by α. This gives a consistent picture because one can deduce that also in all the subsequent rows the full tilding-invariance is achieved when we replace α, β, γ, ... by their renormalized and tilding-invariant forms $\alpha\tilde{\alpha}/\alpha^0$, $\beta\tilde{\alpha}/\alpha$, $\gamma\tilde{\alpha}/\alpha^2, \ldots$, respectively. In the other words, the system (11) must be pre-multiplied by the row of the factors 1, $1/\alpha$, $\tilde{\alpha}/\alpha$, $\tilde{\alpha}/\alpha^2$, $\tilde{\alpha}^2/\alpha^2$, ... obtained, in recurrent manner, by the multiplication by the quotient which depends on the parity, i.e., equals to $1/\alpha$ and to $\tilde{\alpha}$ in subsequent steps. This means that the even and odd items in eq. (11) have a different structure.

This difference may be reflected by the change of the notation. Once we put $\alpha = s_1 = a$, $\beta = s_2 = A$, $\gamma = s_3 = b$, $\delta = s_4 = B$, $\epsilon = s_5 = c$ (while denoting also $\tilde{\alpha} = s_q = \tilde{a}$ etc) etc, equation (4) acquires another formally tilding-symmetric matrix form

$$(12)\qquad \begin{pmatrix} a & 1 & 0 \\ A & a & 2 \\ b & A & a \\ B & b & A \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \tilde{a}/a \\ -\tilde{a} \\ 1 \end{pmatrix} = 0 .$$

The pair of the old variables a and $\tilde{a}$ must be replaced by their tilding-invariant product $\xi = a\,\tilde{a} = \tilde{\xi}$ and its tilding-covariant complement $\rho = a/\tilde{a} = 1/\tilde{\rho}$. In an opposite direction, whenever needed, we may re-construct a and $\tilde{a}$ from the two quadratic relations $a^2 = \rho\,\xi$ and $\tilde{a}^2 = \xi/\rho$, i.e., up to an inessential indeterminacy in sign. After we abbreviate

$$\Sigma_1 = \frac{A}{\rho}, \quad \Sigma_2 = \frac{B}{\rho^2}, \quad \Sigma_3 = \frac{C}{\rho^3}, \quad \ldots, \quad \sigma_1 = \frac{1}{a}\frac{a}{\rho^0}, \quad \sigma_2 = \frac{1}{a}\frac{b}{\rho}, \quad \sigma_3 = \frac{1}{a}\frac{c}{\rho^2}, \quad \ldots$$

and postulate that $\Sigma_0 = 2$ and $\sigma_1 = 1$, this procedure results in the conclusion that our recurrences may be re-written as the following sequence of the coupled pairs of the recurrent relations,

$$(13)\qquad \Sigma_k = \xi\,\sigma_k - \Sigma_{k-1}\,, \qquad \sigma_{k+1} = \Sigma_k - \sigma_k, \qquad k = 1, 2, \ldots .$$

One re-interprets eqs. (13) as the mere recurrent definition of the auxiliary sequence of functions of our auxiliary real variable ξ,

$$\Sigma_1 = \xi - 2, \quad \sigma_2 = \xi - 3, \quad \Sigma_2 = \xi^2 - 4\,\xi + 2, \quad \ldots .$$

We see that the functions $\Sigma_k(\xi)$ and $\sigma_{k+1}(\xi)$ are, by their construction, both polynomials of the same degree k.

Our final change in the notation will prescribe ξ replaced by $\xi = 4x^2$, with Σ_k represented as $\Sigma_k = 2\,T_{2k}(x)$ and with σ_k re-scaled into $\sigma_k = T_{2k-1}(x)/x$. We notice that our recurrences become simpler in the new notation but what is more important is that after such a transformation, our new polynomials $T_n(x)$ *coincide precisely* with the classical orthogonal Chebyshev polynomials of the first kind [12],

$$(14)\qquad T_0(x) = 1, \; T_1(x) = x, \; T_2(x) = 2x^2 - 1, \; \ldots .$$

In this sense, our $N = 3$ recurrences are *solved exactly* in closed form.

3.3. The two tilding-covariant variables at $N = 4$. Using the abbreviations $\alpha = s_1$ and $\tilde{\alpha} = s_q$ etc., let us interpret the $N = 4$ QE recurrences (4) as a tilding-symmetric problem

$$\left(\begin{array}{cccc} \alpha & 1 & & \\ \beta & \alpha & 2 & \\ \gamma & \beta & \alpha & 3 \\ \delta & \gamma & \beta & \alpha \\ \epsilon & \delta & \gamma & \beta \\ \vdots & \vdots & \vdots & \vdots \\ & & & \\ 3 & \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\ & 2 & \tilde{\alpha} & \tilde{\beta} \\ & & 1 & \tilde{\alpha} \end{array} \right) \left(\begin{array}{c} p \\ t \\ \tilde{t} \\ \tilde{p} \end{array} \right) = 0 \,. \tag{15}$$

The first and the last line may be dropped as defining merely $p = -t/\alpha$ and $\tilde{p} = -\tilde{t}/\tilde{\alpha}$. From the next two outer lines we may express t in terms of $\tilde{t}$ or *vice versa.* Once we normalize $\tilde{t} = 1$ and re-parametrize $s_2 = \beta = \beta(b) = \alpha^2 + 2\,b\,\alpha$ and, in parallel, $s_{q-1} = \tilde{\beta} = \tilde{\beta}(\tilde{b}) = \tilde{\alpha}^2 + 2\,\tilde{b}\,\tilde{\alpha}$, the vanishing of the related secular two-by-two determinant may be re-read as the constraint $b\,\tilde{b} = 1$. The rest of eq. (15) reads

$$\left(\begin{array}{cccc} \gamma & \beta(b) & \alpha & 3 \\ \delta & \gamma & \beta(b) & \alpha \\ \epsilon & \delta & \gamma & \beta(b) \\ \vdots & \vdots & \vdots & \vdots \\ & & & \\ 3 & \tilde{\alpha} & \tilde{\beta}(\tilde{b}) & \tilde{\gamma} \end{array} \right) \left(\begin{array}{c} -\tilde{b}\,\tilde{\alpha}/\alpha \\ \tilde{b}\,\tilde{\alpha} \\ \tilde{\alpha} \\ -1 \end{array} \right) = 0 \,. \tag{16}$$

Its first row expresses γ as a function of α and b and $\tilde{\alpha}$ and $\tilde{b}$,

$$\gamma\,\tilde{b}\,\tilde{\alpha}/\alpha = Q = \beta(b)\,\tilde{b}\,\tilde{\alpha} + \alpha\,\tilde{\alpha} - 3\,. \tag{17}$$

Fortunately, the quantity Q may be read as a polynomial in the mere two new, auxiliary variables $Z = \alpha\,\tilde{\alpha} \equiv \tilde{Z}$ and $Y = \alpha\,\tilde{b} = Z/\tilde{Y}$. In the light of the preceding subsection, this function $Q = Q(Z,Y) = Z\,Y + 3\,Z - 3$ of two variables will obviously play the role of a generalized Hermite polynomial. We may only regret that it is not a tilding-invariant function anymore.

The second row of eq. (16) must be multiplied by $Y\,\tilde{\alpha}$ to give the next-order polynomial in the same two variables,

$$R = \delta\tilde{b}\,\tilde{\alpha}^2\,Y/\alpha = Z\,Y\,Q(Z,Y) + Z^2\,Y + 2\,Z^2 - Z\,Y = R(Z,Y)\,. \tag{18}$$

This defines the new quantity (= a rescaled δ) and we may proceed to the third row multiplied by $Y\,\tilde{\alpha}^2$,

$$S = \epsilon\tilde{b}\,\tilde{\alpha}^3\,Y/\alpha = Z\,R(Z,Y) + Z^2\,Q(Z,Y) - Z^2\,Y - 2\,Z^2 = S(Z,Y)\,. \tag{19}$$

After the multiplication by $Y^2\,\tilde{\alpha}^3$ the fourth row reads

$$T = \zeta\tilde{b}\,\tilde{\alpha}^4\,Y^2/\alpha = Z\,Y\,S(Z,Y) + Z^2\,R(Z,Y) - Y\,Z^2\,Q(Z,Y) \tag{20}$$

with the next factor $Y^2\,\tilde{\alpha}^4$ giving the next, fifth row

$$U = \eta\tilde{b}\,\tilde{\alpha}^5\,Y^2/\alpha = Z\,T(Z,Y) + Z^2\,S(Z,Y) - Z^2\,R(Z,Y) \tag{21}$$

etc. Step by step we construct, in this manner, the two sequences of functions denoted as $P_n(Y,Z)$ and $Q_n(Y,Z)$ and defined by the pair of the coupled recurrences,

$$\begin{aligned} P_{n+1} &= Y\,Z\,Q_n + Z^2\,P_n - Y\,Z^2\,Q_{n-1}\,, \\ Q_{n+1} &= Z\,P_{n+1} + Z^2\,Q_n - Z^2\,P_n\,, \qquad n = 0,1,\ldots \end{aligned} \tag{22}$$

from the initial values $Q_{-1} = 1/Z$, $P_0 = Y+2$ and $Q_0 = Q = Y\,Z + 3\,Z - 3$ generating $R = P_1$ etc.

In a way paralleling the previous $N = 3$ case, we might slightly modify the functions and define $P_{n+1} = \sqrt{Y}\,W_{2n+1}$ while $Q_{n+1} = W_{2n+2}$. It is easy to verify that we can now use just the single common recurrence

$$W_{n+1} = \sqrt{Y}\,Z\,W_n + Z^2\,W_{n-1} - \sqrt{Y}\,Z^2\,W_{n-2}\,, \qquad n = 0,1,\ldots \tag{23}$$

with the merely slightly modified initialization by $W_{-2} = Q_{-1} = 1/Z$, $W_{-1} = P_0/\sqrt{Y} = \sqrt{Y} + 2/\sqrt{Y}$ and $W_0 = Q_0 = Q = Y\,Z + 3\,Z - 3$. We may see the clear parallels with the previous $N = 3$ case, noticing that the polynomials W_{3n} and W_{3n-1} are both divisible by Z^{2n} while W_{3n-2} is only divisible by Z^{2n-1}. We shall skip the further technical analyses of this sort here.

4. Matching and secular polynomials at $N = 3$

For the sake of simplicity, let us only pay attention to the choice of $N = 3$. Then, the knowledge of the closed form of the polynomials $\Sigma_k(\xi)$ and $\sigma_{k+1}(\xi)$ enables us to define the explicit values of *all* our coupling constants as functions of the mere *two* parameters a and $\tilde{a}$ entering $\xi = a\,\tilde{a}$ and $\rho = a/\tilde{a}$,

$$\begin{aligned} &a_1 = a = a\,\rho^0\,\sigma_1(\xi), \quad A_1 = A = \rho\,\Sigma_1(\xi), \quad a_2 = b = a\,\rho\,\sigma_2(\xi), \\ &A_2 = B = \rho^2\,\Sigma_2(\xi), \quad a_3 = c = a\,\rho^2\,\sigma_3(\xi), \quad \ldots\,. \end{aligned} \tag{24}$$

One could also have constructed this general solution of our recurrences (10) in an opposite, upward direction. For this purpose, it suffices when all the above formulae are modified by a consequent application of the tilding operation.

We have seen that the $N = 3$ case operates with two unknowns. At the same time, the set of recurrences (10) contains precisely two redundant items. In one extreme example we may read whole this set as a sequence of definitions of $\beta = s_2 = s_2(s_1, s_q)$, $\gamma = s_3 = s_3(s_1, s_q)$, $\ldots$, $s_{q+1} = s_{q+1}(s_1, s_q)$ where the last two lines are redundant since we already knew the outcome, viz., $s_{q+1} = 2$ and $s_q(s_1, s_q) = s_q$. This may be understood as a source of our final pair of boundary conditions determining the QE-compatible values of the pair of the unknown parameters.

In a way paralleling the previous $N = 2$ example, any other two lines of eq. (10) might be selected as boundary conditions. In contrast to the $N = 2$ example, almost all of the non-extreme choices of matching conditions would be preferable in practice, lowering the degree of the resulting secular polynomials.

This observation deserves to be explained in more detail. Indeed, it makes sense to distinguish between the four possible selections of the optimal matching conditions.

4.1. $q = 4K$. Whenever $q = 4K$ where $K = 1, 2, \ldots$, the above-mentioned recurrent construction may be started, simultaneously, at both the upper and lower end of eq. (10). Without any difficulties and using eq. (24), the recipe defines *all* the unknown quantities, i.e., the doublets of pairs of the values

$$(a_j,\ A_j),\ \left(\tilde{a}_j,\ \tilde{A}_j\right), \qquad j = 1, 2, \ldots, K. \tag{25}$$

The two middle lines of eq. (10) define the other two redundant functions (or "non-existent couplings") a_{K+1} and $\tilde{a}_{K+1}$. This induces no real difficulty since the two parameters $a_1 = a$ and $\tilde{a}_1 = \tilde{a}$ are not yet specified. The latter two definitions are not redundant, therefore, as they have to fix these initial values.

The inspection of eq. (10) reveals that our symbol a_{K+1} is an alternative name for another and well defined coupling $\tilde{A}_K$. Similarly, the quantity $\tilde{a}_{K+1}$ is an "alias" for A_K. We determine the missing QE roots a and $\tilde{a}$ via the two redundant equations $a_{K+1} = \tilde{A}_K$ and $\tilde{a}_{K+1} = A_K$ or, in the notation of eq. (24),

$$a\,\sigma_{K+1}\,\rho^K = \Sigma_K\,\tilde{\rho}^K, \qquad \tilde{a}\,\sigma_{K+1}\,\tilde{\rho}^K = \Sigma_K\,\rho^K.$$

Their ratio reads $\rho^{4K+1} = 1$ and gives the unique real root $\rho = 1$. Our first conclusion is that we must have $a = \tilde{a}$. The above two equations coincide and any of them represents our ultimate matching condition or constraint imposed upon $\xi = a^2$,

$$a\,\sigma_{K+1}(a^2) = \Sigma_K(a^2), \qquad q = 4K. \tag{26}$$

A sample of its roots may be found in Table 1 at $q = 4$ and $q = 8$. The inspection of the subsequent Table 2 reveals that with the further growth of K, the determination of these roots becomes purely numerical very quickly.

4.2. $q = 4K + 2$. After a move to $q = 4K + 2$ with $K = (0,)\,1, 2, \ldots$, the previous recipe does not change too much. This time we define all the unknowns in a reversed order,

$$(A_j,\ a_{j+1}),\ \left(\tilde{A}_j,\ \tilde{a}_{j+1}\right), \qquad j = 1, 2, \ldots, K. \tag{27}$$

Mutatis mutandis we find that the central part of eq. (10) defines the other "non-existent" couplings A_{K+1} and $\tilde{A}_{K+1}$ so that the doublet of equations $A_{K+1} = \tilde{a}_{K+1}$ and $\tilde{A}_{K+1} = a_{K+1}$ leads to another set of the selfconsistency conditions,

$$a\,\sigma_{K+1}\,\rho^K = \Sigma_{K+1}\,\tilde{\rho}^{K+1}, \qquad \tilde{a}\,\sigma_{K+1}\,\tilde{\rho}^K = \Sigma_{K+1}\,\rho^{K+1}.$$

Their ratio degenerates to the modified constraint $\rho^{4K+3} = 1$ with the same unique real root as above, $a/\tilde{a} = \rho = 1$. Both our innovated identities coincide,

$$a\,\sigma_{K+1}(a^2) = \Sigma_{K+1}(a^2), \qquad q = 4K + 2 \tag{28}$$

and guarantee the desired matching. Their numerical aspects are sampled again in Table 1 (easily solvable cases at $K = 0, 1$ and 2). The adjacent Table 2 complements this list and facilitates the determination of the explicit form of the secular equation (28) at all the integers K.

4.3. $q = 4K+1$. The subset of odd $q = 4K+1$ with $K = (0,)\,1, 2, \ldots$ requires a more careful analysis. Although we have the same complete list (27) of the definitions of the QE-fixed couplings as above, its last two items are defined twice, in two different ways. Their necessary compatibility represented by the relation $a_{K+1} = \tilde{a}_{K+1}$ or rather

$$a\,\sigma_{K+1}\,\rho^K = \tilde{a}\,\sigma_{K+1}\,\tilde{\rho}^K$$

implies that $\rho^{2K+1} = 1$ so that we must put $a/\tilde{a} = \rho = 1$. In the light of this conclusion, the other two consequences $A_{K+1} = \tilde{A}_K$ and $\tilde{A}_{K+1} = A_K$ of the two other next-to-central rows of eq. (10) coincide and give the same ultimate matching rule

$$\Sigma_{K+1}(\xi) = \Sigma_K(\xi)\,, \qquad q = 4K+1\,. \tag{29}$$

Its numerical performance appears illustrated by the corresponding subset of roots in Table 3.

Marginally, let us note that for the specific exponents $q = 4K+1$, the secular polynomial may be re-written in the compact form

$$R^{(K,-)}(\xi) = (\xi - 4)\left[\begin{pmatrix} 2K \\ 0 \end{pmatrix}\xi^K - \begin{pmatrix} 2K-1 \\ 1 \end{pmatrix}\xi^{K-1} + \begin{pmatrix} 2K-2 \\ 2 \end{pmatrix}\xi^{K-2}\right.$$

$$\left. + \ldots (-1)^{K+2}\begin{pmatrix} K+2 \\ K-2 \end{pmatrix}\xi^2 + (-1)^{K+1}\begin{pmatrix} K+1 \\ K-1 \end{pmatrix}\xi + (-1)^K\begin{pmatrix} K \\ K \end{pmatrix}\right]. \tag{30}$$

The secular polynomials $\sum_{m=0}^{K} \xi^m\, d_m^{[K]}$ contain the $(K+1)-$plets of coefficients $\mathcal{K}^{(K)} = \left(d_K^{[K]},\, d_{K-1}^{[K]}, \ldots, d_0^{[K]}\right)$ such that $\mathcal{K}^{(0)} = (1)$, $\mathcal{K}^{(1)} = (1,-1)$, $\mathcal{K}^{(2)} = (1,-3,1)$, $\mathcal{K}^{(3)} = (1,-5,6,1)$, etc. This rule parallels the even$-q$ recipe of Table 2.

4.4. $q = 4K+3$. The last possible choice of the odd exponents $q = 4K+3$ (with $K = (0,)\,1, 2, \ldots)$ in the potentials $V_{(q,n)}(r)$ of eq. (2) leads to a routine completion of all the above analysis. A marginal modification of the list (25) is needed to specify all the necessary QE couplings, recurrently determined as functions of a and $\tilde{a}$ only,

$$(a_j,\, A_j)\,, \quad \left(\tilde{a}_j,\, \tilde{A}_j\right), \qquad j = 1, 2, \ldots, K+1\,. \tag{31}$$

Nevertheless, the results of the matching become slightly different this time. Although the first, central-line rule $A_{K+1} = \tilde{A}_{K+1}$ prescribes merely

$$\Sigma_{K+1}\,\rho^{K+1} = \Sigma_{K+1}\,\tilde{\rho}^{K+1}$$

its consequence $\rho^{2K+2} = 1$ admits the two alternative signs in the resulting $a/\tilde{a} = \rho = \pm 1$. Under this condition, the other two equations (in detail, $a_{K+2} = \tilde{a}_{K+1}$ and its tilding-conjugate $\tilde{a}_{K+2} = a_{K+1}$) coincide as well, giving the same final condition

$$\sigma_{K+2}(\xi) = \sigma_{K+1}(\xi)\,, \qquad q = 4K+3\,. \tag{32}$$

Curiously enough, this equation is the most easily solvable implicit definition of the QE roots $\xi = a^2 = \tilde{a}^2$ (cf. Table 4).

Even the shortest glimpse at the results of the factorization of the effective secular polynomial $R^{(K,+)}(\xi) = \sigma_{K+2}(\xi) - \sigma_{K+1}(\xi)$ reveals that the sequence of the exponents $q = 4K+3$ might be viewed as the most privileged one. The search for its QE roots becomes by far the easiest. After we omit the roots $\xi = 0$ and $\xi = 4$ as trivial, we encounter another unexpected and purely empirically observed symmetry. Indeed, the secular roots $a_1 = \sqrt{\xi_{\pm n}} = \sqrt{2 \pm \Xi_n^{[K]}}$ listed in Table 4 at the indices $q = 4K+3$ appear to be of the very similar form. It is realy instructive to list a few sample distances $\Xi_n^{[K]}$ of a_1^2 from their median = 2. We have $\Xi_n^{[1]} = 0$, $\Xi_n^{[2]} = 1$, $\Xi_n^{[3]} = 0$ or $\sqrt{2}$, two values of $\Xi_n^{[4]} = (\sqrt{5} \pm 1)/2$, the three values of $\Xi_n^{[5]} = 0$, 1 and $\sqrt{3}$. One may see that the full set of the secular roots $a_1 = \sqrt{\xi}$ exhibits a weird regularity manifested by a reflection symmetry with respect to the center at $\xi_c = 2$. All roots become tractable as certain quasi-conjugate pairs $\xi = \xi_{\pm n} = 2 \pm \Xi_n^{[K]}$. In this sense, the results listed in Table 4 may be tentatively extrapolated to all the values of q. Indeed, once we omit the permanent pair of the minimal and maximal QE-compatiblity roots $s = s_1 = \pm 2$ as a trivial, we may use the auxiliary variable $\Xi^{[K]}$ as defined by the relation $s_1 = \sqrt{2 \pm \Xi^{[K]}}$ at all the integers K. The inspection of Table 4 then reveals that all the complete sets of all the QE-compatiblity roots at all the listed integers $q = 4K + 3$ (i.e., at the set of $K = 1, 2, 3, 4$ and 5) coincide with the complete sets of roots of the Chebyshev polynomials of the second kind,

$$U_K\left(\frac{2 - s_1^2}{2}\right) = 0, \qquad K = 1, 2, \ldots, K_0 \tag{33}$$

with the confirmed $K_0 = 5$ at present. This means that in a way complementing the $N > N_0$ extrapolations performed in our paper [4] we may now tentatively extrapolate also the result (33) and conjecture that this equation determines all the closed and exact solutions of our $N = 3$ QE Magyari-Schrödinger eq. (4) also at *all* the larger integers $q = 4K + 3 > 4K_0 + 3 = 23$.

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The list of the table headings.

- Table 1. Real QE roots $s = s_1$ at the first few even q for $N = 3$.
- Table 2. Double Pascal triangle for coefficients in the reduced secular equations $\sum_{k=0}^{q/2} s_1^k c_k^{(q)} = 0$. This extrapolates Table 1 to *all* the even q's.
- Table 3. Real QE roots $s = s_1$ at the first few odd q for $N = 3$.
- Table 4. Real QE roots $s = s_1$ at the first few $q \equiv 3$ $(mod\ 4)$ for $N = 3$.

Tables

Table 1. Real QE roots $s = s_1$ at the first few even q for $N = 3$.

q	roots $s = s_1$
2	2 $\quad -1$
4	2 $\quad (\sqrt{5}-1)/2 \quad -(\sqrt{5}+1)/2$
6	2 $\quad$ (plus all three roots of $a^3 + a^2 - 2a - 1$)
8	2 $\quad -1 \quad$ (plus all three roots of $a^3 - 3a + 1$)
10	2 $\quad$ (plus all five roots of $a^5 + a^4 - 4a^3 - 3a^2 + 3a + 1$)

Table 2. Double Pascal triangle for coefficients in the reduced secular equations $\sum_{k=0}^{q/2} s_1^k\, c_k^{(q)} = 0$. This extrapolates Table 1 to *all* the even q's.

$k =$ / q	coefficients $c_k^{(q)}$: 0	1	2	3	4	5	...
2	1	1					
4	−1	1	1				
6	−1	−2	1	1			
8	1	−2	−3	1	1		
10	1	3	−3	−4	1	1	
12	−1	3	6	−4	−5	1	⋱
14	−1	−4	6	10	−5	−6	⋱
16	1	−4	−10	10	15	−6	⋱

Table 3. Real QE roots $s = s_1$ at the first few odd q for $N = 3$.

q	roots $s = s_1$				
1	2	−2			
3	2	−2			
5	2	1	−1	−2	
7	2	$\sqrt{2}$	$-\sqrt{2}$	−2	
9	2	$\sqrt{\frac{3+\sqrt{5}}{2}}$	$\sqrt{\frac{3-\sqrt{5}}{2}}$	$-\sqrt{\frac{3-\sqrt{5}}{2}}$	$-\sqrt{\frac{3+\sqrt{5}}{2}}$	−2
11	2	$\sqrt{3}$	1	−1	$-\sqrt{3}$	−2

Table 4. Real QE roots $s = s_1$ at the first few $q \equiv 3$ (*mod* 4) for $N = 3$.

q	roots $s = s_1$		
3	± 2		
7	± 2	$\pm\sqrt{2}$	
11	± 2	$\pm\sqrt{2 \pm 1}$	
15	± 2	$\pm\sqrt{2}$	$\pm\sqrt{2 \pm \sqrt{2}}$
19	± 2	$\pm\sqrt{2 \pm \left(\sqrt{5} \mp 1\right)/2}$	
23	± 2	$\pm\sqrt{2}$	$\pm\sqrt{2 \pm \sqrt{2 \mp 1}}$

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