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In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 24th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2005. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 75. pp. [259]--267.

Persistent URL: http://dml.cz/dmlcz/701752

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 75 (2005), pp. 259-267

HORIZONTAL AND CONTACT FORMS ON CONSTRAINT MANIFOLDS

OLGA KRUPKOVÁ AND MARTIN SWACZYNA

ABSTRACT. Differential forms on constraint submanifolds of jet bundles are investigated. The horizontalization and contactization operators are generalized, and canonical decomposition of forms, arising due to the existence of the constraint structure, is found.

1. INTRODUCTION

Horizontal and contact forms play an essential role in many geometrical constructions on jet bundles. Namely, the *operators of horizontalization and contactizations*, and the arising *canonical decomposition of differential forms* into a sum of horizontal and contact parts of different contact degree are fundamental tools in the calculus of variations and the theory of differential systems on fibered manifolds, as well as in numerous applications in mathematical physics. Recently, there has been an interest in extending results to the case of *non-holonomic systems* which are modelled as differential systems on submanifolds of jet manifolds. Yet, the case of non-holonomic mechanics has been intensively studied (cf. e.g. [1]–[3], [7]–[16], [18], and others), and the corresponding constraint structure has been discovered [7], [13]. This is the case where the underlying fibered manifold is of the form $\pi : Y \to X$, where $\dim X = 1$, and the constraint structure is given by a fibered submanifold Q of $\pi_1 : J^1Y \to Y$, endowed with a naturally arising distribution, called *canonical distribution* or *Chetaev bundle* over Q.

2000 *Mathematics Subject Classification*: 58A10, 58A20, 58A30, 58F05, 70F25, 70H05.

Key words and phrases: jet bundle, horizontal form, contact form, constraint submanifold, canonical distribution, constraint ideal, constraint horizontalization, constraint contactization.

Research supported by grants GAČR 201/03/0512 of the Czech Grant Agency, and MSM 153100011 of the Czech Ministry of Education, Youth and Sports. The first and the second author also acknowledge support of the grants IGS SU 19/2003 of the Silesian University in Opava, and IGA PřF OU 8/2004 of the Ostrava University, respectively.

The paper is in final form and no version of it will be submitted elsewhere.

The aim of this paper is to study the structure of differential forms on constraint submanifolds. We construct operators of horizontalization and contactizations of different degrees, adapted to the constraint structure. As the main result we obtain a *Theorem on canonical decomposition of differential forms on constraint manifolds into a sum of constraint-horizontal and constraint-contact parts.*

2. Horizontal and contact forms on fibered manifolds

Let us recall briefly the calculus of horizontal and contact forms on fibered manifolds, as developed in [4] and [5]. For an exposition we refer also to [6], [17].

In what follows, we consider a smooth fibered manifold $\pi : Y \to X$ with $\dim X = 1$, $\dim Y = m + 1$, and its first (resp. second) jet prolongation, J^1Y (resp. J^2Y). Fibered coordinates on Y, associated coordinates on J^1Y, and those on J^2Y, are denoted by (t, q^σ), $(t, q^\sigma, \dot{q}^\sigma)$, and $(t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$ respectively, where $1 \leq \sigma \leq m$. We use standard notations $\pi_1 : J^1Y \to X$, $\pi_{1,0} : J^1Y \to Y$, $\pi_{2,1} : J^2Y \to J^1Y$, etc. A section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π. A vector field ξ on J^1Y is called π_1-*projectable* (resp. $\pi_{1,0}$-*projectable*) if there exists a vector field ξ_0 on X (resp. on Y), such that $T\pi_1 \cdot \xi = \xi_0 \circ \pi_1$ (resp. $T\pi_{1,0} \cdot \xi = \xi_0 \circ \pi_{1,0}$). ξ is called π_1-*vertical* (resp. $\pi_{1,0}$-*vertical*) if $T\pi_1 \cdot \xi = 0$ (resp. $T\pi_{1,0} \cdot \xi = 0$).

Denote by $\Lambda^q(J^1Y)$ the module of differential q-forms on J^1Y over the ring of functions. $\eta \in \Lambda^q(J^1Y)$ is called π_1-*horizontal* (resp. $\pi_{1,0}$-*horizontal*) if $i_\xi \eta = 0$ for every π_1-vertical (resp. $\pi_{1,0}$-vertical) vector field ξ on J^1Y. The submodule of π_1-horizontal (resp. $\pi_{1,0}$-horizontal) q-forms on J^1Y will be denoted by $\Lambda^q_X(J^1Y)$ (resp. $\Lambda^q_Y(J^1Y)$). If $\eta \in \Lambda^q(J^1Y)$, $q \geq 1$, one sets for every point $y = J^2_x\gamma \in J^2Y$, and every system of vector fields $\xi_1, \dots, \xi_q \in T_yJ^2Y$

$$h\eta(J^2_x\gamma)(\xi_1, \dots, \xi_q) = \eta(J^1_x\gamma)(T_xJ^1\gamma \cdot T\pi_2 \cdot \xi_1, \dots, T_xJ^1\gamma \cdot T\pi_2 \cdot \xi_q)\,. \tag{2.1}$$

For a function f on J^1Y, h is defined simply by $hf(J^2_x\gamma) = f(J^1_x\gamma)$. The mapping $h : \Lambda^q(J^1Y) \to \Lambda^q_X(J^2Y)$ is called *horizontalization* with respect to the projection π. $\eta \in \Lambda^q(J^1Y)$ is called *contact* if $J^1\gamma^*\eta = 0$ for every section γ of π. Consequently, every q-form for $q > \dim X$ is contact. The module of contact q-forms on J^1Y is denoted by $\Omega^q(J^1Y)$. Note that $\Omega^1(J^1Y)$ is locally generated by the following forms,

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma\, dt\,, \quad 1 \leq \sigma \leq m\,, \tag{2.2}$$

called *canonical contact* 1-*forms*. Putting

$$p\eta = \pi^*_{2,1}\eta - h\eta \tag{2.3}$$

one gets a mapping $p : \Lambda^q(J^1Y) \to \Omega^q(J^2Y)$, assigning to every q-form η on J^1Y a contact q-form $p\eta$ on J^2Y. p is called *contactization* with respect to the projection π. For a function f, $pf = 0$.

The mappings h and p have the following properties:

Proposition 2.1 [4]. *Let $\lambda, \eta \in \Lambda^q(J^1Y), \omega \in \Lambda^p(J^1Y)$.*

(1) $h(\lambda + \eta) = h\lambda + h\eta$, $h(\lambda \wedge \omega) = h\lambda \wedge h\omega$.
(2) $p(\lambda + \eta) = p\lambda + p\eta$, $p(\lambda \wedge \omega) = p\lambda \wedge p\omega + p\lambda \wedge h\omega + h\lambda \wedge p\omega$.
(3) $h(f\eta) = (f \circ \pi_{2,1}) \cdot h\eta$, $p(f\eta) = (f \circ \pi_{2,1}) \cdot p\eta$ *for a function* f.
(4) η *is horizontal (resp. contact) if and only if* $p\eta = 0$ *(resp.* $h\eta = 0$*).*
(5) *If* $q > \dim X$ *then* $h\eta = 0$, $p\eta = \pi_{2,1}^* \eta$.
(6) $h\eta$ *is a unique horizontal form such that for every section* γ *of* π *the condition* $J^1\gamma^*\eta = J^2\gamma^* h\eta$ *is satisfied.*
(7) *For every section* γ *of* π *the condition* $J^2\gamma^* p\eta = 0$ *is satisfied.*
(8) *If* η *is* $\pi_{1,0}$*-horizontal then both* $h\eta$ *and* $p\eta$ *are* $\pi_{2,1}$*-projectable.*
(9) $p\eta$ *is* $\pi_{2,1}$*-projectable if and only if* $h\eta$ *is* $\pi_{2,1}$*-projectable.*
(10) *If* ϕ *is an isomorphism of* π *then* $hJ^1\phi^*\eta = J^2\phi^* h\eta$, $pJ^1\phi^*\eta = J^2\phi^* p\eta$.

A contact q-form η is called 1-*contact* if for every π_1-vertical vector field ξ the $(q-1)$-form $i_\xi \eta$ is horizontal; η is called i-*contact*, $i > 1$, if $i_\xi \eta$ is $(i-1)$-contact. Denote by $\Omega^{q-i,i}(J^1Y)$ the module of i-contact q-forms on J^1Y. We have the following *Decomposition Theorem*:

Theorem 2.1 [4]. *Every* $\eta \in \Lambda^q(J^1Y)$ *admits a unique decomposition*

$$\pi_{2,1}^* \eta = \eta_{q-1} + \eta_q \tag{2.4}$$

into a sum of a horizontal and 1*-contact form on* J^2Y *if* $q = 1$*, and into a sum of a* $(q-1)$*-contact and* q*-contact form on* J^2Y *if* $q > 1$*, respectively.*

Obviously, in the above formula, $\eta_0 = h\eta$, and if $q = 1$, $\eta_1 = p\eta$. The operator assigning to η its i-contact part η_i is denoted by p_i, and called i-*contactization.* In fibered coordinates, $p_{q-1}\eta$ (resp. $p_q\eta$) is expressed as a linear combination of q-forms $\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \cdots \omega^{\sigma_{q-1}} \wedge dt$ (resp. $\omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \cdots \omega^{\sigma_{q-1}} \wedge \omega^{\sigma_q}$).

Corollary 2.1. *The mappings* h *and* p_i, $1 \leq i \leq q$, *restricted to* $\Lambda^q_Y(J^1Y)$ *save the order (i.e. they map* q*-forms on* J^1Y *to* q*-forms on* J^1Y*), and*

$$\begin{aligned} \Lambda^1_Y(J^1Y) &= \Lambda^1_X(J^1Y) \oplus \Omega^{0,1}(J^1Y)\,, \\ \Lambda^q_Y(J^1Y) &= \Omega^{1,q-1}(J^1Y) \oplus \Omega^{0,q}(J^1Y)\,, \qquad q > 1\,. \end{aligned} \tag{2.5}$$

3. The non-holonomic constraint structure

Let $k < m$ be an integer. A *non-holonomic constraint of codimension* k *in* J^1Y is defined to be a fibered submanifold $\pi_{1,0}|_Q : Q \to Y$ of codimension k of the fibered manifold $\pi_{1,0} : J^1Y \to Y$ (cf. e. g. [7], [13], [15], [18]). We denote by ι the canonical embedding of Q into J^1Y. Locally, Q is given by equations

$$f^i = 0, \quad 1 \leq i \leq k\,, \quad \text{where} \quad \operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k\,, \tag{3.1}$$

or, in normal form,

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \quad 1 \le i \le k. \tag{3.2}$$

The submanifold Q is naturally endowed with a distribution, called the *canonical distribution* [7], or *Chetaev bundle* [13], denoted by $\mathcal{C}$. It is annihilated by a system of k linearly independent (local) 1-forms, called *canonical constraint 1-forms*, as follows:

$$\varphi^i = \iota^* \phi^i, \quad \text{where} \quad \phi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma}, \quad 1 \le i \le k, \tag{3.3}$$

i.e.,

$$\varphi^i = -\sum_{l=1}^{m-k} \frac{\partial g^i}{\partial \dot{q}^l} \omega^l + \iota^* \omega^{m-k+i}, \quad 1 \le i \le k. \tag{3.4}$$

The ideal in the exterior algebra of forms on Q generated by the annihilator of $\mathcal{C}$ is called the *constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$, or simply $\mathcal{I}$; its elements are called *constraint forms*. The pair $(Q, \mathcal{C})$ is then called a (*non-holonomic*) *constraint structure* on the fibered manifold π [7], [8].

Let $\tilde{Q}$ be the *lift* of Q in J^2Y, i.e. the manifold of all points $J^2_x\gamma \in J^2Y$ such that $J^1_x\gamma \in Q$. If Q is given by (3.2) then equations of $\tilde{Q}$ are

$$\dot{q}^{m-k+i} = g^i(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}), \quad \ddot{q}^{m-k+i} = \frac{dg^i}{dt}. \tag{3.5}$$

We denote by $\rho : \tilde{Q} \to Q$ the corresponding projection (i.e., $\rho = \pi_{2,1}|_{\tilde{Q}}$). The distribution $\tilde{\mathcal{C}}$ on $\tilde{Q}$, defined by

$$T_y\rho(\tilde{\mathcal{C}}(y)) = \mathcal{C}(\rho(y)), \tag{3.6}$$

for every $y \in \tilde{Q}$, is called the *lift* of $\mathcal{C}$. We have $\dim \tilde{Q} = 3m + 1 - 2k$, $\operatorname{rank} \tilde{\mathcal{C}} = 3m + 1 - 3k$, $\operatorname{corank} \tilde{\mathcal{C}} = \operatorname{corank} \mathcal{C} = k$. The annihilator of $\tilde{\mathcal{C}}$ is locally spanned by the 1-forms $\tilde{\varphi}^i = \rho^* \varphi^i$, $1 \le i \le k$, [10]. In what follows, we denote by $\tilde{\mathcal{I}}$ the ideal on $\tilde{Q}$, generated by $\tilde{\mathcal{C}}^0$.

A section γ of π is called *Q-admissible* if $J^1\gamma(x) \in Q$ for all $x \in \operatorname{dom} \gamma$. An isomorphism ϕ of π is called *Q-compatible* if $J^1\phi(Q) \subset Q$. Obviously, Q-compatible isomorphisms transfer Q-admissible sections into Q-admissible sections. Similarly the concept of a $\tilde{Q}$-admissible section and a $\tilde{Q}$-compatible isomorphism of π is defined. By definition, every Q-admissible section is $\tilde{Q}$-admissible, every Q-compatible isomorphism is $\tilde{Q}$-compatible, and $\rho \circ J^2\phi = J^1\phi \circ \rho$.

If ψ is a local diffeomorphism of Q, recall that the canonical distribution $\mathcal{C}$ is said to be *ψ-invariant* if at each point z of Q, $T\psi(\mathcal{C}(z)) \subset \mathcal{C}(\psi(z))$.

Proposition 3.1. *If $\mathcal{C}$ is ψ-invariant then $\psi^*\mathcal{I} \subset \mathcal{I}$. Moreover, if $\psi = J^1\phi$ where ϕ is a Q-compatible isomorphism of π then also $J^2\phi^*\tilde{\mathcal{I}} \subset \tilde{\mathcal{I}}$.*

Proof. It is sufficient to prove the assertions for constraint 1-forms. By assumption, for every constraint 1-form φ one has at each point $z \in Q$ for all vectors $\xi \in \mathcal{C}(z)$ the following: $i_\xi \psi^*\varphi(z) = \psi^*\varphi(z)(\xi) = \varphi(\psi(z))(T\psi \cdot \xi) = 0$. If additionally $\psi = J^1\phi$ where ϕ is a Q-compatible isomorphism of π then for all the canonical constraint 1-forms, $J^2\phi^*\rho^*\varphi^i = (\rho \circ J^2\phi)^*\varphi^i = (J^1\phi \circ \rho)^*\varphi^i = \rho^*J^1\phi^*\varphi^i \in \rho^*\mathcal{I} \subset \tilde{\mathcal{I}}$. Now, for any constraint 1-form φ on $\tilde{Q}$ one has $\varphi = a_i\rho^*\varphi^i$, where a_i are functions on an open set in $\tilde{Q}$, hence $J^2\phi^*\varphi = (a_i \circ J^2\phi)J^2\phi^*\rho^*\varphi^i \in \tilde{\mathcal{I}}$. □

4. Horizontal and contact forms on constraint manifolds

If Q is a constraint in J^1Y, we denote by $\Lambda^q(Q)$ and $\Lambda^q(\tilde{Q})$ the module of q-forms on Q, and $\tilde{Q}$, respectively. The concepts of a π_1-horizontal and contact form directly transfer to forms on Q: $\eta \in \Lambda^q(Q)$ is called *π_1-horizontal* if $i_\xi\eta = 0$ for every π_1-vertical vector field on Q. η is called *contact* if $J^1\gamma^*\eta = 0$ for every Q-admissible section of π. Notice that contact 1-forms on Q are locally generated by 1-forms $\iota^*\omega^\sigma$, $1 \le \sigma \le m$, i.e. $dq^l - \dot{q}^l dt$, $1 \le l \le m-k$, $dq^{m-k+i} - g^i dt$, $1 \le i \le k$. We denote by $\Lambda^q_X(Q)$, resp. $\Lambda^q_Y(Q)$, resp. $\Omega^q(Q)$ the submodule of π_1-horizontal, resp. $\pi_{1,0}$-horizontal, resp. contact q-forms on Q. Similar definitions and notations are used for $\tilde{Q}$. Apparently, it holds

$$\Lambda^1_Y(Q) = \Lambda^1_X(Q) \oplus \Omega^1(Q)\,, \quad \Lambda^q_Y(Q) = \Omega^q(Q)\,, \quad q > 1\,, \tag{4.1}$$

and similarly for $\tilde{Q}$. This enables one to define the mappings h and p for forms on Q in a similar way as in the unconstrained case, making use of the projection $\rho : \tilde{Q} \to Q$. For $\eta \in \Lambda^q(Q)$, $h\eta$ and $p\eta$ are defined on $\tilde{Q}$.

In what follows, we shall study the structure of forms on Q and $\tilde{Q}$ which is connected with the existence of the canonical constraint structure defined by the constraint ideal $\mathcal{I}$. We denote by $\Lambda^q(\mathcal{I})$ the module of constraint q-forms on Q, and by $\Lambda^q_Y(\mathcal{I})$ the submodule of $\pi_{1,0}$-horizontal constraint q-forms. Similar notations are used if $\tilde{\mathcal{I}}$, the constraint ideal on $\tilde{Q}$ is considered. By $\Lambda^0(\mathcal{I})$, resp. $\Lambda^0(\tilde{\mathcal{I}})$ we understand $\{0\}$. Note that $\Lambda^q(\mathcal{I}) \subset \Omega^q(Q)$.

We can construct quotient modules $\Lambda^q(Q)/\Lambda^q(\mathcal{I})$, resp. $\Lambda^q(\tilde{Q})/\Lambda^q(\tilde{\mathcal{I}})$, the elements of which are equivalence classes $[\alpha]_{\Lambda^q(\mathcal{I})}$, resp. $[\alpha]_{\Lambda^q(\tilde{\mathcal{I}})}$ of q-forms modulo constraint q-forms. The corresponding module operations, as well as the wedge product of classes are defined as usual.

Definition 4.1. $\eta \in \Lambda^q(Q)$, resp. $\eta \in \Lambda^q(\tilde{Q})$ is called *constraint-horizontal* if $i_\xi\eta \in \mathcal{I}$ for every π_1-vertical vector field $\xi \in \mathcal{C}$, resp. $i_\xi\eta \in \tilde{\mathcal{I}}$ for every π_2-vertical vector field $\xi \in \tilde{\mathcal{C}}$.

In particular, a 1-form η on Q (resp. on $\tilde{Q}$) is constraint-horizontal if $i_\xi\eta = 0$ for every π_1-vertical vector field $\xi \in \mathcal{C}$ (resp. for every π_2-vertical vector field $\xi \in \tilde{\mathcal{C}}$). Note that constraint-horizontal 1-forms take the form $\eta = \eta_0 + \varphi$, where η_0 is a horizontal form and φ is a constraint form. Constraint-horizontal q-forms, $q > 1$, coincide with constraint forms.

If $h : \Lambda^q(Q) \to \Lambda^q_X(\tilde{Q})$, and $p : \Lambda^q(Q) \to \Omega^q(\tilde{Q})$ are the "unconstrained" horizontalization and contactization mappings, we can define corresponding mappings between quotient modules:

(4.2)
$$\bar{h} : \Lambda^q(Q)/\Lambda^q(\mathcal{I}) \to (\Lambda^q_X(\tilde{Q}) \oplus \Lambda^q(\tilde{\mathcal{I}}))/\Lambda^q(\tilde{\mathcal{I}}), \quad \bar{h}[\eta]_{\Lambda^q(\mathcal{I})} = [h\eta]_{\Lambda^q(\tilde{\mathcal{I}})} = h\eta + \varphi,$$

$$\bar{p} : \Lambda^q(Q)/\Lambda^q(\mathcal{I}) \to \Omega^q(\tilde{Q})/\Lambda^q(\tilde{\mathcal{I}}), \quad \bar{p}[\eta]_{\Lambda^q(\mathcal{I})} = [p\eta]_{\Lambda^q(\tilde{\mathcal{I}})} = p\eta + \varphi, \tag{4.3}$$

which are defined on equivalence classes modulo constraint q-forms (above, φ runs over constraint q-forms defined on $\tilde{Q}$).

Definition 4.2. The mappings $\bar{h}$ and $\bar{p}$ will be called *constraint horizontalization* and *constraint contactization.*

Obviously, if $\bar{p}[\eta]_{\Lambda^q(\mathcal{I})} = [0]_{\Lambda^q(\mathcal{I})}$ then the equivalence class $[\eta]_{\Lambda^q(\mathcal{I})}$ is *constraint-horizontal.* We say that the equivalence class $[\eta]_{\Lambda^q(\mathcal{I})}$ is *constraint-contact* if $\bar{h}[\eta]_{\Lambda^q(\mathcal{I})} = [0]_{\Lambda^q(\mathcal{I})}$. Notice that constraint forms are both horizontal and contact with respect to these mappings.

Using the properties of the unconstraint horizontalization and contactization we immediately get the following properties of $\bar{h}$ and $\bar{p}$:

Proposition 4.1. *Let $\eta, \lambda \in \Lambda^q(Q)$, $\omega \in \Lambda^p(Q)$, f a function on Q. Consider the projection $\rho : \tilde{Q} \to Q$. It holds*

$$\begin{aligned}
\bar{h}[f\eta]_{\Lambda^q(\mathcal{I})} &= (f \circ \rho) \cdot \bar{h}[\eta]_{\Lambda^q(\mathcal{I})}, \\
\bar{p}[f\eta]_{\Lambda^q(\mathcal{I})} &= (f \circ \rho) \cdot \bar{p}[\eta]_{\Lambda^q(\mathcal{I})}, \\
\bar{h}([\lambda]_{\Lambda^q(\mathcal{I})} + [\eta]_{\Lambda^q(\mathcal{I})}) &= \bar{h}[\lambda]_{\Lambda^q(\mathcal{I})} + \bar{h}[\eta]_{\Lambda^q(\mathcal{I})}, \\
\bar{h}([\lambda]_{\Lambda^q(\mathcal{I})} \wedge [\omega]_{\Lambda^p(\mathcal{I})}) &= \bar{h}[\lambda]_{\Lambda^q(\mathcal{I})} \wedge \bar{h}[\omega]_{\Lambda^p(\mathcal{I})}, \\
\bar{p}([\lambda]_{\Lambda^q(\mathcal{I})} + [\eta]_{\Lambda^q(\mathcal{I})}) &= \bar{p}[\lambda]_{\Lambda^q(\mathcal{I})} + \bar{p}[\eta]_{\Lambda^q(\mathcal{I})}, \\
\bar{p}([\lambda]_{\Lambda^q(\mathcal{I})} \wedge [\omega]_{\Lambda^p(\mathcal{I})}) &= \bar{p}[\lambda]_{\Lambda^q(\mathcal{I})} \wedge \bar{p}[\omega]_{\Lambda^p(\mathcal{I})} \\
&\quad + \bar{p}[\lambda]_{\Lambda^q(\mathcal{I})} \wedge \bar{h}[\omega]_{\Lambda^p(\mathcal{I})} + \bar{h}[\lambda]_{\Lambda^q(\mathcal{I})} \wedge \bar{p}[\omega]_{\Lambda^p(\mathcal{I})}.
\end{aligned} \tag{4.4}$$

Proposition 4.2.

(1) *The equivalence class $[\eta]_{\Lambda^q(\mathcal{I})}$ is constraint-horizontal (resp. contact) if and only if $\bar{p}[\eta]_{\Lambda^q(\mathcal{I})} = [0]_{\Lambda^q(\mathcal{I})}$ (resp. $\bar{h}[\eta]_{\Lambda^q(\mathcal{I})} = [0]_{\Lambda^q(\mathcal{I})}$).*

(2) *If $q > \dim X$ then $\bar{h}[\eta]_{\Lambda^q(\mathcal{I})} = [0]_{\Lambda^q(\tilde{\mathcal{I}})}$, $\bar{p}[\eta]_{\Lambda^q(\mathcal{I})} = [\rho^*\eta]_{\Lambda^q(\tilde{\mathcal{I}})}$.*

(3) *$\bar{h}[\eta]_{\Lambda^q(\mathcal{I})}$ is a unique class of constraint-horizontal q-forms such that for every Q-admissible section γ of π the condition $J^1\gamma^*[\eta]_{\Lambda^q(\mathcal{I})} = J^2\gamma^*\bar{h}[\eta]_{\Lambda^q(\mathcal{I})}$ is satisfied.*

(4) *For every Q-admissible section γ of π the condition $J^2\gamma^*\bar{p}[\eta]_{\Lambda^q(\mathcal{I})} = 0$ is satisfied.*

(5) *If* $[\eta]_{\Lambda^q_Y(\mathcal{I})}$ *is* $\pi_{1,0}$*-horizontal then both* $\bar{h}[\eta]_{\Lambda^q_Y(\mathcal{I})}$ *and* $\bar{p}[\eta]_{\Lambda^q_Y(\mathcal{I})}$ *are* ρ*-projectable.*

(6) $\bar{p}[\eta]_{\Lambda^q(\mathcal{I})}$ *is* ρ*-projectable iff* $\bar{h}[\eta]_{\Lambda^q(\mathcal{I})}$ *is* ρ*-projectable.*

(7) *If* ϕ *is a* Q*-compatible isomorphism of* π*, leaving invariant the canonical distribution then*

$$\bar{h}J^1\phi^*[\eta]_{\Lambda^q(\mathcal{I})} = J^2\phi^*\bar{h}[\eta]_{\Lambda^q(\mathcal{I})}\,, \quad \bar{p}J^1\phi^*[\eta]_{\Lambda^q(\mathcal{I})} = J^2\phi^*\bar{p}[\eta]_{\Lambda^q(\mathcal{I})}\,. \tag{4.5}$$

Proof. (1)–(6) follow directly from the definitions. (7) is a consequence of Proposition 3.1. and definition of $\bar{h}$ and $\bar{p}$. Indeed, since ϕ transforms constraint forms into constraint forms, we obtain

$$\begin{aligned}\bar{h}J^1\phi^*[\eta]_{\Lambda^q(\mathcal{I})} &= \bar{h}[J^1\phi^*\eta]_{\Lambda^q(\mathcal{I})} = [hJ^1\phi^*\eta]_{\Lambda^q(\tilde{\mathcal{I}})} \\ &= [J^2\phi^*h\eta]_{\Lambda^q(\tilde{\mathcal{I}})} = J^2\phi^*[h\eta]_{\Lambda^q(\tilde{\mathcal{I}})} = J^2\phi^*\bar{h}[\eta]_{\Lambda^q(\mathcal{I})}\,,\end{aligned} \tag{4.6}$$

and similarly for $\bar{p}$. □

Now, we can see that the following decomposition theorems take place:

Theorem 4.1. *Every equivalence class of* 1*-forms admits a unique decomposition*

$$\rho^*[\eta]_{\Lambda^1(\mathcal{I})} = \bar{h}[\eta]_{\Lambda^1(\mathcal{I})} + \bar{p}[\eta]_{\Lambda^1(\mathcal{I})}\,. \tag{4.7}$$

Corollary 4.1. *For* $\eta = df$ (4.7) *gives a unique decomposition of* df *into a constraint-horizontal and constraint-contact part (constraint-horizontal and constraint-contact differential), respectively:*

$$\rho^*[df]_{\Lambda^1(\mathcal{I})} = \bar{h}[df]_{\Lambda^1(\mathcal{I})} + \bar{p}[df]_{\Lambda^1(\mathcal{I})}\,. \tag{4.8}$$

It holds

$$\bar{h}[df]_{\Lambda^1(\mathcal{I})} = \frac{d_{\mathcal{C}}f}{dt}dt + \varphi\,, \quad \bar{p}[df]_{\Lambda^1(\mathcal{I})} = \frac{\partial_{\mathcal{C}}f}{\partial q^l}\omega^l + \frac{\partial f}{\partial \dot{q}^l}\dot{\omega}^l + \varphi\,,$$

where φ *runs over constraint* 1*-forms on* Q*, and*

$$\begin{aligned}\frac{d_{\mathcal{C}}f}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^l}\dot{q}^l + \frac{\partial f}{\partial q^{m-k+i}}g^i + \frac{\partial f}{\partial \dot{q}^l}\ddot{q}^l\,, \\ \frac{\partial_{\mathcal{C}}f}{\partial q^s} &= \frac{\partial f}{\partial q^s} + \frac{\partial g^i}{\partial \dot{q}^s}\frac{\partial f}{\partial q^{m-k+i}}\end{aligned} \tag{4.9}$$

(above, summation over $l = 1, \ldots, m-k$*, and* $i = 1\ldots, k$ *is understood).*

The operators $d_{\mathcal{C}}/dt$ and $\partial_{\mathcal{C}}/\partial q^l$ will be called *constraint total derivative* and *constraint partial derivative* respectively. Notice that both they are directional derivatives with respect to vector fields belonging to the lift of the canonical distribution $\mathcal{C}$; in particular, $d_{\mathcal{C}}/dt = \partial_\Gamma$, the Lie derivative along a *semispray* $\Gamma \in \tilde{\mathcal{C}}$.

Restricting considerations to classes of $\pi_{1,0}$-horizontal 1-forms the following decomposition of quotient modules follows:

Theorem 4.2. *It holds*

$$\Lambda^1_Y(Q)/\Lambda^1(\mathcal{I}) = \big((\Lambda^1_X(Q) \oplus \Lambda^1(\mathcal{I}))/\Lambda^1(\mathcal{I})\big) \oplus \big(\Omega^1(Q)/\Lambda^1(\mathcal{I})\big). \tag{4.10}$$

A contact form $\eta \in \Lambda^q(Q)$ is called *1-contact* if for every π_1-vertical vector field ξ on Q the $(q-1)$-form $i_\xi \eta$ is horizontal. Let $i > 1$. $\eta \in \Lambda^q(Q)$ is called *i-contact* if for every π_1-vertical vector field ξ on Q, $i_\xi \eta$ is $(i-1)$-contact. Denote by $\Omega^{q-i,i}(Q)$ the submodule of i-contact q-forms on Q. Similar definitions and notations take place for forms on $\tilde{Q}$.

If $q > 1$ and $p_i : \Lambda^q(Q) \to \Omega^{q-i,i}(\tilde{Q})$ is the "unconstrained" i-contactization mapping, we can define a corresponding mapping between quotient modules:

$$\begin{aligned} &\bar{p}_i : \Lambda^q(Q)/\Lambda^q(\mathcal{I}) \to \big(\Omega^{q-i,i}(\tilde{Q}) + \Lambda^q(\tilde{\mathcal{I}})\big)/\Lambda^q(\tilde{\mathcal{I}}), \\ &\bar{p}_i[\eta]_{\Lambda^q(\mathcal{I})} = [p_i\eta]_{\Lambda^q(\tilde{\mathcal{I}})} = p_i\eta + \varphi, \end{aligned} \tag{4.11}$$

where φ runs over constraint q-forms. Note that above, $+$ is not a direct sum.

Definition 4.3. $\bar{p}_i$ is called *constraint i-contactization.*

Similarly as above, we get:

Theorem 4.3. *Every equivalence class of q-forms* $(q > 1)$ *admits a unique decomposition*

$$\rho^*[\eta]_{\Lambda^q(\mathcal{I})} = \bar{p}_{q-1}[\eta]_{\Lambda^q(\mathcal{I})} + \bar{p}_q[\eta]_{\Lambda^q(\mathcal{I})}. \tag{4.12}$$

Obviously if $\bar{p}_q[\eta]_{\Lambda^q(\mathcal{I})} = 0$, then the class $[\eta]_{\Lambda^q(\mathcal{I})}$ is constraint $(q-1)$-contact, and if $\bar{p}_{q-1}[\eta]_{\Lambda^q(\mathcal{I})} = 0$, then $[\eta]_{\Lambda^q(\mathcal{I})}$ is constraint q-contact. Notice that since constraint q-forms belong to the zero class, they are both $(q-1)$-contact and q-contact with respect to these mappings.

Theorem 4.4. *The following decomposition takes place:*

$$\Lambda^q_Y(Q)/\Lambda^q_Y(\mathcal{I}) = \big((\Omega^{1,q-1}(Q)\Lambda^q_Y(\mathcal{I}))/\Lambda^q_Y(\mathcal{I})\big) \oplus \big((\Omega^{0,q}(Q) + \Lambda^q_Y(\mathcal{I}))/\Lambda^q_Y(\mathcal{I})\big). \tag{4.13}$$

References

[1] J. F. Cariñena and M. F. Rañada, *Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers*, J. Phys. A: Math. Gen. **26** (1993), 1335–1351.

[2] G. Giachetta, *Jet methods in nonholonomic mechanics*, J. Math. Phys. **33** (1992), 1652–1665.

[3] W. S. Koon and J. E. Marsden, *The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems*, Rep. Math. Phys. **40** (1997), 21–62.

[4] D. Krupka, *Some geometric aspects of variational problems in fibered manifolds*, Folia Fac. Sci. Nat. UJEP Brunensis **14** (1973), 1–65.

[5] D. Krupka, *Lepagean forms in higher order variational theory*, in: Modern Developments in Analytical Mechanics I: Geometrical Dynamics, Proc. IUTAM-ISIMM Symposium, Torino, Italy 1982, S. Benenti, M. Francaviglia and A. Lichnerowicz, eds. (Accad. delle Scienze di Torino, Torino, 1983), 197–238.

[6] O. Krupková, *The Geometry of Ordinary Variational Equations*, Lecture Notes in Math. **1678**, Springer, Berlin, 1997.

[7] O. Krupková, *Mechanical systems with nonholonomic constraints*, J. Math. Phys. **38** (1997), 5098–5126.

[8] O. Krupková, *On the geometry of non-holonomic mechanical systems*, in: Proc. Conf. Diff. Geom. Appl., Brno, August 1998 (Masaryk University, Brno, 1999), 533–546.

[9] O. Krupková, *Recent results in the geometry of constrained systems*, Rep. Math. Phys. **49** (2002), 269–278.

[10] O. Krupková and M. Swaczyna, *The non-holonomic variational principle*, Preprint 8/2002, Inst. Theor. Phys. and Astrophys., Masaryk University, Brno, Czech Republic (2002), 34pp.

[11] M. de León, J. C. Marrero and D. M. de Diego, *Non-holonomic Lagrangian systems in jet manifolds*, J. Phys. A: Math. Gen. **30** (1997), 1167–1190.

[12] E. Massa and E. Pagani, *Classical mechanics of non-holonomic systems: a geometric approach*, Ann. Inst. Henri Poincaré **55** (1991), 511-544.

[13] E. Massa and E. Pagani, *A new look at classical mechanics of constrained systems*, Ann. Inst. Henri Poincaré **66** (1997), 1–36.

[14] M. F. Rañada, *Time-dependent Lagrangian systems: A geometric approach to the theory of systems with constraints*, J. Math. Phys. **35** (1994), 748–758.

[15] W. Sarlet, *A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems*, Extracta Mathematicae **11** (1996), 202–212.

[16] W. Sarlet, F. Cantrijn and D. J. Saunders, *A geometrical framework for the study of non-holonomic Lagrangian systems*, J. Phys. A: Math. Gen. **28** (1995), 3253–3268.

[17] D. J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lecture Notes Series **142**, Cambridge University Press, Cambridge, 1989.

[18] D. J. Saunders, W. Sarlet and F. Cantrijn, *A geometrical framework for the study of non-holonomic Lagrangian systems: II*, J. Phys. A: Math. Gen. **29** (1996), 4265–4274.

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