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HASSE GRAPHS AND PARABOLIC SUBALGEBRAS OF EXCEPTIONAL LIE ALGEBRA f_4

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ABSTRACT. We study, from the point of view of abstract representation theory, graded parabolic subalgebras $\mathfrak{p}$ of exceptional Lie algebra f_4. For standard parabolic subalgebras of f_4 up to grading $|4|$ we identify explicitly representation structure of all graded parts and we also construct corresponding Hasse diagrams.

1. INTRODUCTION

The following definitions are standard and can be found in [5, 3, 1].

Definition 1.1. *Let $\mathbb{K}$ be one of the fields $\mathbb{R}, \mathbb{C}$ and $k \in \mathbb{N}$. A $|k|$-graded Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is a Lie algebra over $\mathbb{K}$ equipped with a decomposition (grading)*

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \mathfrak{g}_k \tag{1}$$

fulfilling $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, together with the requirement that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ of $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1}$. We shall use the notation

$$\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \,, \; \mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

for the corresponding graded parabolic Lie subalgebra (resp. its nilpotent part) of $\mathfrak{g}$.

In this article we shall stick to the case $\mathbb{K} = \mathbb{C}$, i.e. all Lie groups and Lie algebras are considered to be defined over the field of complex numbers and all representations live in complex vector spaces. In the case of real Lie groups and algebras the situation is more difficult and for example the pattern of BGG sequence depends on a particular weight, which is being resolved.

We shall restrict to the semisimple $|k|$-graded Lie algebras $\mathfrak{g}$. In this case $\mathfrak{g}_0$ is the reductive Lie subalgebra, i.e. it decomposes on the direct sum of abelian Lie subalgebra $Z(\mathfrak{g}_0)$ and semisimple Lie subalgebra $\mathfrak{g}_0^s$, $\mathfrak{g}_0 \simeq Z(\mathfrak{g}_0) \oplus \mathfrak{g}_0^s$. In what follows we use the notation $Z(\mathfrak{g}_0) := \mathbb{C}E_1 \oplus \mathbb{C}E_2 \oplus \ldots$, where $E_1, E_2, \ldots$ are suitably normalized generators of $Z(\mathfrak{g}_0)$.

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Let us denote a Lie group with Lie algebra $\mathfrak{g}$ resp. $\mathfrak{p}$ by G resp. P, and let us denote by $\mathbb{V}_\lambda$ the irreducible representation of parabolic subalgebra $\mathfrak{p}$ with highest weight λ.

We shall give the definition of invariant differential operators in the homogeneous case.

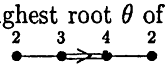
Definition 1.2. *Let $\mathbb{V}_{\lambda_1}$ resp. $\mathbb{V}_{\lambda_2}$ be two irreducible P-modules with dominant weights λ_1 resp. λ_2. Let $V_1 = G \times_P \mathbb{V}_{\lambda_1}, V_2 = G \times_P \mathbb{V}_{\lambda_2}$ be the associated homogeneous vector bundles. The differential operator*

$$D : \Gamma(V_1) \longrightarrow \Gamma(V_2) \tag{2}$$

is called invariant differential operator if it commutes with canonical action of G on sections of associated vector bundles.

The definition of invariant differential operators is not so simple in the non-homogeneous (curved) case and can be found in [2, 3]. Because we shall study only abstract representation theory hidden behind differential geometrical applications, we shall suppress the word differential and in what follows shall write ($\mathfrak{g}-$) invariant operators only.

A special wide class of invariant operators associated to the couples $(\mathfrak{g}, \mathfrak{p})$ are so called **standard** invariant operators. Standard operators are, by definition, nontrivial on pull-back of sections over G/B induced from the natural fibration $G/B \overset{P/B}{\longrightarrow} G/P$, where B is a Borel subgroup of G ($B \subset P$). These operators are coming in sequences called Bernstein-Gelfand-Gelfand (BGG) sequences. The graph structure of BGG sequence is exactly the (combinatorial) graph structure of Hasse diagram for the given parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, [1]. From the differential geometric point of view, the structure of BGG sequence has been recently discussed in [3].

Using pure (and standard) techniques of representation theory the present paper analyzes the graded structure and the Hasse diagrams of standard parabolic subalgebras $\mathfrak{p}$ of the exceptional Lie algebra $\mathfrak{g} = f_4$. For the representations of algebras in question we use the well known notation of crossed Dynkin diagrams, [1]. The grading on corresponding parabolic subalgebra is the sum of coefficients over the crossed nodes given by decomposition of the highest root θ of f_4 into simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. In particular, the coefficients are 

, i.e. $\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$ and the grading of parabolic subalgebra, generated by crossed simple roots, is given by the sum of the coefficients $(2, 3, 4, 2)$ superscribed over them.

We identify explicitly the representation spaces $\mathfrak{g}_i, i \geq 1$ as $\mathfrak{g}_0$-modules and construct their Hasse diagrams $W^\mathfrak{p}$ using the methods [1], p.49 for all parabolic subalgebras $\mathfrak{p}$ up to grading four of $\mathfrak{g}$. This restriction comes from our ineffectiveness to present the Hasse graphs in a reasonable graphic form for more (than four) graded cases. As will be seen, the cases of four graded parabolic subalgebras are already sufficiently combinatorially complicated for this exceptional Lie algebra. We would like to remark that, due to the exceptionality of the Lie algebra f_4, it is useless to consider various inductive (with respect to the rank of the Lie algebra) "tricks" of the construction of Hasse diagrams. In other words, their construction in this article is just based on the tedious and cumbersome computation of corresponding orbits of relative Weyl groups in the weight spaces, [1]. Also the arrows, presented in the Hasse diagrams, are just the arrows corresponding to the reflections along simple roots in the orbit of relative

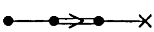
Weyl groups in the weight spaces. The missing arrows correspond to the reflections along non-simple roots. The general rules for standard series of Lie algebras and their parabolic subalgebras clearly indicate, where such arrows should be supplemented, but we did not check it explicitly. It is clear that such results correspond from the point of view of abstract representation theory to the existence of homomorphisms of generalized Verma modules.

For a fixed parabolic subalgebra $\mathfrak{p} \subset f_4$, the vertices of Hasse graph are the representations of $\mathfrak{p}$ and the edges represent f_4-invariant morphisms between source and target representation spaces. The number $i \in \{1,2,3,4\}$ over a particular arrow corresponds to the reflection along the underlying simple root $\alpha_i \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. In particular a given vertex is represented in $W^{\mathfrak{p}}$ by a word composed of reflections, superscripted over the arrows, over the whole path starting in the root vertex of the Hasse graph and ending in a given vertex. We follow the convention of juxtaposing reflections as given in [1].

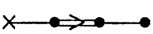
In this way we supplement the results [6] concerning finite dimensional representations of the classical series of Lie algebras and construct Hasse diagrams and representation decomposition of graded parts of parabolic subalgebras of the exceptional Lie algebra f_4 which, to our knowledge, have not yet been discussed. Secondly, these results can serve as a starting point for the study of invariant operators associated with couples $(\mathfrak{f}_4, \mathfrak{p})$.

The results of this article can be summarized in the following theorem.

Theorem 1.3. *The representation structure of graded parts and associated Hasse diagrams for the couples $(f_4, \mathfrak{p})$ for parabolic Lie subalgebras $\mathfrak{p} \subset f_4$ are, up to grading* 4 *of $\mathfrak{g}$, exhausted by following possibilities:*

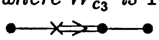
1.  *(see Figure 1.)*

$$\mathfrak{g}_{\pm 1} \simeq S_{b_3}, \quad \mathfrak{g}_{\pm 2} \simeq V_{b_3}, \quad \dim \mathfrak{g}_{\pm 1} = 8, \quad \dim \mathfrak{g}_{\pm 2} = 7. \tag{3}$$

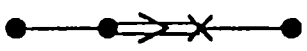
2.  *(see Figure 1.)*

$$\mathfrak{g}_{\pm 1} \simeq W_{c_3}, \quad \mathfrak{g}_{\pm 2} \simeq 1_{c_3}, \quad \dim \mathfrak{g}_{\pm 1} = 14, \quad \dim \mathfrak{g}_{\pm 2} = 1, \tag{4}$$

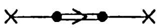
where W_{c_3} is 14*-dimensional irreducible c_3-module, $W_{c_3} \subset \Lambda^3 V_{c_3}$.*

3.  *(see Figure 2.)*

$$\begin{aligned} &\mathfrak{g}_1 \simeq \odot^2 V_{a_2} \otimes V_{a_1}, \quad \mathfrak{g}_{-1} \simeq \odot^2 V^\star_{a_2} \otimes V^\star_{a_1}, \\ &\mathfrak{g}_2 \simeq \odot^2 V_{a_2} \otimes 1_{a_1}, \quad \mathfrak{g}_{-2} \simeq \odot^2 V^\star_{a_2} \otimes 1_{a_1}, \\ &\mathfrak{g}_3 \simeq 1_{a_2} \otimes V_{a_1}, \quad \mathfrak{g}_{-3} \simeq 1_{a_2} \otimes V^\star_{a_1}, \\ &\dim \mathfrak{g}_{\pm 1} = 12, \quad \dim \mathfrak{g}_{\pm 2} = 6, \quad \dim \mathfrak{g}_{\pm 3} = 2. \end{aligned} \tag{5}$$

4.  , *(see Figure 2.)*

$$\begin{aligned}
&\mathfrak{g}_1 \simeq V_{a_2} \otimes V_{a_1}, && \mathfrak{g}_{-1} \simeq V^\star_{a_2} \otimes V^\star_{a_1}, \\
&\mathfrak{g}_2 \simeq V_{a_2} \otimes \odot^2 V_{a_1}, && \mathfrak{g}_{-2} \simeq V^\star_{a_2} \otimes \odot^2 V^\star_{a_1}, \\
&\mathfrak{g}_3 \simeq V_{a_2} \otimes 1_{a_1}, && \mathfrak{g}_{-3} \simeq V^\star_{a_2} \otimes 1_{a_1}, \\
&\mathfrak{g}_4 \simeq 1_{a_2} \otimes V_{a_1}, && \mathfrak{g}_{-4} \simeq 1_{a_2} \otimes V^\star_{a_1}, \\
&\dim \mathfrak{g}_{\pm 1} = 6, \quad \dim \mathfrak{g}_{\pm 2} = 9, && \dim \mathfrak{g}_{\pm 3} = 3, \quad \dim \mathfrak{g}_{\pm 4} = 2.
\end{aligned} \tag{6}$$

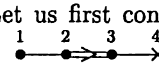
5.  , *(see Figure 3.)*

$$\begin{aligned}
&\mathfrak{g}_{\pm 1} \simeq V_{b_2} \oplus S_{b_2}, \quad \mathfrak{g}_{\pm 2} \simeq S_{b_2} \oplus 1_{b_2}, \quad \mathfrak{g}_{\pm 3} \simeq V_{b_2}, \quad \mathfrak{g}_{\pm 4} \simeq 1_{b_2}, \\
&\dim \mathfrak{g}_{\pm 1} = 9, \quad \dim \mathfrak{g}_{\pm 2} = 5, \quad \dim \mathfrak{g}_{\pm 3} = 5, \quad \dim \mathfrak{g}_{\pm 4} = 1.
\end{aligned} \tag{7}$$

Remark 1.4. The notation used in Theorem is such that V denotes the fundamental vector representation, S the spinor representation, $\odot$ the symmetric power, $\wedge$ the anti-symmetric power, * the dual representation etc. The subscript by any representation denotes the corresponding Lie algebra of this representation.

Proof. Every item is explicitly discussed in one particular subsection:

2. Standard (graded) parabolic subalgebras of f_4

2.1. **|2|-graded case.** Note that $\dim f_4 = 52$. Let us first consider the |2|-graded parabolic subalgebra with crossed Dynkin diagram 

,

$$\mathfrak{g}_0^s \simeq b_3, \quad \mathfrak{g}_0 \simeq b_3 \oplus \mathbb{C}E.$$

We shall use the notation $e_i \in \mathfrak{h}^\star \subset \mathfrak{g}^\star$ for the basis of dual of the Cartan subalgebra and α_i $(i = 1, \dots, 4)$ for simple roots of f_4. The set of all roots of f_4 is (see for example, [4])

$$\Delta_{f_4} = \{\pm e_i\}_{i=1}^4 \cup \{\pm e_i \pm e_j\}_{i<j,i,j=1}^4 \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$

and the simple ones are

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \frac{e_1 - e_2 - e_3 - e_4}{2}.$$

The set of all roots of b_3 is

$$\Delta_{b_3} = \{\pm e_i\}_{i=2}^4 \cup \{\pm e_i \pm e_j\}_{i<j,i,j=2}^4,$$

with simple ones given by

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4.$$

The set of remaining roots $\Delta_{f_4} \setminus \Delta_{b_3}$ is

$$\{\pm e_1\} \cup \{\pm e_1 \pm e_i\}_{i=2}^4 \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

The set of weights of $\mathfrak{g}_1$ resp. $\mathfrak{g}_{-1}$ (realized in the root space of f_4)

$$\left\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\} \quad \text{resp.} \quad \left\{\frac{1}{2}(-e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$

form irreducible $\mathfrak{g}_0^s = b_3$-modules with highest weights $\frac{1}{2}(e_1+e_2+e_3+e_4)$ and $\frac{1}{2}(-e_1+e_2+e_3+e_4)$ respectively, where $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} = 8$.

The set of remaining weights,

$$\{\pm e_1\} \cup \{\pm e_1 \pm e_i\}_{i=2}^{4}$$

decomposes on two vector representations of b_3 with highest weights $e_1 + e_2$ resp. $-e_1 + e_2$, containing the weights

$$\{e_1 \pm e_2, e_1 \pm e_3, e_1 \pm e_4, e_1\} \quad \text{resp.} \quad \{-e_1 \pm e_2, -e_1 \pm e_3, -e_1 \pm e_4, -e_1\}.$$

These irreducible $\mathfrak{g}_0^s = b_3$-modules correspond to $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ respectively.

Finally, we must identify b_3-module $\mathfrak{g}_2$ inside the second exterior power of b_3-module $\mathfrak{g}_1$ via relation $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_2$ in $\mathfrak{g}$ (and similarly for the couple $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$). This is easy, because the second tensor power of the spinor module S_{b_3} decomposes as

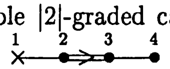
$$S_{b_3} \otimes S_{b_3} \simeq \wedge^0 V_{b_3} \oplus \wedge^1 V_{b_3} \oplus \wedge^2 V_{b_3} \oplus \wedge^3 V_{b_3},$$
$$\dim \wedge^0 V_{b_3} = 1, \quad \dim \wedge^1 V_{b_3} = 7, \quad \dim \wedge^2 V_{b_3} = 21, \quad \dim \wedge^2 V_{b_3} = 35$$

and so it follows from the representation decomposition of the action of permutation group S_2 on $S_{b_3} \otimes S_{b_3}$ (by permutation of factors of the tensor product)

$$S_{b_3} \otimes S_{b_3} \simeq \odot^2 S_{b_3} \oplus \wedge^2 S_{b_3}, \quad \dim(\wedge^2 S_{b_3}) = 28, \quad \dim(\odot^2 S_{b_3}) = 36$$

the inclusion $V_{b_3} \subset \wedge^2 S_{b_3}$, i.e. $\mathfrak{g}_2 \simeq V_{b_3}$ appears in $\wedge^2 \mathfrak{g}_1 \simeq \wedge^2 S_{b_3}$.

Because this case corresponds to $|2|$-graded parabolic subalgebra and $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_2 = 15$, the decomposition is (from the dimensional reasons) complete.

2.2. **$|2|$-graded case.** The second possible $|2|$-graded case is parabolic subalgebra associated to the crossed Dynkin diagram 

,

$$\mathfrak{g}_0^s \simeq c_3, \quad \mathfrak{g}_0 \simeq c_3 \oplus \mathbb{C}E.$$

The standard set of all roots of c_3 is

$$\Delta_{c_3} = \{\pm 2e_i\}_{i=1}^{3} \cup \{\pm e_i \pm e_j\}_{i<j,i,j=1}^{3},$$

with simple roots given by

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = 2e_3.$$

In what follows, we shall construct an embedding

$$\Delta_{c_3} \hookrightarrow \Delta_{f_4}.$$

Note that in the previous $|2|$-graded case the analog of inverse of the previous map (restricted to the image of embedding) was trivial (the identity), i.e. it was simply the restriction map. However, in this case it is not the identity map and so the next our aim is its explicit form.

Let us consider the standard vector space $\mathbb{R}^4$ with the basis $\{e_1, e_2, e_3, e_4\}$. Let us first multiply (renormalize) all root vectors in $\Delta_{c_3} \subset \mathbb{R}^4$ by $\frac{1}{\sqrt{2}}$, i.e. we introduce linear normalization map (given on simple roots and extended by linearity)

$$(8) \qquad \{e_1 - e_2, e_2 - e_3, 2e_3\} \to \left\{\frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3), \sqrt{2}e_3\right\},$$

and then follow by $T \in O(4)$-transformation

$$(9)\quad \left\{\frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3), \sqrt{2}e_3\right\} \xrightarrow{T} \left\{\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, e_3 - e_4\right\}$$

whose explicit matrix realization is

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Its inverse $T^{-1} \in O(4)$ is orthogonal transformation

$$(10)\quad \left\{\frac{1}{2}(e_1 - e_2 - e_3 - e_4), e_4, e_3 - e_4, e_2 - e_3\right\}$$
$$\xrightarrow{T^{-1}} \left\{\frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3), \sqrt{2}e_3, \frac{1}{\sqrt{2}}(-e_1 - e_2 - e_3 - e_4)\right\}$$

and it is of the form

$$T^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

Let us make a suitable abbreviation in the notation — in what follows, we shall use the same notation T resp. T^{-1} for the (right and left respectively) compositions of orthogonal transformations T resp. T^{-1} with normalization map.

The image $= \mathrm{Im}T(\Delta_{c_3})$ under the map T of the (normalized) root system Δ_{c_3} is

$$\frac{1}{\sqrt{2}}(\pm e_1 \pm e_2) : \frac{1}{2}(+e_1 - e_2 + e_3 + e_4), \frac{1}{2}(+e_1 - e_2 - e_3 - e_4),$$
$$\frac{1}{2}(-e_1 + e_2 + e_3 + e_4), \frac{1}{2}(-e_1 + e_2 - e_3 - e_4);$$
$$\frac{1}{\sqrt{2}}(\pm e_1 \pm e_3) : \frac{1}{2}(+e_1 - e_2 + e_3 - e_4), \frac{1}{2}(+e_1 - e_2 - e_3 + e_4),$$
$$\frac{1}{2}(-e_1 + e_2 + e_3 - e_4), \frac{1}{2}(-e_1 + e_2 - e_3 + e_4);$$
$$\frac{1}{\sqrt{2}}(\pm e_2 \pm e_3) : \pm e_3, \pm e_4;$$
$$\sqrt{2}(\pm e_1) : +e_1 - e_2, -e_1 + e_2;$$
$$\sqrt{2}(\pm e_2) : +e_3 + e_4, -e_3 - e_4;$$
$$\sqrt{2}(\pm e_3) : +e_3 - e_4, -e_3 + e_4.$$

For the sake of completeness, we add the list of roots $\Delta_{f_4} \setminus \mathrm{Im}T(\Delta_{c_3})$:

$$\Delta_{f_4} \setminus \mathrm{Im}T(\Delta_{c_3}) = \Big\{ \pm e_1, \pm e_2, e_1 + e_2, -e_1 - e_2, \pm e_1 \pm e_3, \pm e_1 \pm e_4, \pm e_2 \pm e_3, \pm e_2 \pm e_4, \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4), \frac{1}{2}(-e_1 - e_2 \pm e_3 \pm e_4)\Big\}.$$

The remaining task is the identification of c_3-modules $\mathfrak{g}_1, \mathfrak{g}_2$ (and their duals $\mathfrak{g}_{-1}, \mathfrak{g}_{-2}$ respectively). Using the orthogonal transformation T, let us consider the image $\mathrm{Im}T(W) \subset \Delta_{f_4}$ of (normalized) weight spaces of the fundamental 14-dimensional representation W of c_3 (W appears in the third wedge power $\wedge^3 V$ of the fundamental vector representation V, see the Remark 2.1). We have

$$\mathrm{Im}T(W) = \Big\{\frac{1}{2}(e_1 - e_2 \pm 2e_3), \frac{1}{2}(e_1 - e_2 \pm 2e_4), \frac{1}{2}(-e_1 + e_2 \pm 2e_3), \frac{1}{2}(-e_1 + e_2 \pm 2e_4), \frac{1}{2}(e_1 - e_2), \frac{1}{2}(-e_1 + e_2), \frac{1}{2}(\pm e_3 \pm e_4)\Big\}. \tag{11}$$

A closer inspection reveals that adding the vector $\frac{1}{2}(e_1 + e_2)$ (which is itself the half of the root $(e_1 + e_2) \in \Delta_{f_4}$) to all weight spaces, we recover the realization of the weights of c_3-module $\mathfrak{g}_1 \simeq W$ inside of Δ_{f_4}:

$$\mathfrak{g}_1 \simeq W = \Big\{e_1 \pm e_3, e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4, e_1, e_2, \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\Big\} \tag{12}$$

with highest weight $e_1 + e_3$. Then the module $\mathfrak{g}_{-1} \simeq W^\star$ corresponds to

$$\mathfrak{g}_{-1} \simeq W = \Big\{-e_1 \pm e_3, -e_1 \pm e_4, -e_2 \pm e_3, -e_2 \pm e_4, -e_1, -e_2, \frac{1}{2}(-e_1 - e_2 \pm e_3 \pm e_4)\Big\} \tag{13}$$

with highest weight $-e_2 + e_3$. The second exterior power of $\mathfrak{g}_1$ is one dimensional, $\wedge^2 \mathfrak{g}_1 \simeq \wedge^6 \mathbb{C}^6 \simeq \mathbb{C}$ and the only nonzero products of weight spaces are

$$[e_1 + e_4, e_2 - e_4] = [e_1 + e_3, e_2 - e_3] = [e_1 - e_4, e_2 + e_4] = [e_1 - e_3, e_2 + e_3] = e_1 + e_2$$

and similarly in the case of $\mathfrak{g}_{-1}$. In other words, $\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = 1$ and $\mathfrak{g}_2$ resp. $\mathfrak{g}_{-2}$ are generated by $e_1 + e_2$ resp. $-e_1 - e_2$. The structure of all these c_3-modules can be also directly verified using simple roots (8).

Notice that this grading on parabolic subalgebra corresponds to the example of the so called **contact** structure.

Remark 2.1. Let us recall the weight structure of fundamental representation W of c_3. Let $\mathrm{V} = \mathbb{C}^6$ be the fundamental vector representation of c_3 with the weights $\pm e_1, \pm e_2, \pm e_3$ (and the highest weight e_1). W is 14-dimensional c_3-module and it can be realized as irreducible summand of

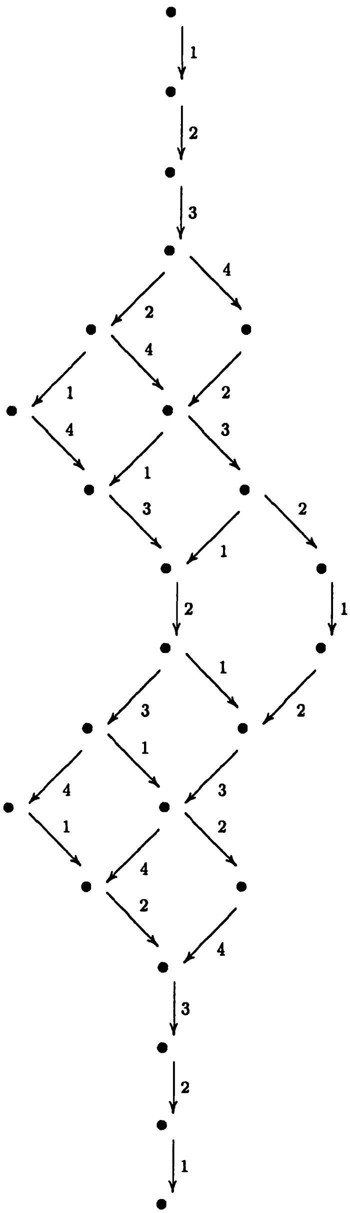
$$\wedge^3 \mathrm{V} \simeq W \oplus \mathrm{V},$$

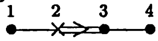
consisting of weights

$$\pm e_1 \pm e_2 \pm e_3, \pm e_1, \pm e_2, \pm e_3,$$

with highest weight $e_1 + e_2 + e_3$.

Figure 1: Hasse diagram for parabolic Lie subalgebras of 2.1, 2.2:



2.3. |3|-graded case. This is the generalized quaternionic case with crossed Dynkin diagram  ,

$$\mathfrak{g}_0^s \simeq a_1 \oplus a_2 , \quad \mathfrak{g}_0 \simeq (a_1 \oplus a_2) \oplus \mathbb{C}E .$$

Because $\dim \mathfrak{g}_0 = 12$, we have $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_2 + \dim \mathfrak{g}_3 = 20$.

The root spaces Δ_{a_1} resp. Δ_{a_2} are

$$\begin{aligned} \Delta_{a_1} &= \{e_1 - e_2, e_2 - e_1\} , \qquad \Delta_{a_1}^+ = \{e_1 - e_2\} , \\ \Delta_{a_2} &= \{e_1 - e_2, e_2 - e_1, e_1 - e_3, e_3 - e_1, e_2 - e_3, e_3 - e_2\} , \\ \Delta_{a_2}^+ &= \{e_1 - e_2, e_1 - e_3, e_2 - e_3\} , \end{aligned}$$

with simple roots of two simple factors a_1, a_2 given by

$$\begin{aligned} a_1 &: \alpha_1 = e_1 - e_2 , \\ a_2 &: \alpha_1 = e_1 - e_2 , \quad \alpha_2 = e_2 - e_3 . \end{aligned}$$

Note that, similarly to the previous case, we should construct an embedding

$$\Delta_{a_1 \oplus a_2} \hookrightarrow \Delta_{f_4} .$$

We first multiply (renormalize) the roots in Δ_{a_2} by $\frac{1}{\sqrt{2}}$, i.e. (on simple roots)

$$(14) \quad \{a_1 : e_1 - e_2 \mid a_2 : (e_1 - e_2), (e_2 - e_3)\} \\ \to \Big\{a_1 : e_1 - e_2 \mid a_2 : \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3)\Big\} ,$$

then we suitably embed $\Delta_{a_1 \oplus a_2}$ into $\mathbb{R}^4$,

$$(15) \quad \Big\{a_1 : e_1 - e_2 \mid a_2 : \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3)\Big\} \\ \hookrightarrow \Big\{\frac{1}{\sqrt{6}}(e_1 + e_2 + e_3 - 3e_4) \Big| \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3)\Big\}$$

and follow it by orthogonal transformation $U \in O(4)$

$$(16) \quad \Big\{\frac{1}{\sqrt{6}}(e_1 + e_2 + e_3 - 3e_4), \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3)\Big\} \\ \overset{\mathrm{U}}{\to} \Big\{e_2 - e_3, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\Big\}$$

in such a way that the inverse $U^{-1} \in O(4)$ generates orthogonal transformation

$$(17) \quad \Big\{e_2 - e_3, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\Big\} \\ \overset{\mathrm{U}^{-1}}{\to} \Big\{\frac{1}{\sqrt{6}}(e_1 + e_2 + e_3 - 3e_4), \frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_2 - e_3)\Big\}$$

and has the matrix form

$$U^{-1} = \begin{pmatrix} \frac{\sqrt{2}+\sqrt{6}}{6} & \frac{-\sqrt{2}+\sqrt{6}}{6} & -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}+\sqrt{6}}{6} & \frac{-\sqrt{2}+\sqrt{6}}{6} & -\frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{-2\sqrt{2}+\sqrt{6}}{6} & \frac{2\sqrt{2}+\sqrt{6}}{6} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & 0 \end{pmatrix} .$$

The set of simple roots of $\mathfrak{g}_0$ is

$$a_1 : \alpha_1 = e_2 - e_3 \,,$$
$$a_2 : \alpha_3 = e_4 \,, \quad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 + e_4)$$

and the graded parts $\mathfrak{g}_i$, $i \neq 0$ contain the following subset of roots of Δ_{f_4}:

$$\begin{aligned} & \{\pm e_i\}_{i=1}^3 \cup \{\pm e_1 \pm e_2\} \cup \{\pm e_1 \pm e_3\} \cup \{\pm e_1 \pm e_4\} \cup \{\pm e_2 \pm e_4)\} \\ & \cup \{\pm e_3 \pm e_4)\} \cup \{\pm(e_2 + e_3)\} \cup \left\{\frac{1}{2}(e_1 - e_2 + e_3 \pm e_4)\right\} \\ & \cup \left\{\frac{1}{2}(e_1 + e_2 - e_3 \pm e_4)\right\} \cup \left\{\frac{1}{2}(e_1 + e_2 + e_3 \pm e_4)\right\} \\ & \cup \left\{\frac{1}{2}(-e_1 - e_2 + e_3 \pm e_4)\right\} \cup \left\{\frac{1}{2}(-e_1 + e_2 - e_3 \pm e_4)\right\} \\ & \cup \left\{\frac{1}{2}(-e_1 - e_2 - e_3 \pm e_4)\right\}. \end{aligned} \tag{18}$$

The first graded part $\mathfrak{g}_1 \simeq \odot^2 V_{a_2} \otimes V_{a_1}$ includes the following root spaces:

$$\begin{array}{r} \{e_1 - e_2, e_1 - e_3\}\left\{\frac{1}{2}(e_1 - e_2 + e_3 + e_4), \frac{1}{2}(e_1 + e_2 - e_3 + e_4)\right\}\{e_3 + e_4, e_2 - e_4\} \\ \left\{\frac{1}{2}(e_1 - e_2 + e_3 - e_4), \frac{1}{2}(e_1 + e_2 - e_3 - e_4)\right\}\{e_3, e_2\} \\ \{e_3 - e_4, e_2 + e_4\} \end{array} \tag{19}$$

where any of the six couples in parenthesis denotes V_{a_1}-part and the triangle of couples describes $\odot^2 V_{a_2}$-part of the tensor product. The graded part $\mathfrak{g}_{-1} \simeq \odot^2 V^\star_{a_2} \otimes V^\star_{a_1}$ consists of

$$\begin{array}{c} \{-e_1 + e_2, -e_1 + e_3\}\left\{\frac{1}{2}(-e_1 + e_2 - e_3 - e_4), \frac{1}{2}(-e_1 - e_2 + e_3 - e_4)\right\} \\ \{-e_3 - e_4, -e_2 + e_4\}\left\{\frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \frac{1}{2}(-e_1 - e_2 + e_3 + e_4)\right\} \\ \{-e_3, -e_2\}\{-e_3 + e_4, -e_2 - e_4\} \end{array} \tag{20}$$

The second part of the grading, $\mathfrak{g}_2 \simeq \odot^2 V_{a_2} \otimes 1_{a_1}$ contains the following weights:

$$\begin{array}{ccc} \{e_1 + e_4\}\left\{\frac{1}{2}(e_1 + e_2 + e_3 + e_4)\right\}\{e_2 + e_3\} & & \\ & \{e_1\} & \left\{\frac{1}{2}(e_1 + e_2 + e_3 - e_4)\right\} \\ & & \{e \quad - e_4\} \end{array} \tag{21}$$

and its dual $\mathfrak{g}_{-2} \simeq \odot^2 V^\star_{a_2} \otimes 1_{a_1}$

$$\begin{array}{ccc} \{-e_1 - e_4\}\left\{\frac{1}{2}(-e_1 - e_2 - e_3 - e_4)\right\}\{-e_2 - e_3\} & & \\ & \{-e_1\} & \left\{\frac{1}{2}(-e_1 - e_2 - e_3 + e_4)\right\} \\ & & \{-e_1 + e_4\} \end{array} \tag{22}$$

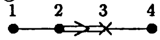
The parts $\mathfrak{g}_3 \simeq 1_{a_2} \otimes V_{a_1}, \mathfrak{g}_{-3} \simeq 1_{a_2} \otimes V^\star_{a_1}$ carry the root spaces

$$\begin{aligned} &\mathfrak{g}_3 : e_1 + e_2, e_1 + e_3 \,, \\ &\mathfrak{g}_{-3} : -e_1 - e_2, -e_1 - e_3 \,. \end{aligned} \tag{23}$$

The highest weights of the corresponding irreducible representations are

$$\begin{aligned} &\mathfrak{g}_{\pm 1} : (e_1 - e_2), (-e_3 + e_4) \,, \\ &\mathfrak{g}_{\pm 2} : (e_1 + e_4), (-e_2 - e_3) \,, \\ &\mathfrak{g}_{\pm 3} : (e_1 + e_2), (-e_1 - e_3) \,. \end{aligned} \tag{24}$$

It is now straightforward but tedious to verify all the relations $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $i, j = 1, \ldots, 4$.

2.4. **|4|-graded case.** The analysis of representations proceeds in both |4|-graded cases similarly as in the previous ones and so we shall content ourselves with brief comments of all results. The first |4|-graded case is also generalized quaternionic structure with crossed Dynkin diagram  ,

$$\mathfrak{g}^s_0 \simeq a_2 \oplus a_1 \,, \; \mathfrak{g}_0 \simeq (a_2 \oplus a_1) \oplus \mathbb{C}E \,.$$

The structure of $\mathfrak{g}_0$ is the same as in the previous |3|-graded case, but the set of simple roots

$$\begin{aligned} &a_1 : \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4) \,, \\ &a_2 : \alpha_1 = e_2 - e_3, \; \alpha_2 = e_3 - e_4 \,, \end{aligned}$$

is such that the graded parts $\mathfrak{g}_i$, $i \neq 0$, will contain the following subset of roots of Δ_{f_4}:

$$\begin{aligned} &\{\pm e_i\}_{i=1}^4 \cup \{\pm e_1 \pm e_2\} \cup \{\pm e_1 \pm e_3\} \cup \{\pm e_1 \pm e_4\} \cup \{\pm(e_2 + e_3)\} \\ &\cup \{\pm(e_2 + e_4)\} \cup \{\pm(e_3 + e_4)\} \cup \left\{\frac{1}{2}(e_1 - e_2 - e_3 + e_4)\right\} \\ &\cup \left\{\frac{1}{2}(e_1 - e_2 + e_3 \pm e_4)\right\} \cup \left\{\frac{1}{2}(e_1 + e_2 - e_3 \pm e_4)\right\} \\ &\cup \left\{\frac{1}{2}(e_1 + e_2 + e_3 \pm e_4)\right\} \cup \left\{\frac{1}{2}(-e_1 + e_2 + e_3 - e_4)\right\} \\ &\cup \left\{\frac{1}{2}(-e_1 + e_2 - e_3 \pm e_4)\right\} \cup \left\{\frac{1}{2}(-e_1 - e_2 + e_3 \pm e_4)\right\} \\ &\cup \left\{\frac{1}{2}(-e_1 - e_2 - e_3 \pm e_4)\right\}. \end{aligned} \tag{25}$$

The first graded part $\mathfrak{g}_1 \simeq V_{a_2} \otimes V_{a_1}$ of the grading contains the following weights:

$$\begin{aligned} &\left\{\frac{1}{2}(e_1 - e_2 - e_3 + e_4), e_4\right\} \quad \left\{\frac{1}{2}(e_1 - e_2 + e_3 - e_4), e_3\right\} \\ &\left\{\frac{1}{2}(e_1 + e_2 - e_3 - e_4), e_2\right\} \end{aligned} \tag{26}$$

where any of the three couples in parenthesis denote V_{a_1}-part and the triangle of couples describes V_{a_2}-representation. Quite analogous structure emerges for $\mathfrak{g}_{-1} \simeq V^\star_{a_2} \otimes V^\star_{a_1}$:

$$\begin{aligned} &\left\{\frac{1}{2}(-e_1+e_2+e_3-e_4), -e_4\right\} \quad \left\{\frac{1}{2}(-e_1+e_2-e_3+e_4), -e_3\right\} \\ &\left\{\frac{1}{2}(-e_1-e_2+e_3+e_4), -e_2\right\} \end{aligned} \tag{27}$$

The second part $\mathfrak{g}_2 \simeq V_{a_2} \otimes \odot^2 V_{a_1}$ of the grading contains the following weights:

$$\begin{aligned} &\left\{e_1-e_2, \frac{1}{2}(e_1-e_2+e_3+e_4), e_3+e_4\right\} \\ &\left\{e_1-e_3, \frac{1}{2}(e_1+e_2-e_3+e_4), e_2+e_4\right\} \\ &\left\{e_1-e_4, \frac{1}{2}(e_1+e_2+e_3-e_4), e_2+e_3\right\} \end{aligned} \tag{28}$$

and its dual $\mathfrak{g}_{-2} \simeq V^\star_{a_2} \otimes \odot^2 V^\star_{a_1}$ is realized by

$$\begin{aligned} &\left\{-e_1+e_2, \frac{1}{2}(-e_1+e_2-e_3-e_4), -e_3-e_4\right\} \\ &\left\{-e_1+e_3, \frac{1}{2}(-e_1-e_2+e_3-e_4), -e_2-e_4\right\} \\ &\left\{-e_1+e_4, \frac{1}{2}(-e_1-e_2-e_3+e_4), -e_2-e_3\right\}. \end{aligned} \tag{29}$$

The parts $\mathfrak{g}_3 \simeq V_{a_2} \otimes 1_{a_1}, \mathfrak{g}_{-3} \simeq V^\star_{a_2} \otimes 1_{a_1}$ are carried by the roots

$$\begin{aligned} &\mathfrak{g}_3 : e_1+e_2, e_1+e_3, e_1+e_4 \\ &\mathfrak{g}_{-3} : -e_1-e_2, -e_1-e_3, -e_1-e_4 \end{aligned} \tag{30}$$

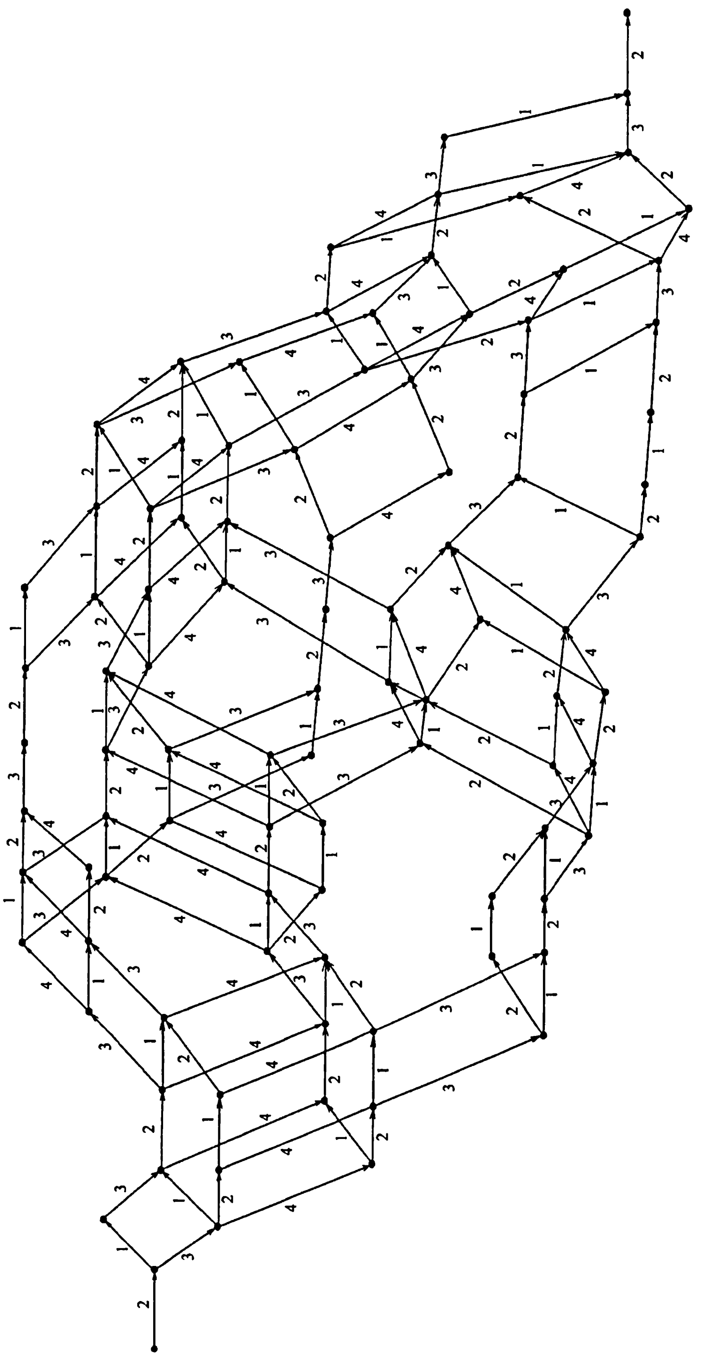
and finally $\mathfrak{g}_4 \simeq 1_{a_2} \otimes V_{a_1}, \mathfrak{g}_{-4} \simeq 1_{a_2} \otimes V^\star_{a_1}$ is

$$\begin{aligned} &\mathfrak{g}_4 : e_1, \frac{1}{2}(e_1+e_2+e_3+e_4) \\ &\mathfrak{g}_{-4} : -e_1, -\frac{1}{2}(e_1+e_2+e_3+e_4). \end{aligned} \tag{31}$$

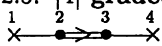
The highest weights of corresponding irreducible representations are:

$$\begin{aligned} &\mathfrak{g}_{\pm 1} : \frac{1}{2}(e_1+e_2-e_3-e_4), \frac{1}{2}(-e_1+e_2+e_3-e_4), \\ &\mathfrak{g}_{\pm 2} : (e_1-e_2), (-e_3-e_4), \\ &\mathfrak{g}_{\pm 3} : (e_1+e_2), (-e_1-e_4), \\ &\mathfrak{g}_{\pm 4} : e_1, -\frac{1}{2}(e_1+e_2+e_3+e_4). \end{aligned} \tag{32}$$

Figure 2: Hasse diagram of parabolic Lie subalgebra 2.3, 2.4:



2.5. **|4|-graded case.** This case corresponds to the crossed Dynkin diagram



,

$$\mathfrak{g}_0^s \simeq b_2\,, \quad \mathfrak{g}_0 \simeq b_2 \oplus \mathbb{C}E_1 \oplus \mathbb{C}E_2\,.$$

The set of all roots of $\mathfrak{g}_0^s \simeq b_2$ is

$$\Delta_{b_2} = \{\pm e_3, \pm e_4, \pm e_3 \pm e_4\}$$

with simple roots given by

$$\alpha_2 = e_3 - e_4\,, \quad \alpha_3 = e_4\,.$$

The set of remaining roots $\Delta_{f_4} \setminus \Delta_{b_3}$ is

$$\{\pm e_1, \pm e_2\} \cup \{\pm e_1 \pm e_2, \pm e_1 \pm e_3, \pm e_1 \pm e_4, \pm e_2 \pm e_3, \pm e_2 \pm e_4\}$$
$$\cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

The structure of the graded parts looks as follows. The part of degree one $\mathfrak{g}_1$ is reducible and decomposes on two irreducible components $\mathfrak{g}_1 \simeq V_{b_2} \oplus S_{b_2}$,

$$\mathfrak{g}_1 \simeq V_{b_2} \oplus S_{b_2} \simeq \{e_2 \pm e_3, e_2 \pm e_4, e_2\} \cup \left\{\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)\right\}, \tag{33}$$

and similarly for its dual $\mathfrak{g}_{-1}$,

$$\mathfrak{g}_{-1} \simeq V_{b_2} \oplus S_{b_2} \simeq \{-e_2 \pm e_3, -e_2 \pm e_4, -e_2\} \cup \left\{\frac{1}{2}(-e_1 + e_2 \pm e_3 \pm e_4)\right\}. \tag{34}$$

The degree two parts $\mathfrak{g}_{\pm 2}$, $[\mathfrak{g}_{\pm 1}, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 2}$, consist of

$$\begin{aligned} \mathfrak{g}_2 &\simeq 1_{b_2} \oplus S_{b_2} \simeq \{e_1 - e_2\} \cup \left\{\frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\right\}, \\ \mathfrak{g}_{-2} &\simeq 1_{b_2} \oplus S_{b_2} \simeq \{-e_1 + e_2\} \cup \left\{\frac{1}{2}(-e_1 - e_2 \pm e_3 \pm e_4)\right\}. \end{aligned} \tag{35}$$

Note that the trivial representation 1_{b_2} comes from the decomposition of $\wedge^2 S_{b_2}$, while the spinor representation S_{b_2} is an irreducible summand of the tensor product of basic vector and spinor representations, $V_{b_2} \otimes S_{b_2}$.

The representation spaces appearing in $\mathfrak{g}_{\pm 3}$ are isomorphic to the fundamental vector representation, $\mathfrak{g}_{\pm 3} \simeq V_{b_2}$,

$$\begin{aligned} \mathfrak{g}_3 &\simeq V_{b_2} \simeq \{e_1 + e_3, e_1 + e_4, e_1, e_1 - e_4, e_1 - e_3\}\,, \\ \mathfrak{g}_{-3} &\simeq V_{b_2} \simeq \{-e_1 + e_3, -e_1 + e_4, -e_1, -e_1 - e_4, -e_1 - e_3\}\,. \end{aligned} \tag{36}$$

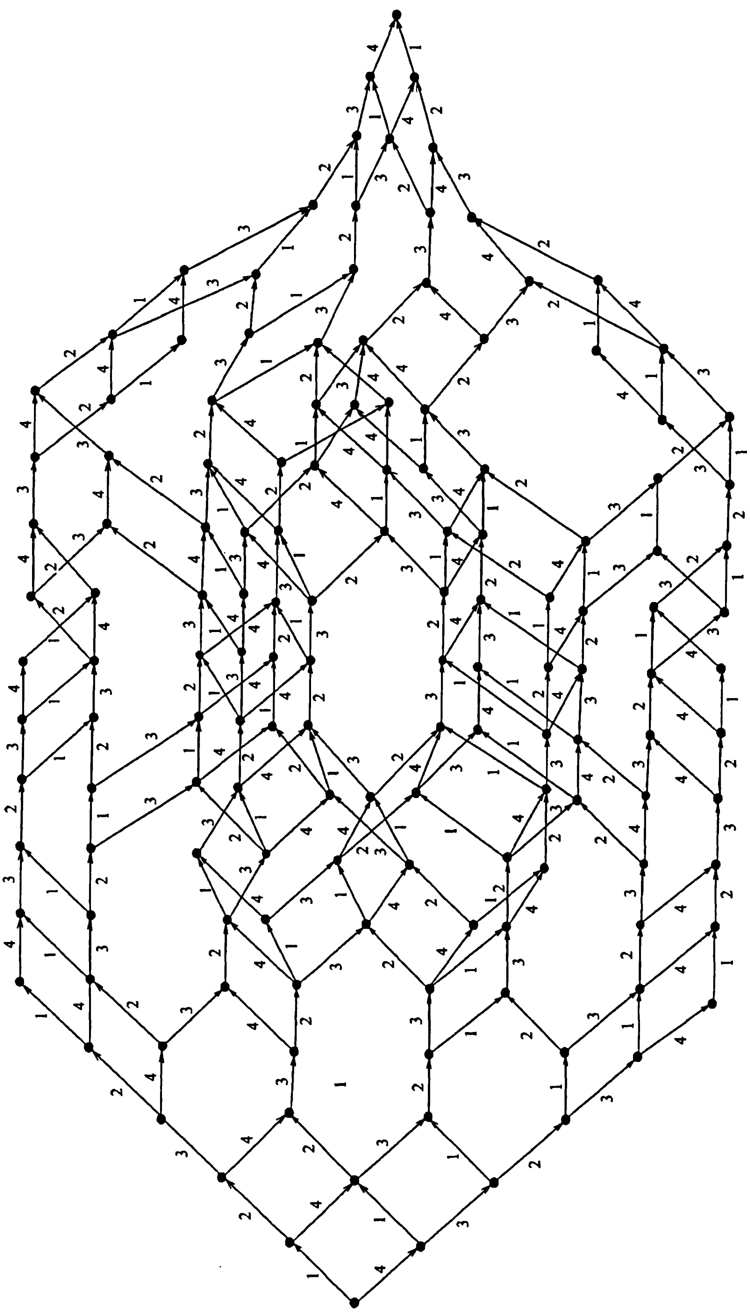
Note that the vector representation $\mathfrak{g}_{\pm 3}$ appears as irreducible summand of both tensor products of the spinor representations $S_{b_2} \simeq \mathfrak{g}_{\pm 2}$ with $S_{b_2} \subset \mathfrak{g}_{\pm 1}$ and the trivial representation $1_{b_2} \subset \mathfrak{g}_{\pm 2}$ with the vector representation $V_{b_2} \subset \mathfrak{g}_{\pm 1}$.

The parts $\mathfrak{g}_{\pm 4}$ (i.e. homogeneity grading four) are trivial 1-dimensional representation spaces,

$$\begin{aligned} \mathfrak{g}_4 &\simeq 1_{b_2} \simeq \{e_1 + e_2\}\,, \\ \mathfrak{g}_{-4} &\simeq 1_{b_2} \simeq \{-e_1 - e_2\}\,. \end{aligned} \tag{37}$$

The trivial representations $\mathfrak{g}_{\pm 4}$ appear as irreducible summands of both second wedge product of the spinor representation $S_{b_2} \simeq \mathfrak{g}_{\pm 2}$ and wedge product of the vector representation $V_{b_2} \subset \mathfrak{g}_{\pm 1}$ with $V_{b_2} \simeq \mathfrak{g}_{\pm 3}$.

Figure 3: Hasse diagram of parabolic Lie subalgebra 2.5:



We add the list of corresponding highest weights of all b_2-modules:

$$\begin{aligned} \mathfrak{g}_{\pm 1} &: \lambda_V = \pm e_2 + e_3\,, \quad \lambda_S = \frac{1}{2}(\pm e_1 \mp e_2 + e_3 + e_4)\,, \\ \mathfrak{g}_{\pm 2} &: \lambda_1 = \pm e_1 \mp e_2\,, \quad \lambda_S = \frac{1}{2}(\pm e_1 \pm e_2 + e_3 + e_4)\,, \\ \mathfrak{g}_{\pm 3} &: \lambda_V = \pm e_1 + e_3\,, \\ \mathfrak{g}_{\pm 4} &: \lambda_1 = \pm e_1 \pm e_2\,. \end{aligned} \tag{38}$$

Remark 2.2. This case can be in some sense considered as a generalization of the contact structure - instead of having $|2|$-graded Lie algebra with $\dim \mathfrak{g}_{\pm 2} = 1$, we arrived at the structure of $|4|$-graded Lie algebra on $\mathfrak{g}_-$ with $\dim \mathfrak{g}_{\pm 4} = 1$.

3. Comments

It is perhaps worth to emphasize that Hasse diagrams are the same for the parabolic subalgebras given either by crossing the second simple root or the third simple root. The lower dimensional examples of generalized quaternionic structures for classical Lie algebra series b_n and c_n, whose Dynkin diagrams differ by the orientation between the last two simple roots, indicate and conjecturally suggest that this could hold true generally. This problem is just under current investigation.

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