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CONFORMAL DE RHAM HODGE THEORY AND OPERATORS GENERALISING THE Q-CURVATURE

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ABSTRACT. We look at several problems in even dimensional conformal geometry based around the de Rham complex. A leading and motivating problem is to find a conformally invariant replacement for the usual de Rham harmonics. An obviously related problem is to find, for each order of differential form bundle, a "gauge" operator which completes the exterior derivative to a system which is both elliptically coercive and conformally invariant. Treating these issues involves constructing a family of new operators which, on the one hand, generalise Branson's celebrated Q-curvature and, on the other hand, compose with the exterior derivative and its formal adjoint to give operators on differential forms which generalise the critical conformal power of the Laplacian of Graham-Jenne-Mason-Sparling. We prove here that, like the critical conformal Laplacians, these conformally invariant operators are not strongly invariant. The construction draws heavily on the ambient metric of Fefferman-Graham and its relationship to the conformal tractor connection and exploring this relationship will be a central theme of the lectures.

These notes draw on recent collaborative work with Tom Branson. There is also significant input from recent joint work with Andi Čap and Larry Peterson. Among the results that are completely new here is Proposition 1.1 which proves that the conformally invariant differential operators between forms, that we construct here, have the curious property that thay are not strongly invariant. Also it is shown, in the final section, that the "Q-operators", which were first developed in [6], can be recovered by a polynomial continuation argument that parallels and generalises Branson's original construction of the Q-curvature. These notes were presented as a series of three lectures at the 24^{th} Winter School on Geometry and Physics, Srní Czech Republic, January 2004.

1. LECTURE 1 – SOME PROBLEMS RELATED TO THE DE RHAM COMPLEX

The de Rham complex and its cohomology are among the most fundamental of tools for relating local differential geometric information to global topology. Over

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these lectures we shall explore the de Rham complex and related issues in the setting of conformal geometry.

On a smooth n-manifold M, let us write $\mathcal{E}$ or $\mathcal{E}^0$ for $C^\infty(M)$ and $\mathcal{E}^k$ for the space of k-forms, i.e., the smooth sections of the k$^{\text{th}}$ exterior power of the cotangent bundle $\Lambda^k T^*M$. Recall that the exterior derivative on functions takes values in T^*M and is defined by $df(v) = vf$ where we view the smooth tangent vector field v as a derivation (so in terms of local coordinates x^i we have $vf = \sum v^i \partial f/\partial x^i$). This is extended to a differential operator $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ by requiring $d^2 f := d(df) = 0$, for $f \in \mathcal{E}$, and the Leibniz rule $dfw = df \wedge w + fdw$, $w \in \mathcal{E}^k$. It follows that d^2 vanishes on k-forms and so we obtain the de Rham complex,

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^n .$$

We are interested in the additional operators between the form bundles that arise when M is equipped with a conformal structure. Recall that a *conformal structure* is an equivalence class of metrics $[g]$ where two metrics are equivalent if they are related by multiplication by a smooth positive function, i.e., $\widehat{g} \sim g$ means there is $\omega \in \mathcal{E}$ such that $\widehat{g} = e^{2\omega} g$. We may equivalently view the conformal class as being given by a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$, whose fibre at x is formed by the values of g_x for all metrics g in the conformal class. By construction, $\mathcal{Q}$ has fibre $\mathbb{R}_+$ and the metrics in the conformal class are in bijective correspondence with smooth sections of $\mathcal{Q}$.

The bundle $\pi : \mathcal{Q} \to M$ is a principal bundle with structure group $\mathbb{R}_+$, and we denote by $E[w]$ the line bundle induced from the representation of $\mathbb{R}_+$ on $\mathbb{R}$ given by $s \mapsto s^{-w/2}$. Sections of $E[w]$ are called a *conformal densities of weight* w and may be identified with functions on $\mathcal{Q}$ that are homogeneous of degree w, i.e., $f(s^2 g_x, x) = s^w f(g_x, x)$ for any $s \in \mathbb{R}_+$. We write $\mathcal{E}[w]$ for the space of sections of the bundle and, for example, $\mathcal{E}^k[w]$ is the space of sections of $(\Lambda^k T^*M) \otimes E[w]$. (Here and elsewhere all sections are taken to be smooth.)

There is a tautological function $\boldsymbol{g}$ on $\mathcal{Q}$ taking values in $\mathcal{E}_{(ab)} := S^2T^*M$. It is the function which assigns to the point $(g_x, x) \in \mathcal{Q}$ the metric g_x at x. This is homogeneous of degree 2 since $\boldsymbol{g}(s^2 g_x, x) = s^2 g_x$. If σ is any positive function on $\mathcal{Q}$ homogeneous of degree 1 then $\sigma^{-2}\boldsymbol{g}$ is independent of the action of $\mathbb{R}_+$ on the fibres of $\mathcal{Q}$, and so $\sigma^{-2}\boldsymbol{g}$ descends to give a metric from the conformal class. Thus $\boldsymbol{g}$ determines and is equivalent to a canonical section of $\mathcal{E}_{ab}[2]$ (called the conformal metric) that we also denote $\boldsymbol{g}$ (or $\boldsymbol{g}_{ab}$). Then, for $\sigma \in \mathcal{E}_+[1]$, $\sigma^{-2}\boldsymbol{g}$ is a metric from the conformal class and we term σ a *conformal scale.* We will use the conformal metric to raise and lower indices.

Recall that the Levi Civita connection is the unique torsion free connection on tensor bundles which preserves a given metric. So on a conformal manifold a choice of conformal scale σ determines a Levi Civita connection that we will denote ∇. The scale σ also determines a connection (that we also denote ∇ and term the Levi Civita connection) on densities by the formula $\nabla\mu = \sigma^w d\sigma^{-w}\mu$, for $\mu \in \mathcal{E}[w]$. (Note that $\sigma^w d\sigma^{-w}\mu$ means $\sigma^w(d(\sigma^{-w}\mu))$. The default is that in such expressions all symbols, except the one at the extreme right, are to be interpreted as operators and parentheses are usually omitted.) For $g = \sigma^{-2}\boldsymbol{g}$ the *conformal rescaling* $g \mapsto \widehat{g} = e^{2\omega} g$ corresponds to $\sigma \mapsto \hat{\sigma} = e^{-\omega}\sigma$ and so it follows at once that

$$\widehat{\nabla}\mu = \nabla\mu + w\Upsilon\mu, \tag{1}$$

where $\widehat{\nabla}$ is the connection for $\hat{\sigma}$ and $\Upsilon := d\omega$. It is similarly easy to show that, for example on 1-forms, the Levi Civita connection transforms conformally according to

$$\widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + g_{ab}\Upsilon^c u_c, \tag{2}$$

where abstract indices are used in an obvious way and the inverse of g is used to raise the index on Υ^c.

To simplify our subsequent discussion let us assume that M is connected, compact and orientable, and that the conformal structure is Riemannian (i.e., $g \in [g]$ has Riemannian signature). Via the conformal metric, the bundle of volume densities can be canonically identified with $E[-n]$ and so the Hodge star operator (for each metric from the conformal class) induces a conformally invariant isomorphism that we shall also term the Hodge star operator: $\star : \mathcal{E}^k \cong \mathcal{E}^{n-k}[n-2k]$. Let us write $\mathcal{E}_{n-k}$ as an alternative notation for the image space here, so we have $\star : \mathcal{E}^k \cong \mathcal{E}_{n-k}$ and more generally $\mathcal{E}_k[w] := \mathcal{E}^k[w+2k-n]$. This notation is suggested by the duality between the section spaces $\mathcal{E}^k$ and $\mathcal{E}_k$. For $\varphi \in \mathcal{E}^k$ and $\psi \in \mathcal{E}_k$, there is the natural conformally invariant global pairing

$$\varphi, \psi \mapsto \langle \varphi, \psi \rangle := \int_M \varphi{\cdot}\psi \, d\mu_g = \int_M \varphi \wedge \star\psi,$$

where $\varphi{\cdot}\psi \in \mathcal{E}[-n]$ denotes a complete contraction between φ and ψ.

In even dimensions $\mathcal{E}^{n/2} = \mathcal{E}_{n/2}$ and the de Rham complex may be written in the more symmetric form

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n/2-2} \xrightarrow{d} \mathcal{E}^{n/2-1} \xrightarrow{d} \mathcal{E}^{n/2} \xrightarrow{\star d} \mathcal{E}_{n/2-1} \xrightarrow{\delta} \mathcal{E}_{n/2-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0,$$

where δ is the composition $\star d \star$. (Or one could alternatively replace the $\xrightarrow{d} \mathcal{E}^{n/2} \xrightarrow{\star d}$ with $\xrightarrow{\star d} \mathcal{E}^{n/2} \xrightarrow{\delta}$.) Of course these operators are all conformally invariant since the exterior derivative is well defined on any smooth manifold and the conformal structure is used here only to give the isomorphisms $\star : \mathcal{E}^k \to \mathcal{E}_{n-k}$ and $\star : \mathcal{E}_k \to \mathcal{E}^{n-k}$. Since $\star$ maps $\mathcal{E}^{n/2}$ to itself we also have the conformally invariant operator $\delta : \mathcal{E}^{n/2} \to \mathcal{E}_{n/2-1}$. This does not annihilate the exact forms; in fact in terms of the global pairing introduced above we have $\langle \varphi, \delta d\varphi \rangle = \langle d\varphi, d\varphi \rangle$ and so, in the compact setting, $\delta d\varphi = 0$ implies $d\varphi = 0$. Rather the composition $\delta d : \mathcal{E}^{n/2-1} \to \mathcal{E}_{n/2-1}$ is the well known conformally invariant *Maxwell operator*. (In dimension 4 and Lorentzian signature this gives the equations of electromagnetism.) Thus we have the Maxwell *detour complex*,

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n/2-2} \xrightarrow{d} \mathcal{E}^{n/2-1} \xrightarrow{\delta d} \mathcal{E}_{n/2-1} \xrightarrow{\delta} \mathcal{E}_{n/2-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0.$$

This really is symmetric since, for each k, the operator $\delta : \mathcal{E}_{k+1} \to \mathcal{E}_k$ is (up to a sign) the formal adjoint of $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$. It is convenient to absorb this sign and redefine δ to be exactly the formal adjoint, i.e., so that $\langle \varphi, \delta\psi \rangle = \langle d\varphi, \psi \rangle$ for $\varphi \in \mathcal{E}^k$ and $\psi \in \mathcal{E}_{k+1}$. The Maxwell operator and this detour complex are a feature of even dimensional conformal geometry that we wish to study and generalise. There are not analogues in odd dimensions and so, for the remainder of this lecture (except where otherwise indicated), let us suppose $n \geq 4$ is even.

As a point on notation. We will use $\iota(\cdot)$ and $\varepsilon(\cdot)$ as the notation for interior and exterior multiplication by 1-forms on differential forms. For a 1-form u and a k-form v the conventions are

$$(\varepsilon(u)v)_{a_0\dots a_k} = (k+1)u_{a_0}v_{a_1\dots a_k}, \quad \text{and} \quad (\iota(u)v)_{a_2\dots a_k} = u^{a_1}v_{a_1a_2\dots a_k}.$$

Here, and below, sequentially labelled indices are implicitly skewed over.

1.1. **The problems.** We are now set to state and consider a series of fundamental problems concerning the de Rham complex on conformal manifolds.

Problem 1: Conformal Hodge theory. Let us write $H^k(M)$ for the k^{th} cohomology space of the de Rham complex. If M is equipped with a Riemannian metric g, then de Rham Hodge theory exhibits an isomorphism between $H^k(M)$ and the space of *harmonics* $\mathcal{H}^k(M)$. The latter is the null space of the form Laplacian $\not{\Delta} = \delta d + d\delta$ on k-forms or, alternatively, it is recovered by

$$\mathcal{H}^k(M) = \mathcal{N}(d : \mathcal{E}^k \to \mathcal{E}^{k+1}) \cap \mathcal{N}(\delta : \mathcal{E}^k \to \mathcal{E}^{k-1}).$$

Here δ is not, in general, the conformally invariant operator described above but rather just the Riemannian formal adjoint of d. For $u = u_{a_1\dots a_k}$ a k-form we have

$$(du)_{a_0\dots a_k} = (k+1)\nabla_{a_0}u_{a_1\dots a_k},$$

since the Levi Civita connection is torsion free. Thus δ is given by

$$v_{a_0\dots a_k} \mapsto -\nabla^{a_0}v_{a_0\dots a_k}.$$

This agrees with the conformally invariant δ only if $v \in \mathcal{E}_k$; since only then is the integration by parts a conformally invariant operation. (A point on notation: we will write du to mean $(k+1)\nabla_{a_0}u_{a_1\dots a_k}$ and δv to mean $-\nabla^{a_0}v_{a_0\dots a_k}$ even when the density weights of u and v are such that these are not conformally invariant.) Note that $\mathcal{E}^k = \mathcal{E}_k \otimes \mathcal{E}[n-2k]$. Thus from the Leibniz rule for ∇ and (1) one immediately has that, on $\mathcal{E}^k$,

$$\hat{\delta} = \delta - (n-2k)\iota(\Upsilon).$$

Thus this is conformally invariant if $k = n/2$ and also if $k = 0$ (since both sides then act trivially) but not otherwise. So $\mathcal{H}^0$, which is just the space $\mathcal{C}^0$ of constant functions, is conformally invariant and so also is $\mathcal{H}^{n/2}$. Otherwise the harmonics $\mathcal{H}^k$ move in $\mathcal{E}^k$ depending on the choice of metric. The problem is to find a replacement space which is both isomorphic to $H^k(M)$ and stable under conformal transformations.

Without drawing on the details of Hodge theory we can see at the outset that the Riemannian system (d, δ) has a finite dimensional null space, since it is an elliptically coercive system. The notion of (graded) ellipticity we are using here, and below, is that the operator concerned is a right factor of an operator with leading term a power of the Laplacian. In this case the power is one:

$$(\delta, d)\begin{pmatrix} d \\ \delta \end{pmatrix} = -\Delta + \text{LOT},$$

where Δ denotes the Bochner Laplacian $\nabla^a\nabla_a$ and LOT indicates lower order terms. This analysis suggests another problem.

FIGURE 1. The conformal de Rham diagram in even dimensions

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n/2-2} \xrightarrow{d} \mathcal{E}^{n/2-1} \xrightarrow{d} \mathcal{E}^{n/2} \xrightarrow{\star d} \mathcal{E}_{n/2-1} \xrightarrow{\delta} \mathcal{E}_{n/2-2} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0 ,$$

$$L_{n/2-1} = \delta d \qquad L_{n/2-2} \qquad L_0$$

Problem 2: Gauge companion operators. For each k attempt to find a differential operator G_k satisfying the following:
1. G_k is conformally invariant on the null space $\mathcal{C}^k := \mathcal{N}(d : \mathcal{E}^k \to \mathcal{E}^{k+1})$.
2. (In a choice of scale) the system (d, G_k) is elliptic.

On even dimensional manifolds there is another family of conformally invariant operators between differential forms. These are operators:

$$L_k : \mathcal{E}^k \to \mathcal{E}_k \quad L_k = (\delta d)^{n/2-k} + \text{LOT} \quad k \in \{0, 1, \ldots, n/2\}$$

which we will term the *long* operators since they complete the de Rham complex to the picture in figure 1. While for $k \geq 1$ the existence of operators between the bundles concerned can be concluded from the general results of Eastwood and Slovák in [16], these are not unique and it turns out that our problems above are related to the possibility of a special class of such operators.

Problem 3: A preferred class of long operators. The problem here is to attempt to establish existence of, or even better give a construction for, conformally invariant operators $L_k : \mathcal{E}^k \to \mathcal{E}_k$ which factor through δ and d in the sense,

$$L_k = \delta M_{k+1} d . \tag{3}$$

Such operators exist in the conformally flat case and we want the L_k to generalise these flat case operators. The operators sought should also be natural, that is in a choice of scale they should be given by a formula polynomial in the Levi Civita connection ∇ and its curvature R.

For $k = n/2 - 1$ the solution to this problem is the Maxwell operator δd mentioned above. Somewhat more compelling evidence that there could be a positive solution to this problem, in general, dates back to Branson's [2] which provides direct constructions of such operators at orders 4 and 6 (i.e. on, respectively, $\mathcal{E}^{n/2-2}$ and $\mathcal{E}^{n/2-3}$). At the other extreme of order Graham et al (GJMS) [24] give an order n conformally invariant differential operator $P_n : \mathcal{E}^0 \to \mathcal{E}_0$ which has the desired form (3).

A powerful, and well understood, tool for generating conformally invariant operators from other appropriate conformally invariant operators is the *curved translation principle* of Eastwood and Rice [13, 12]. However this does not predict the operators P_n. The existence of these is subtle and the construction of GJMS uses the ambient metric of Fefferman-Graham. We will see below that the curved translation principle cannot in general yield operators of the form (3). Before we discuss these difficulties, we draw in the final main problem which is linked to the GJMS operators.

For each even integer $n_0 \geq 4$, the construction of GJMS gives, not just an operator of order n_0 in dimension n_0, but an order n_0 conformally invariant differential operator $P_{n_0} = \Delta^{n_0/2} + \text{LOT}$ for all odd dimensions and for even dimensions $n \geq n_0$. It is an observation of Branson [3, 4] that, in any choice of scale, these take the form $\delta M_1 d + \frac{1}{2}(n-n_0)Q(n,n_0)$, also that there exists a universal expression for the order zero part $Q(n,n_0)$ which is rational in n (without singularity at n_0) and that setting $n=n_0$ in this yields a remarkable curvature quantity $Q := Q(n_0,n_0)$. This has the conformal transformation

(I) $\quad Q^{\hat{g}} = Q^g + P_{n_0}\omega \quad$ where $\quad \hat{g} := e^{2\omega}g.$

In dimension n_0, $P_{n_0} = \delta M_1 d$ and so $\mathcal{R}(P_{n_0}) \subseteq \mathcal{R}(\delta)$. Thus Q gives a conformally invariant operator

(II) $\quad Q : \mathcal{C}^0 \to H_0(M) \cong H^n(M),$

by $c \mapsto [cQ]$. Also Q has density weight $-n_0$ and so, since $\mathcal{R}(P_{n_0}) \subseteq \mathcal{R}(\delta)$ and Q transforms conformally as in (I), it follows that $\int Q$ is a conformal invariant. In the conformally flat case $\int Q$ recovers a non-zero multiple of the Euler characteristic and so the map (II) is in general non-trivial. In fact the operators P_{n_0} are (formally) self-adjoint so more generally we have,

(III) $\quad c \in \mathcal{C}^0$ and $u \in \mathcal{N}(P_{n_0}) \Rightarrow \int uQc$ is conformally invariant.

It turns out that the Q-curvature has a serious role in geometric analysis and low dimensional topology [10, 11]. There are also connections with the AdS/CFT correspondence of quantum gravity [18] and scattering theory [25]. There have been recent alternative direct constructions of Q via tractor calculus [22], and the ambient construction [19], which avoid dimensional continuation. However there are still many mysteries. In particular an important question is whether there are other similar quantities.

Problem 4: Understand/generalise Branson's Q-curvature. Broadly the problem here is to find an analogue or "closest relative" of the Q-curvature for forms. Aside from shedding light on the Q-curvature itself, the idea is to investigate the existence of other objects which are not conformally invariant locally and yet, by some analogy with (I), (II) and (III) above, yield new global conformal invariants.

1.2. **Earlier work.** The 4th order GJMS operator P_2 is first due to Paneitz [27]. In dimension 4 this acts on functions and (given a choice of metric g) has the formula

$$\delta(d\delta + 2\mathsf{J} - 4\mathsf{P}\sharp)d\,.$$

Here P is the Schouten (or Rho) tensor, viewed as weighted section of $\mathrm{End}(T^*M)$, J its trace and $\sharp$ is the obvious tensorial action. Recall these are related to the Ricci tensor for g by $\mathrm{Ric}(g) = (n-2)\mathsf{P} + \mathsf{J}g$. Thus, in dimension 4, the operator $G = \frac{1}{2}\delta(d\delta + 2\mathsf{J} - 4\mathsf{P}\sharp)$ is conformally invariant on exact 1-forms. It is easily verified by direct calculation that G is also conformally invariant on the space of closed 1-forms and hence also on the null space of the Maxwell operator. Eastwood and Singer made this observation and proposed G as a gauge operator for the Maxwell operator [14]. Note that

$$(\delta d\delta, d)\begin{pmatrix} d \\ \delta d\delta + \mathrm{LOT} \end{pmatrix} = \Delta^2 + \mathrm{LOT}\,.$$

and so $(d,\ G)$ gives a solution to problem 2 for d on 1-forms in dimension 4.

It was observed by the author and Branson [5] that this gauge operator can be recovered from an adaption of the curved translation principle. This idea also provides a conceptual framework and practical approach to constructing gauge operators for many other conformally invariant operators. First we need the basic idea of the tractor connection and its associated calculus.

Any conformal n-manifold, such that $n \geq 3$, admits a unique normal tractor bundle and connection. The tractor connection is a connection on a vector bundle that we term the standard conformal tractor bundle $\mathbb{T}$. We write $\mathcal{T}$ for the space of sections of $\mathbb{T}$. For a choice of metric g from the conformal class this bundle can be identified with the direct sum $[\mathbb{T}]_g = E[1] \oplus E^1[1] \oplus E[-1]$, where $E^1[1]$ means $T^*M \otimes E[1]$. Assigning abstract indices we could instead write $[\mathbb{T}^A]_g = E[1] \oplus E_a[1] \oplus E[-1]$. Thus a section $V \in \mathcal{T}$ then corresponds to a triple (α, μ, ρ) of sections from the direct sum according to $V^A = Y^A\alpha + Z^{Ab}\mu_b + X^A\rho$ (where this defines the "projectors" X, Z and Y). Under a conformal rescaling $g \mapsto \widehat{g} = e^{2\omega}g$, this triple transforms according to

$$(4) \qquad [V]_g = \begin{pmatrix} \alpha \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_b & \delta^a_b & 0 \\ -\frac{1}{2}\Upsilon^a\Upsilon_a & -\Upsilon^a & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_a \\ \rho \end{pmatrix} = [V]_{\hat{g}}$$

where $\Upsilon := d\omega$. It is easily verified that this determines an equivalence relation on the triples over the equivalence relation on metrics and hence the quotient gives $\mathbb{T}$ as a well defined vector bundle on $(M, [g])$ with a composition series $\mathbb{T} = E[1] \oplus\!\!\!\!- E^1[1] \oplus\!\!\!\!- E[-1]$ (meaning that $E[-1]$ is a subbundle of $\mathbb{T}$ and $E^1[1]$ is a subbundle of the quotient $\mathbb{T}/E[-1]$).

In terms of this splitting for g the conformally invariant tractor metric is given by $h(V, V) = g^{ab}\mu_a\mu_b + 2\alpha\rho$. The tractor connection [1] is given by

$$(5) \qquad [\nabla_a V^B]_g = \begin{pmatrix} \nabla_a\alpha - \mu_a \\ \nabla_a\mu_b + g_{ab}\rho + \mathsf{P}_{ab}\alpha \\ \nabla_a\rho - \mathsf{P}_{ab}\mu^b \end{pmatrix}.$$

In terms of this formula, the conformal invariance of the connection is recognised by the fact that the components on the right-hand-side transform, under conformal rescaling, according to (4). In subsequent calculations we will often omit the $[\cdot]_g$ which emphasises the choice of splitting, since, in any case, this should be clear by the context. For the purposes of calculations it is often more convenient (see [22]) to use that the connection is determined by

$$(6) \qquad \nabla_a X_A = Z_{Aa}\,, \quad \nabla_a Z_{Ab} = -\mathsf{P}_{ab}X_A - Y_A g_{ab}\,, \quad \nabla_a Y_A = \mathsf{P}_{ab}Z_A{}^b\,,$$

and the Leibniz rule. The tractor bundle and connection are induced by, and are equivalent to, the normal conformal Cartan connection, see [9].

The bundle of k-form tractors $\mathbb{T}^k$ is the $k^{\underline{\mathrm{th}}}$ exterior power of the bundle of standard tractors. This has a composition series which, in terms of section spaces, is given by

$$(7) \qquad \mathcal{T}^k = \Lambda^k\mathcal{T} \cong \mathcal{E}^{k-1}[k] \oplus\!\!\!\!- \left(\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\right) \oplus\!\!\!\!- \mathcal{E}^{k-1}[k-2]\,.$$

Given a choice of metric g from the conformal class there is a splitting of this composition series corresponding to the splitting of $\mathbb{T}$ as mentioned above. Relative to this, a typical k-form tractor field F corresponds to a 4-tuple $(\alpha, \mu, \varphi, \rho)$ of sections of the direct sum (obtained by replacing each $\oplus\!\!\!\!-$ with $\oplus$ in (7)) and we write

$$F = \mathbb{Y}^k\cdot\alpha + \mathbb{Z}^k\cdot\mu + \mathbb{W}^k\cdot\varphi + \mathbb{X}^k\cdot\rho,$$

where '·' is the usual pointwise form inner product in the tensor arguments,

$$\varphi\cdot\psi = \frac{1}{p!}\varphi^{a_1\ldots a_p}\psi_{a_1\ldots a_p} \text{ for } p\text{-forms},$$

and for $k > 1$, if $\underline{\wedge}$ denotes the wedge product in the tractor arguments, then we have

$$\mathbb{Z}^k = Z\underline{\wedge}\mathbb{Z}^{k-1},\ \ \mathbb{X}^k = X\underline{\wedge}\mathbb{Z}^{k-1},\ \ \mathbb{Y}^k = Y\underline{\wedge}\mathbb{Z}^{k-1},\ \ \mathbb{W}^k = X\underline{\wedge}Y\underline{\wedge}\mathbb{Z}^{k-2}. \tag{8}$$

By convention, $\mathbb{Z}^0 = 1$ and $\mathbb{Z}^{-1} = 0$. The connection on $\mathbb{T}$ gives a (conformally invariant) connection on $\mathbb{T}^k$ by the Leibniz rule. Under a change of scale $\hat{g} = e^{2\omega}g$, it follows from the transformation law for the standard tractor bundle (4), that

$$\begin{aligned} \widehat{\mathbb{X}} &= \mathbb{X}, \\ \widehat{\mathbb{Z}} &= \mathbb{Z} + \varepsilon(\Upsilon)\mathbb{X}, \\ \widehat{\mathbb{W}} &= \mathbb{W} - \iota(\Upsilon)\mathbb{X}, \\ \widehat{\mathbb{Y}} &= \mathbb{Y} - \iota(\Upsilon)\mathbb{Z} - \varepsilon(\Upsilon)\mathbb{W} + \tfrac{1}{2}(\varepsilon(\Upsilon)\iota(\Upsilon) - \iota(\Upsilon)\varepsilon(\Upsilon))\mathbb{X}, \end{aligned}$$

where again $\Upsilon = d\omega$ and the interior and exterior multiplication apply to the tensor indices of the $\mathbb{X}$, $\mathbb{Z}$, $\mathbb{W}$ and $\mathbb{Y}$ projectors.

We are now ready to investigate gauge operators for the Maxwell operator on $\mathcal{E}^{n/2-1}$. By (7) the forms $\mathcal{E}^k$ turn up at the $\mathbb{Z}^k$ slot of $\mathcal{T}^k[-k]$. It is straightforward to verify, using the formulae above, that

$$\mu \mapsto \begin{pmatrix} 0 \\ \mu \\ (n-2)^{-1}\delta\mu \end{pmatrix},\ k=1 \text{ and } \mu \mapsto \begin{pmatrix} & 0 & \\ \mu & & 0 \\ & (n-2k)^{-1}\delta\mu & \end{pmatrix},\ \mathrm{k} \geq 2 \tag{9}$$

are conformally invariant differential splitting operators $S_k : \mathcal{E}^k \to \mathcal{T}^k[-k]$.

Now recall the Yamabe operator (or conformal Laplacian)

$$\Box = -\nabla^a\nabla_a - (1 - n/2)\mathsf{J}$$

is conformally invariant on the densities $\mathcal{E}[1 - n/2]$. In fact this same formula also gives an invariant operator on the sections of $E[1 - n/2] \otimes \mathbb{U}$, where $\mathbb{U}$ is any vector bundle with connection, provided we view ∇ as the coupled Levi Civita vector bundle connection. Operators with this property are said to be *strongly invariant.* (This is just a slight variation of the notion introduced in [12].) To see this one can simply observe that the direct calculation, using (1) and (2), which verifies the invariance of $\Box$ does not require the commutation of covariant derivatives. In particular we may couple with the tractor bundle. As an immediate application, note that $S_{n/2-1}$ takes values in $\mathcal{T}^{n/2-1}[1-n/2]$ and so the composition $\Box S_{n/2-1} : \mathcal{E}^{n/2-1} \to \mathcal{T}^{n/2-1}[-1-n/2]$ is conformally invariant.

In dimension 4, for example, μ is a 1-form and we should (following [5]) calculate $(-\nabla^a\nabla_a + \mathsf{J})(Z\cdot\mu + X\frac{1}{2}\delta\mu)$ using (6). This yields

$$\begin{pmatrix} 0 \\ \delta d\mu \\ \frac{1}{2}\delta(d\delta\mu + 2\mathsf{J} - 4\mathsf{P}\sharp)\mu \end{pmatrix} = \begin{pmatrix} 0 \\ \delta d\mu \\ G\mu \end{pmatrix}.$$

We have recovered exactly the Maxwell operator and Eastwood-Singer gauge pair. Since this construction is conformally invariant it is immediate from this final formula that G is invariant on the null space of the Maxwell operator δd. This construction is in the spirit of the *curved translation principle* of Eastwood et al. [13, 12]; we have obtained the (operator, gauge) pair by translating from the Yamabe operator.

Buoyed by this success we are drawn to immediately try the same idea to obtain a gauge operator for the Maxwell operator in higher even dimensions. However this time we find that, for $\mu \in \mathcal{E}^{n/2-1}$, we get

$$\square\begin{pmatrix} & 0 & \\ \mu & & 0 \\ & (n-2k)^{-1}\delta\mu & \end{pmatrix} = \begin{pmatrix} & 0 & \\ & (\delta d + C\sharp\sharp)\mu & 0 \\ & \frac{1}{2}\delta d\delta\mu + \delta(\mathsf{J}\mu) - 2\mathsf{P}\sharp\delta\mu - 2\delta(\mathsf{P}\sharp\mu) + \frac{1}{2}C\sharp\sharp\delta\mu & \end{pmatrix},$$

where C is the Weyl curvature of the conformal structure (recall the Weyl curvature is conformally invariant) and in the $C\sharp\sharp$ action we view this as a (weighted) section of the tensor square of $\mathrm{End}(T^*M)$. We see here that, unfortunately, the Maxwell operator does not automatically turn up as the leading slot. We did not encounter this in dimension 4 because when μ is a 1-form $C\sharp\sharp\mu$ vanishes (since C is trace-free). Of course $C\sharp\sharp\mu$ is conformally invariant but we cannot simply subtract this and maintain conformal invariance without also adjusting the $\mathbb{X}$-slot. The existence of a correction, in such circumstances, is a delicate matter. It turns out that in this instance there is a fix. The output above decomposes into a sum of conformally invariant tractors according to

$$(10)\qquad \begin{pmatrix} & 0 & \\ & \delta d\mu & 0 \\ & \frac{1}{2}\delta d\delta\mu + \delta(\mathsf{J}\mu) - 2\delta(\mathsf{P}\sharp\mu) & \end{pmatrix} + \begin{pmatrix} & 0 & \\ & C\sharp\sharp\mu & 0 \\ & Y\bullet\mu + \frac{1}{2}C\sharp\sharp\delta\mu & \end{pmatrix},$$

where $Y = Y^{ab}{}_c := \nabla^a\mathsf{P}^b{}_c - \nabla^b\mathsf{P}^a{}_c$ and $Y\bullet\mu$ means $-\sum_{s=2}^{k} Y^{a_1 b}{}_{a_s}\mu_{a_1\ldots b\ldots a_k}$.

So $G_{n/2-1} := \frac{1}{2}\delta d\delta + \delta(\mathsf{J}) - 2\delta(\mathsf{P}\sharp)$ is a gauge operator for the Maxwell operator (and so also for d) in all even dimensions. By construction it is conformally invariant on $\mathcal{N}(d) \subseteq \mathcal{N}(\delta d)$ (in fact this is an equality in the compact Riemannian setting) and from our earlier observations it combines with d to give an elliptic system. Finally note that if we apply the Maxwell operator plus gauge system to an exact $\mu = d\nu$ we obtain

$$\begin{pmatrix} & 0 & \\ 0 & & 0 \\ & G_{n/2-1}d\nu & \end{pmatrix},$$

since δd annihilates exact forms. In the alternative notation this is

$$\mathbb{X}\cdot G_{n/2-1}d\nu = \mathbb{X}\cdot\big(\frac{1}{2}\delta d\delta\mu + \delta(\mathsf{J}\mu) - 2\delta(\mathsf{P}\sharp\mu)d\nu\big).$$

Since this is conformally invariant by construction, $\widehat{\mathbb{X}} = \mathbb{X}$, and the coefficients in the other slots are all zero, it follows that

$$(11)\qquad \big(G_{n/2-1}d = \frac{1}{2}\delta(d\delta + 2\mathsf{J} - 4\mathsf{P}\sharp)d\big) : \mathcal{E}^{n/2-2} \to \mathcal{E}_{n/2-2}$$

is a conformally invariant long operator of the form proposed in problem 3.

Although we have succeeded in pushing this calculation through there are are two main problems which suggest that this approach would be difficult, if not impossible, to generalise sufficiently to deal with our problems 2 and 3 in general. One is that even at the low order of example treated, the calculations leading to the results presented were non-trivial and involved, for example, the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$. More seriously the decomposition in (10) involved inspecting the explicit formulae and solving equations to extend the $C\sharp\sharp\mu$ term to a conformally invariant tractor operator. Ab initio one does not know that this will succeed. The weight of $C\sharp\sharp\mu$ is exactly such that the standard tools using Lie algebra cohomology [7] or central character as in the theory of Verma modules [16] fail to indicate the existence of this extension.

As a final point, in this lecture, let us note that the operator (11), and the proposed higher order analogues, are <u>not</u> strongly invariant. We can easily see this directly for (11). Let us write $G := G_{n/2-1}$ and note first that, from the transformation formula $\hat{\mathbb{Z}} = \mathbb{Z} + \varepsilon(\Upsilon)\mathbb{X}$ and the invariant splitting (10), it follows that $\widehat{G} = G - \iota(\Upsilon)\delta d$. This is also readily verified by direct calculation using the conformal transformation formulae (1) and (2) for the Levi Civita connection. Now consider a coupled variant of G acting on a vector bundle valued $(n/2-1)$-form μ. Suppose that the vector bundle has a connection A, with curvature F, and G^A is given by the formula above for G, except that d and δ are replaced by their connection coupled variants d^A and δ^A. Now the direct computation of the conformal transform of this, $\widehat{G}^A\mu$, is the same as for the case of forms except that now vector bundle curvature terms may enter from the commutation of derivatives. Given that G^A is just a 3[rd] order operator one easily sees that

$$\widehat{G}^A\mu = G^A\mu - \iota(\Upsilon)\delta^A d^A\mu + \Upsilon{\cdot}F{\cdot}\mu\,,$$

where $\Upsilon{\cdot}F{\cdot}\mu$ indicates a sum of terms linear in Υ, F and μ. Now let us suppose that $\mu = d^A\nu$ where the vector bundle valued $(n/2-2)$-form ν satisfies $d^A\nu(p) = 0$ for some point $p \in M$. Note that d^A is conformally invariant on ν. Then, at p, μ vanishes and we have

$$\widehat{G}^A d^A\nu = G^A d^A\nu - \iota(\Upsilon)\delta^A d^A d^A\nu = G^A d^A\nu - \iota(\Upsilon)\delta^A F \wedge \nu\,,$$

where $F \wedge \nu$ includes an implicit curvature action on ν. It is an elementary matter to verify by example that the term $\iota(\Upsilon)\delta^A F \wedge \nu$ does not vanish in general. Thus $G^A d^A$ is not conformally invariant, and so (11) is not strongly invariant.

The operator (11) is not unique. For example we could add to it the conformally invariant term $\delta C\sharp\sharp d$. It is natural to wonder if there is some modification which still has the form $\delta M d$ but which is strongly invariant. One needs to be careful considering such arguments since strong invariance is really a property of the *formulae* for operators rather the operators themselves. Nevertheless we will show that in fact, apart from the 2[nd]-order Maxwell operators δd, none of the operators sought in problem 3 can be strongly invariant. (That is there are not strongly invariant formulae for these operators which have the form $\delta M d$.) This means that they cannot be obtained by the usual use of the curved translation principle since that procedure involves composing strongly invariant operators to obtain new strongly invariant operators. This is an important feature of the desired operators, so we state the result as a proposition. (In fact we give a stronger result.) In proving this we will use, what is now a well known result (which can be deduced from the results in [16]) as follows. On the conformally flat sphere one has the invariant operators $\mathcal{E}^{n/2-1} \xrightarrow{\star d} \mathcal{E}^{n/2}$ and $\mathcal{E}^{n/2} \xrightarrow{\delta} \mathcal{E}_{n/2-1}$. These with the operators indicated in figure 1, give, up to linear combinations, all of the conformally invariant operators between the differential form bundles in figure 1. It follows, for example, that the composition of operators from the figure always yields a trivial operator.

Proposition 1.1. *Suppose that for $k \in \{0, 1, \ldots, n/2 - 2\}$, the composition*

$$Sd : \mathcal{E}^k \to \mathcal{E}_k$$

is a strongly invariant natural conformally invariant operator. Then this operator vanishes on conformally flat structures.

Proof. Suppose first that $k \geq 1$. Let $\mathcal{V}$ be a trivial bundle, with fibre V. Let us equip this with a family of connections which differ from the trivial connection by tA, where t is a real parameter and A is any field of $\mathrm{End}(V)$-valued 1-forms. Since Sd is strongly invariant we can couple to the connection corresponding to tA, for each t, to obtain the conformally invariant operator $S^{tA}d^{tA}$ on $\mathcal{V}$-valued k-forms. Since also the exterior derivative is strongly invariant it follows that the composition $S^{tA}d^{tA}d^{tA}$ is also conformally invariant on $\mathcal{V}$-valued $(k-1)$-forms. But this is a non-zero multiple of

$$S^{tA}F^{tA}$$

where F^{tA} is the curvature of the connection tA. Of course the F^{tA} acts by the exterior product, via its form indices, as well as the usual $\mathrm{End}(\mathcal{V})$-action of a curvature. Viewing t as a parameter, it is clear that the displayed operator can be expressed by a formula polynomial in t and so its derivatives, with respect to t, are also conformally invariant. In particular if we write $F_0 := dF^{tA}/dt|_{t=0}$ then, by differentiating and evaluating at 0, we obtain that

$$SF_0$$

is conformally invariant on $\mathcal{V}$-valued $(k-1)$-forms. Now F_0 is an $\mathrm{End}(\mathcal{V})$-valued 2-form. Since $\mathrm{End}(V)$ is canonically isomorphic to its dual, we may view F_0 instead as map from $\mathrm{End}(\mathcal{V}) \to \mathcal{E}^2$. It is easily verified that, by suitable choice of V and the field A, one can arrange that this map is surjective. So let us assume this. We have stated that if H is any $\mathcal{V}$-valued $(k-1)$-form then SF_0H is conformally invariant. Now suppose W is a section of $\mathcal{V}^*$ that is parallel for the trivial connection on $\mathcal{V}^*$. Then $W{\cdot}SF_0H = S(F_0H){\cdot}W$, where the '$\cdot$' indicates that the section W is contracted into the free $\mathcal{V}$-index of (F_0H). Thus S is conformally invariant on the $(k+1)$-form $(F_0H){\cdot}W$. Since S is linear, it is also conformally invariant on sums of $(k+1)$-forms constructed this way and so, by the surjectivity of F_0, we can conclude that $S : \mathcal{E}^{k+1} \to \mathcal{E}_k$ is conformally invariant. But then it follows that this is trivial in the flat case because, for k in the range assumed (from the classification described above), there are no non-trivial conformally invariant operators, on the conformal sphere, of the form $\mathcal{E}^{k+1} \to \mathcal{E}_k$. This does the cases $k \neq 0$.

Now suppose, with a view to contradiction that $L : \mathcal{E}^0 \to \mathcal{E}_0$ is natural, strongly invariant, and is non-trivial on the conformal sphere. Then the leading term is $\Delta^{n/2}$, at least up to a constant non-zero multiple. Coupling to the standard tractor bundle and connection we may conclude the existence of a conformally invariant operator

$$H^B{}_A : \mathcal{T}^A \to \mathcal{T}^B$$

with principal part $\Delta^{n/2}$ (where now Δ is the tractor-coupled Laplacian). Now there is the so-called tractor-D operator [1] which is (strongly) conformally invariant and given by the formula

$$D^A f := (n+2w-2)wY^A f + (n+2w-2)Z^{Aa}\nabla_a f - X^A(\Delta + w\mathrm{J})f$$

for f any weight w tractor or density field. Composing first with this on the right it is easily verified (or see [20] or [22]) that, in the conformally flat case, we have

$$H^B{}_A D^B f = -X^B \Delta^{n/2+1} f$$

for f any weight 1 density field. From this it follows easily that, in the general curved setting, the conformally invariant composition

$$D_B H^B{}_A D^A : \mathcal{E}[1] \to \mathcal{E}[-1-n].$$

has leading term a non-zero constant multiple of $\Delta^{n/2+1}$. However this is a contradiction as there is no such operator [21]. □

Notice that we have proved a little more than what is claimed in the proposition. We have shown that there is no strongly conformally invariant curved analogue of the operator $P_n : \mathcal{E}^0 \to \mathcal{E}_0$ on the sphere, regardless of its form.

2. Lecture 2 – Operators like Q and the ambient connection

In the first half of this lecture we show that the 4 problems are solved simultaneously (with some mild qualifications) by a sequence of remarkable operators which include and generalise the Q-curvature. The construction of these operators uses the Fefferman-Graham ambient metric construction and its relationship to the tractor calculus. In the second half of the lecture we set up the background for this.

2.1. The solution. We collect the main points into a theorem which includes some of the central results in [6]:

Theorem 2.1. *In each even dimension n there exist natural Riemannian differential operators*

$$Q_k : \mathcal{E}^k \to \mathcal{E}_k, \quad \textit{non-zero for} \quad k = 0, 1, \ldots, n/2,$$

(and for other k we take these to be zero) with the following properties:
(i) $Q_0 1$ *is the Branson Q-curvature.*
(ii) *As an operator on closed k-forms* Q_k *has the conformal transformation law*

$$\widehat{Q}_k = Q_k + \delta Q_{k+1} d\omega$$

where $\hat{g} = e^{2\omega} g$*, for a smooth function* ω*, and on the right-hand-side we view* ω *as a multiplication operator.*
(iii)

$$\delta Q_{k+1} d : \mathcal{E}^k \to \mathcal{E}_k \quad k = 0, 1, \ldots, n/2-1$$

is conformally invariant (from (ii)), formally self-adjoint and has leading term a non-zero multiple of $(\delta d)^{n/2-k}$.
(iv) *The system* $(\delta Q_{k+1} d,\ \delta Q_k)$ *is elliptic* ($k \in \{0, 1, \ldots, n/2-1\}$) *and, for each k,* δQ_k *has the conformal transformation*

$$\widehat{\delta Q_k} = \delta Q_k + c\iota(d\omega)\delta Q_{k+1} d,$$

where $\hat{g} = e^{2\omega} g$ *and c is a constant. In particular* δQ_k *is conformally invariant on* $\mathcal{N}(\delta Q_{k+1} d)$.

By part (iii), our problem 3 is solved by taking $L_k = \delta Q_{k+1} d$. It follows immediately that in each dimension there is a family of conformally invariant *detour complexes*,

$$\mathcal{E}^0 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{\delta} \mathcal{E}_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_0 , \tag{12}$$

which generalise the Maxwell detour complex. Since L_k has leading term $(\delta d)^{n/2-k}$ these are elliptic (i.e. exact at the symbol level).

Clearly $\mathcal{N}(d) \subseteq \mathcal{N}(\delta Q_{k+1} d)$ and so δQ_k is conformally invariant on $\mathcal{N}(d)$. Since $\delta Q_{k+1} d$ has leading term $(\delta d)^{n/2-k}$, it follows that the ellipticity of the system $(\delta Q_{k+1} d,\ \delta Q_k)$, as asserted in *(iv)*, implies that $(d, \delta Q_k)$ is elliptic. So setting $G_k := \delta Q_k$ gives a solution to problem 2. Thus, writing $\mathcal{C}^k$ for the space of closed k-forms, we propose

$$\mathcal{H}^k_G := \mathcal{N}(G_k : \mathcal{C}^k \to \mathcal{E}_{k-1})$$

as the space of *conformal harmonics*, for $k = 0, 1, \ldots, n/2$. Since G_k is conformally invariant on $\mathcal{C}^k$ it follows that this space is conformally invariant.

Of course property (ii) generalises the transformation formula (I) of the Q-curvature (where we view Q as a multiplication operator on the constant functions). Then note that if $c \in \mathcal{C}^k$, and $u \in \mathcal{E}^k$ then

$$\int_M (u, Q^{\hat{g}}_k c) d\mu_g = \int_M (u, Q^g_k c + L_k \omega c) d\mu_g = \int_M (u, Q^g_k c) d\mu_g + \int_M (L_k u, c) d\mu_g$$

as L_k is formally self-adjoint. Here we are using $(\cdot,\cdot)$ for the complete contraction, via g^{-1}, of forms. So if $u \in \mathcal{N}(L_k)$ then the last term vanishes and we have

$$c \in \mathcal{C}^k \text{ and } u \in \mathcal{N}(L_k) \ \Rightarrow \ \int (u, Q_k c) d\mu_g \ \text{ is conformally invariant.}$$

which generalises property (III) of the Q-curvature.

The transformation law in (i) implies that Q_k gives a conformally invariant map $Q_k : \mathcal{C}^k \to \mathcal{E}^k / \mathcal{R}(\delta)$. If $u \in \mathcal{H}^k_G$, then u is both closed and in the null space of G_k, and so $\delta Q_k u = 0$ since $G_k = \delta Q_k$. Thus Q_k fives a conformally invariant map

$$Q_k : \mathcal{H}^k_G \to H_k(M) = \mathcal{N}(\delta : \mathcal{E}_k \to \mathcal{E}_{k-1}) / \mathcal{R}(\delta : \mathcal{E}_{k+1} \to \mathcal{E}_k) \cong H^k(M) . \tag{13}$$

Note that since Q_0 takes values in densities, $G_0 = \delta Q_0$ is trivial and $\mathcal{H}^0_G = \mathcal{C}^0$ and so the result displayed generalises to the Q_k property (II) of the Q-curvature. Examples where the maps (13) are non-trivial are given in [6].

It remains to check to how accurately the defined conformal harmonics reflect the de Rham cohomology. We have already observed that $\mathcal{H}^0_G = \mathcal{C}^0 \cong H^0(M)$. It turns out that $Q_{n/2}$ is a non-vanishing constant (as a multiplication operator) and so at middle forms we recover the usual harmonics, $\mathcal{H}^{n/2}_G = \mathcal{H}^{n/2} \cong H^{n/2}(M)$. Between these extremes, it is easy to obtain an estimate on the size of the space $\mathcal{H}^k_G$. Note there is a map $\mathcal{H}^k_G \to H^k(M)$ given by mapping closed forms to their class in cohomology. If $w \in \mathcal{H}^k_G$ is mapped to the class of 0 in $H^k(M)$ then w is exact. Say $w = d\varphi$. Since, in addition, $\delta Q_k w = 0$, it follows that $\delta Q_k d\varphi = 0$, that is $L_{k-1}\varphi = 0$. Now $L_{k-1} = \delta Q_k d$, so $\mathcal{C}^{k-1} \subseteq \mathcal{N}(L_{k-1})$ and it follows that there is an exact sequence

$$0 \to \mathcal{C}^{k-1} \to \mathcal{N}(L_{k-1}) \xrightarrow{d} \mathcal{H}^k_G \to H^k(M) .$$

Now $H^{k-1}(M)$ is the image of $\mathcal{C}^{k-1}$ under the composition $\mathcal{C}^{k-1} \to \mathcal{N}(L_{k-1}) \to H_L^{k-1}(M)$, so finally we obtain

$$0 \to H^{k-1}(M) \to H_L^{k-1}(M) \to \mathcal{H}_G^k \to H^k(M) \quad \text{for } k = 1, \dots, n/2-1,$$

and so

$$\dim \mathcal{H}_G^k \le b^k + \dim(H_L^{k-1}(M)/H^{k-1}(M))$$

where $\dim H_L^k(M)$ is the cohomology at $\mathcal{E}^k$ of the sequence (12) and b^k is the k^{th}-Betti number, i.e. $b^k = \dim H^k(M)$. Obtaining a lower bound is not so straightforward. Nevertheless, using Hodge theory and the map (13) it can be shown [6] that $b^k \le \dim \mathcal{H}^k$. Thus to have $\dim \mathcal{H}_G^k = b^k$ it is sufficient for the conformal regularity condition $H_L^{k-1}(M) = H^{k-1}(M)$ (or equivalently $\mathcal{N}(L_{k-1}) = \mathcal{C}^{k-1}$) to be satisfied. For $n = 4$ and $k = 1$ this is the notion of strong regularity proposed by [15]. Although, for each k, the regularity should hold generically, in some appropriate sense, for compact conformal Riemannian manifolds there are counter-examples to strong regularity on 4-manifolds [28]. In [6] it is shown that there is a condition weaker than $H_L^{k-1}(M) = H^{k-1}(M)$ which is necessary and sufficient for $\dim \mathcal{H}_G^k = b^k$.

2.2. **The Fefferman-Graham ambient construction.** Recall that $\pi : \mathcal{Q} \to M$ is the conformal bundle of metrics. Let us use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(n+2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$-action ρ and an $\mathbb{R}_+$-equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$-action, that is for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$ we have $\boldsymbol{X} f(u) = (d/dt) f(\rho(e^t)u)|_{t=0}$. If $i : \mathcal{Q} \to \tilde{M}$ is an ambient manifold, then an *ambient metric* is a pseudo-Riemannian metric $\boldsymbol{h}$ of signature $(n+1, 1)$ on $\tilde{M}$ such that the following conditions hold:
(i) The metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$-action, i.e. if $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$, then we have $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$. (I.e. $\boldsymbol{X}$ is a homothetic vector field for h.)
(ii) For $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$. Henceforth we will identify $\mathcal{Q}$ with its image in $\tilde{M}$ and suppress the embedding map i.

In [17] Fefferman and Graham treat the problem of constructing a formal power series solution along $\mathcal{Q}$ for the Goursat problem of finding an ambient metric $\boldsymbol{h}$ satisfying (i) and (ii) and the condition that it be Ricci flat, i.e. $\text{Ric}(\boldsymbol{h}) = 0$. From their results and some minor subsequent observations [22, 24] we have the following: there is a formal solution for $\boldsymbol{h}$ satisfying (i), (ii) and with

$$\text{(iii)} \quad \text{Ric}(\boldsymbol{h}) = 0 \begin{cases} \text{to all orders if } n \text{ is odd,} \\ \text{up to the addition of terms vanishing} \\ \text{to order } n/2 - 1 \text{ if } n \text{ is even,} \end{cases}$$

with $\boldsymbol{Q} := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ a defining function for $\mathcal{Q}$ and $\boldsymbol{h}(\boldsymbol{X}, \cdot) = \frac{1}{2} d\boldsymbol{Q}$ to all orders in both dimension parities. We will use the term ambient metric to mean an ambient manifold with metric satisfying all these conditions. Note that if M is locally conformally flat then the flat ambient metric is a (canonical) solution to the ambient metric problem. It is straightforward to check [6] that this is forced in odd dimensions while in even dimensions this extends the solution. When discussing the conformally flat case we assume this solution.

We should point out that Fefferman and Graham give uniqueness statements for their metric, but we do not need these here. The uniqueness of the operators we will construct is a consequence of the fact that they can be uniquely expressed in terms of the underlying conformal structure as we shall explain later.

We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection determined by $\boldsymbol{h}$ and use upper case abstract indices $A, B, \ldots$ for tensors on $\tilde{M}$. For example, if v^B is a vector field on $\tilde{M}$, then the ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^C{}_D$ and defined by $[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B]v^C = \boldsymbol{R}_{AB}{}^C{}_D v^D$. In this notation the ambient metric is denoted $\boldsymbol{h}_{AB}$ and with its inverse this is used to raise and lower indices in the usual way. Most often we will use an index free notation and will not distinguish tensors related in this way. Thus for example we shall use $\boldsymbol{X}$ to mean both the Euler vector field $\boldsymbol{X}^A$ and the 1-form $\boldsymbol{X}_A = \boldsymbol{h}_{AB}\boldsymbol{X}^B$.

The condition $\mathcal{L}_{\boldsymbol{X}}\boldsymbol{h} = 2\boldsymbol{h}$ is equivalent to the statement that the symmetric part of $\boldsymbol{\nabla X}$ is $\boldsymbol{h}$. On the other hand, since $\boldsymbol{X}$ is exact, $\boldsymbol{\nabla X}$ is symmetric. Thus

$$\boldsymbol{\nabla X} = \boldsymbol{h}\,, \tag{14}$$

which in turn implies

$$\boldsymbol{X} \lrcorner \boldsymbol{R} = 0\,. \tag{15}$$

Equalities without qualification, as here, indicate that the results hold either to all orders or identically on the ambient manifold.

Let $\tilde{\mathcal{E}}(w)$ denote the space of functions on $\tilde{M}$ which are homogeneous of degree $w \in \mathbb{R}$ with respect to the action ρ. Recall that densities in $\mathcal{E}[w]$ are equivalent to functions in $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$. More generally (weighted) tractor fields correspond to the restriction (to $\mathcal{Q}$) of homogeneous tensor fields on $\tilde{M}$. A tensor field F on $\tilde{M}$ is said to be *homogeneous of degree* w if $\rho(s)^*F = s^w F$, or equivalently $\mathcal{L}_{\boldsymbol{X}}F = wF$. The relationship between the Fefferman-Graham ambient metric construction and the tractor connection was established in [8]. Following this treatment we will sketch how the conformal tractor bundle, metric and connection are related to the ambient metric.

On the ambient tangent bundle $T\tilde{M}$ we define an action of $\mathbb{R}_+$ by $s{\cdot}\xi := s^{-1}\rho(s)_*\xi$. The sections of $T\tilde{M}$ which are fixed by this action are those which are homogeneous of degree -1. Let us denote by $\boldsymbol{\mathcal{T}}$ the space of such sections and write $\boldsymbol{\mathcal{T}}(w)$ for sections in $\boldsymbol{\mathcal{T}} \otimes \tilde{\mathcal{E}}(w)$, where the $\otimes$ here indicates a tensor product over $\tilde{\mathcal{E}}(0)$. Along $\mathcal{Q}$ the $\mathbb{R}_+$ action on $T\tilde{M}$ agrees with the $\mathbb{R}_+$ action on $\mathcal{Q}$, and so the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$, yields a rank $n+2$ vector bundle $\tilde{\mathbb{T}}$ over $\mathcal{Q}/\mathbb{R}_+ = M$. By construction, sections of $p : \tilde{\mathbb{T}} \to M$ are equivalent to sections from $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. We write $\tilde{\mathcal{T}}$ to denote the space of such sections.

Since the ambient metric $\boldsymbol{h}$ is homogeneous of degree 2 it follows that for vector fields ξ and η on $\tilde{M}$ which are homogeneous of degree -1, the function $\boldsymbol{h}(\xi, \eta)$ is homogeneous of degree 0 and thus descends to a smooth function on M. Hence $\boldsymbol{h}$ descends to a smooth bundle metric h of signature $(n+1, 1)$ on $\tilde{\mathbb{T}}$.

Next we show that the space $\boldsymbol{\mathcal{T}}$ has a filtration reflecting the geometry of $\tilde{M}$. First observe that for $\varphi \in \tilde{\mathcal{E}}(-1)$, $\varphi\boldsymbol{X} \in \boldsymbol{\mathcal{T}}$. Restricting to $\mathcal{Q}$ this determines a canonical inclusion $E[-1] \hookrightarrow \tilde{\mathbb{T}}$ with image denoted by $\mathbf{V}$. Since X generates the fibres of

$\pi : \mathcal{Q} \to M$ the smooth distinguished line subbundle $\mathbb{V} \subset \tilde{\mathbb{T}}$ reflects the inclusion of the vertical bundle in $T\tilde{M}|_{\mathcal{Q}}$. We write X for the canonical section in $\tilde{\mathcal{T}}[1]$ giving this inclusion. We define $\mathbb{F}$ to be the orthogonal complement of $\mathbb{V}$ with respect to h. Since $\boldsymbol{Q} = \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is a defining function for $\mathcal{Q}$ it follows that X is null and so $\mathbb{V} \subset \mathbb{F}$. Clearly $\mathbb{F}$ is a smooth rank $n+1$ subbundle of $\tilde{\mathbb{T}}$. Thus $\tilde{\mathbb{T}}/\mathbb{F}$ is a line bundle and it is immediate from the definition of $\mathbb{F}$ that there is a canonical isomorphism $E[1] \cong \tilde{\mathbb{T}}/\mathbb{F}$ arising from the map $\tilde{\mathbb{T}} \to E[1]$ given by $V \mapsto h(X, V)$. Now recall $2\boldsymbol{h}(\boldsymbol{X}, \cdot) = d\boldsymbol{Q}$, so the sections of $\mathcal{T}|_{\mathcal{Q}}$ which correspond to sections of $\mathbb{F}$ are exactly those that take values in $T\mathcal{Q} \subset T\tilde{M}|_{\mathcal{Q}}$. Finally we note that if $\tilde{\xi}$ and $\tilde{\xi}'$ are two lifts to $\mathcal{Q}$ of $\xi \in \mathfrak{X}(M)$ then they are sections of $T\mathcal{Q}$ which are homogeneous of degree 0 and with difference $\tilde{\xi} - \tilde{\xi}'$ taking values in the vertical subbundle. Since $\pi : \mathcal{Q} \to M$ is a submersion it follows immediately that $\mathbb{F}[1]/\mathbb{V}[1] \cong TM \cong T^*M[2]$ (where recall by our conventions $\mathbb{F}[1]$ means $\mathbb{F} \otimes E[1]$ etc.). Tensoring this with $E[-1]$ and combining this observation with our earlier results we can summarise the filtration of $\tilde{\mathbb{T}}$ by the composition series

$$\tilde{\mathbb{T}} = E[1] \oplus\!\!\!\!+ T^*M[1] \oplus\!\!\!\!+ E[-1]\,. \tag{16}$$

Next we show that the Levi-Civita connection $\boldsymbol{\nabla}$ of $\boldsymbol{h}$ determines a linear connection on $\tilde{\mathbb{T}}$. Since $\boldsymbol{\nabla}$ preserves $\boldsymbol{h}$ it follows easily that if $U \in \mathcal{T}(w)$ and $V \in \mathcal{T}(w')$ then $\boldsymbol{\nabla}_U V \in \mathcal{T}(w+w'-1)$. The connection $\boldsymbol{\nabla}$ is torsion free so $\boldsymbol{\nabla}_{\boldsymbol{X}} U - \boldsymbol{\nabla}_U \boldsymbol{X} - [\boldsymbol{X}, U] = 0$ for any tangent vector field U. Now $\boldsymbol{\nabla}_U \boldsymbol{X} = U$, so this simplifies to $\boldsymbol{\nabla}_{\boldsymbol{X}} U = [\boldsymbol{X}, U] + U$. Thus if $U \in \mathcal{T}$, or equivalently $[\boldsymbol{X}, U] = -U$, then $\boldsymbol{\nabla}_{\boldsymbol{X}} U = 0$. The converse is clear and it follows that sections of $\mathcal{T}$ may be characterised as those sections of $T\tilde{M}$ which are covariantly parallel along the integral curves of $\boldsymbol{X}$ (which on $\mathcal{Q}$ are exactly the fibres of π). These two results imply that $\boldsymbol{\nabla}$ determines a connection ∇ on $\tilde{\mathbb{T}}$. For $U \in \mathcal{T}$, let $\tilde{U}$ be the corresponding section of $\mathcal{T}|_{\mathcal{Q}}$. Similarly a tangent vector field ξ on M has a lift to a field $\tilde{\xi} \in \mathcal{T}(1)$, on $\mathcal{Q}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f \in \tilde{\mathcal{E}}(0)$. We extend $\tilde{U}$ and $\tilde{\xi}$ smoothly and homogeneously to fields on $\tilde{M}$. Then we can form $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$; this is clearly independent of the extensions. Since $\boldsymbol{\nabla}_{\boldsymbol{X}} \tilde{U} = 0$, the section $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$ is also independent of the choice of $\tilde{\xi}$ as a lift of ξ. Finally, $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{U}$ is a section of $\mathcal{T}(0)$ and so determines a section $\nabla_\xi U$ of $\tilde{\mathbb{T}}$ which only depends on U and ξ. It is easily verified that this defines a covariant derivative on $\tilde{\mathbb{T}}$ which, by construction, is compatible with the bundle metric h.

The ambient metric is conformally invariant; no choice of metric from the conformal class on M is involved in solving the ambient metric problem. Thus the bundle, metric and connection $(\tilde{\mathbb{T}}, h, \nabla)$ are by construction conformally invariant. On the other hand the ambient metric is not unique (there is some diffeomorphism freedom and, even allowing for this, recall that in even dimensions the construction is only determined by the underlying conformal manifold to finite order). Nevertheless it is straightforward to verify that ∇ satisfies the required non-degeneracy condition and curvature normalisation condition [9] that show that the bundle and connection pair $(\tilde{\mathbb{T}}, \nabla)$, induced by $\boldsymbol{h}$, is a normal standard (tractor bundle, connection) pair. So although the ambient metric is not unique the induced tractor bundle structure $(\tilde{\mathbb{T}}, h, \nabla)$ is equivalent to a normal Cartan connection, and so is unique up to bundle isomorphisms preserving the filtration structure of $\tilde{\mathbb{T}}$, and preserving h and ∇. Hence we may drop the tildes and identify $\tilde{\mathbb{T}}$ with $\mathbb{T}$ and $\tilde{\mathcal{T}}$ with $\mathcal{T}$.

Since $\mathcal{T}$ corresponds to the ambient space $\mathcal{T}|_{\mathcal{Q}}$ and $\mathcal{E}[w]$ corresponds to $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$ it follows, by taking tensor powers, that homogeneous ambient tensors along $\mathcal{Q}$ are equivalent to weighted tractor fields in the corresponding tensor power of $\mathcal{T}$. In particular this is true for exterior powers. The subspace of $\Gamma(\wedge^k T^*\tilde{M})$ consisting of ambient k-forms F satisfying $\boldsymbol{\nabla}_{\boldsymbol{X}} F = wF$ for a given $w \in \mathbb{R}$ will be denoted $\mathcal{T}^k(w)$. We say such forms are (homogeneous) of *weight* w. Then we have that each section $V \in \mathcal{T}^k[w]$, is equivalent to a section $\tilde{V} \in \mathcal{T}^k(w)|_{\mathcal{Q}}$.

2.3. **Exterior calculus on the ambient manifold.** We need to identify which operators on the ambient manifold correspond to, or determine, conformal differential operators on M. In particular for our problems it turns out that operators on ambient differential forms have a primary role.

We will use $\iota(\cdot)$ and $\varepsilon(\cdot)$ as the notation for interior and exterior multiplication by 1-forms on ambient forms, i.e. the same notation as on M and with the same conventions. Thus for example on differential forms, the Lie derivative with respect to $\boldsymbol{X}$ is given by $\mathcal{L}_{\boldsymbol{X}} = \iota(\boldsymbol{X})\boldsymbol{d} + \boldsymbol{d}\iota(\boldsymbol{X})$ and so its formal adjoint is $\mathcal{L}^*_{\boldsymbol{X}} = \boldsymbol{\delta}\varepsilon(\boldsymbol{X}) + \varepsilon(\boldsymbol{X})\boldsymbol{\delta}$. Note that $\mathcal{Q} := \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ may be alternatively expressed

$$\mathcal{Q} = \iota(\boldsymbol{X})\varepsilon(\boldsymbol{X}) + \varepsilon(\boldsymbol{X})\iota(\boldsymbol{X}) .$$

It is useful for our calculations to extend the notation for interior and exterior multiplication, in an obvious way, to operators which increase the rank by one. For example, writing $\boldsymbol{d}$ and $\boldsymbol{\delta}$ for respectively the ambient exterior and its formal adjoint, we have $\boldsymbol{d}\varphi = \varepsilon(\boldsymbol{\nabla})\varphi$ and $\boldsymbol{\delta}\varphi = -\iota(\boldsymbol{\nabla})\varphi$, since the ambient connection is symmetric. Later on these notations and conventions for the use of $\iota(\cdot)$ and $\varepsilon(\cdot)$ are also used for form tractors, and related objects.

We write $\boldsymbol{\Delta}$ for the ambient *form Laplacian* $\boldsymbol{\delta d} + \boldsymbol{d\delta}$. Using this with ambient form operators just introduced generates a closed system of anti-commutators and commutators as given in Tables 1 and 2. (In fact the graded system is isomorphically the Lie superalgebra $\mathfrak{sl}(2|1)$ and extends the $\mathfrak{sl}(2)$ which played a role in [24]. Some of the results below could be rephrased as identities of $\mathfrak{sl}(2|1)$ representation theory, but we have not taken that point of view. We also note that in [26], which concerns powers of the ambient Dirac operator, the authors recover a 5-dimensional superalgebra isomorphic to the orthosymplectic algebra $\mathfrak{osp}(2|1)$. This may be realised as a subalgebra of $\mathfrak{sl}(2|1)$.) Note that the relations in Table 1 are essentially just definitions and standard identities. The relations in the table of commutators follow from the anticommutator results, $\boldsymbol{d}\mathcal{Q} = 2\boldsymbol{X}$, (14), and the usual identities of exterior calculus on pseudo-Riemannian manifolds. In particular, they hold in all dimensions and to all orders.

Now since each section $V \in \mathcal{T}^k[w]$ is equivalent to $\tilde{V} \in \mathcal{T}^k(w)|_{\mathcal{Q}}$, it follows that operators along $\mathcal{Q}$ that correspond directly to operators on $\mathcal{T}^k[w]$ should not depend on how $\tilde{V}$ is extended off $\mathcal{Q}$. We say a differential operator acts *tangentially* along $\mathcal{Q}$, if $P\mathcal{Q} = \mathcal{Q}P'$ (or $[P, \mathcal{Q}] = \mathcal{Q}(P' - P)$) for some operator P', since then

$$P(\tilde{V} + \mathcal{Q}U) = P\tilde{V} + \mathcal{Q}P'U$$

$\{\cdot,\cdot\}$	d	δ	$\varepsilon(X)$	$\iota(X)$
d	0	$\not\Delta$	0	$\mathcal{L}_X$
δ	$\not\Delta$	0	$\mathcal{L}_X^*$	0
$\varepsilon(X)$	0	$\mathcal{L}_X^*$	0	$\not{Q}$
$\iota(X)$	$\mathcal{L}_X$	0	$\not{Q}$	0

TABLE 1. Anticommutators $\{\mathfrak{g}_1, \mathfrak{g}_1\}$

$[\cdot,\cdot]$	d	δ	$\varepsilon(X)$	$\iota(X)$	$\not\Delta$	$\mathcal{L}_X$	$\mathcal{L}_X^*$	$\not{Q}$
$\not\Delta$	0	0	$-2d$	2δ	0	$2\not\Delta$	$-2\not\Delta$	$-2\mathcal{K}_X$
$\mathcal{L}_X$	0	-2δ	$2\varepsilon(X)$	0	$-2\not\Delta$	0	0	$2\not{Q}$
$\mathcal{L}_X^*$	$2d$	0	0	$-2\iota(X)$	$2\not\Delta$	0	0	$-2\not{Q}$
$\not{Q}$	$-2\varepsilon(X)$	$2\iota(X)$	0	0	$2\mathcal{K}_X$	$-2\not{Q}$	$2\not{Q}$	0

TABLE 2. Commutators $[\mathfrak{g}_0, \mathfrak{g}_1]$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$, where $\mathcal{K}_X := \mathcal{L}_X - \mathcal{L}_X^*$

and so $P\tilde{V}|_Q$ is independent of how $\tilde{V}$ is extended off Q. Note that compositions of tangential operators are tangential. If tangential operators are suitably homogeneous then they descend to operators on M and, since the ambient manifold does not depend on any choice of metric from the conformal class, the resulting operators are conformally invariant. Of course they may depend on choices involved in the ambient metric, in which case they would fail to be natural. We will return to this point shortly.

Consider the form Laplacian. From the commutator table we have $[\not\Delta, \not{Q}] = -2\mathcal{K}_X$ where $\mathcal{K}_X$ is a shorthand for $\mathcal{L}_X - \mathcal{L}_X^*$. Thus in general $\not\Delta$ is not tangential. However via a standard pseudo-Riemannian identity and (14) one has

$$\mathcal{K}_X = \mathcal{L}_X - \mathcal{L}_X^* = n + 2\nabla_X + 2,$$

and so $\mathcal{K}_X$ acts as the zero operator on ambient forms homogeneous of weight $-1-n/2$. If U is homogeneous of weight $-1-n/2$ then $\not{Q}U$ is homogeneous of weight $1-n/2$, and so $\not\Delta$ does act tangentially on $\mathcal{T}^k(1-n/2)$. The form Laplacian is homogeneous of weight -2 in the sense that $[\mathcal{L}_X, \not\Delta] = -2\not\Delta$ and so $\not\Delta$ determines a well-defined operator

$$\not\Delta : \mathcal{T}^k(1-n/2)|_Q \to \mathcal{T}^k(-1-n/2)|_Q,$$

which is clearly equivalent to an operator between tractor bundles, that we shall denote $\not\Box$:

$$\not\Box : \mathcal{T}^k[1-n/2] \to \mathcal{T}^k[-1-n/2].$$

This example generalises. On any ambient form field one has

$$[\Delta\!\!\!/^m, Q\!\!\!/] = \sum_{p=0}^{m-1} \Delta\!\!\!/^{m-1-p}[\Delta\!\!\!/, Q\!\!\!/]\Delta\!\!\!/^p = -2\sum_{p=0}^{m-1} \Delta\!\!\!/^{m-1-p}\mathcal{K}_X \Delta\!\!\!/^p .$$

From the homogeneity of $\Delta\!\!\!/$ it follows that, acting on $\mathcal{T}^k(w)$, the $p^{\underline{\text{th}}}$ term on the right acts as $-2[2(w-2p)+n+2]\Delta\!\!\!/^{m-1}$. Summing terms we get that $[\Delta\!\!\!/^m, Q\!\!\!/]$ acts as $-2m(2w-2m+n+4)\Delta\!\!\!/^{m-1}$ on $\mathcal{T}^k(w)$ and so

(17) $$\Delta\!\!\!/^m : \mathcal{T}^k(m-n/2) \to \mathcal{T}^k(-m-n/2) \quad \text{is tangential.}$$

We write $\Delta\!\!\!/_m$ for the corresponding conformally invariant operator on M: $\Delta\!\!\!/_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2]$. Note that because of the weights involved the formal adjoint operator maps between the same spaces, $\Delta\!\!\!/^*_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2]$. It seems likely that these operators agree. (For example they do if $k=0$ [25, 18].) However we do not need to investigate this since we can simply work with the formally self-adjoint average of these

$$\tfrac{1}{2}(\Delta\!\!\!/_m + \Delta\!\!\!/^*_m) =: \Box\!\!\!/_m : \mathcal{T}^k[m-n/2] \to \mathcal{T}^k[-m-n/2] \quad m \in \{0,1,2,\dots\}.$$

It is straightforward to verify that these have leading term $(-1)^m\Delta^m$.

The above conformally invariant powers of the Laplacian arise from ambient operators which are tangential only for a specific weight. There are also ambient operators which act tangentially on forms without any assumptions of homogeneity. From the tables we have $[\mathcal{K}_X, Q\!\!\!/] = [\mathcal{L}_X - \mathcal{L}_X^*, Q\!\!\!/] = 4Q\!\!\!/$ and so $(\mathcal{K}_X - 4)Q\!\!\!/ = Q\!\!\!/\mathcal{K}_X$. Then $[\delta, Q\!\!\!/] = -2\iota(\boldsymbol{X})$. On the other hand $[\iota(\boldsymbol{X})\Delta\!\!\!/, Q\!\!\!/] = \iota(\boldsymbol{X})[\Delta\!\!\!/, Q\!\!\!/] = -2\iota(\boldsymbol{X})\mathcal{K}_X$. So

$$\iota(D\!\!\!/) := -\delta(\mathcal{L}_X - \mathcal{L}_X^* - 4) + \iota(\boldsymbol{X})\Delta\!\!\!/$$

satisfies

$$[\iota(D\!\!\!/), Q\!\!\!/] = -4Q\!\!\!/\delta ,$$

which shows that $\iota(D\!\!\!/)$ acts tangentially on any ambient form. Similarly for $\varepsilon(D\!\!\!/) := \boldsymbol{d}(\mathcal{L}_X - \mathcal{L}_X^* - 4) + \varepsilon(\boldsymbol{X})\Delta\!\!\!/$. These are each homogeneous of weight -1 so for each $w \in \mathbb{R}$ we have tangential operators

$$\varepsilon(D\!\!\!/) : \mathcal{T}^k(w) \to \mathcal{T}^{k+1}(w-1), \quad \iota(D\!\!\!/) : \mathcal{T}^k(w) \to \mathcal{T}^{k-1}(w-1).$$

Via the identities of the tables and the others discussed, there are many alternative ways to write these operators. For example we have

(18) $$\varepsilon(D\!\!\!/) = (\mathcal{L}_X - \mathcal{L}_X^*)\boldsymbol{d} + \Delta\!\!\!/\varepsilon(\boldsymbol{X}) = (n + 2\nabla_X + 2)\boldsymbol{d} + \Delta\!\!\!/\varepsilon(\boldsymbol{X}).$$

The corresponding conformally invariant operators on form tractors are denoted respectively $\varepsilon(D\!\!\!/)$ and $\iota(D\!\!\!/)$,

$$\varepsilon(D\!\!\!/) : \mathcal{T}^k[w] \to \mathcal{T}^{k+1}[w-1], \quad \iota(D\!\!\!/) : \mathcal{T}^k[w] \to \mathcal{T}^{k-1}[w-1].$$

Using the tables it is straightforward to show that these satisfy many surprising and useful useful identities, for example

$$\iota(D\!\!\!/)\iota(D\!\!\!/) = 0, \quad \varepsilon(D\!\!\!/)\varepsilon(D\!\!\!/) = 0, \quad \iota(D\!\!\!/)\varepsilon(D\!\!\!/) + \varepsilon(D\!\!\!/)\iota(D\!\!\!/) = 0.$$

The main key to our constructions in the next lecture is the result that, as an operators on $\mathcal{T}^k[1+\ell-n/2]$, we have the remarkable identity

$$\not{\Box}_\ell \varepsilon(\not{\mathbb{D}}) = \varepsilon(X)\not{\Box}_{\ell+1}\,, \tag{19}$$

which generalises significantly earlier known identities for low order and the conformally flat case [20]. Even at low orders, verifying this using explicit formulae for the operators on M would be a daunting task. At the ambient level this an almost trivial consequence of the fact that the form Laplacian $\not\Delta$ commutes with both $\boldsymbol{d}$ and $\boldsymbol{\delta}$. The relevant result there is worthy of some emphasis so we write it as a proposition.

Proposition 2.2. *If $V \in \mathcal{T}^k(\ell-n/2+1)$ and $U \in \mathcal{T}^{k+1}(\ell-n/2)$ then for $\ell = 0,1,\ldots$ we have*

$$\not\Delta^\ell \varepsilon(\not{\mathbb{D}})V = \varepsilon(\boldsymbol{X})\not\Delta^{\ell+1}V\,, \quad \iota(\not{\mathbb{D}})\not\Delta^\ell U = \not\Delta^{\ell+1}\iota(\boldsymbol{X})U\,.$$

Here $\not\Delta^0$ means 1.

Proof. We will prove the first identity; the proof of the other is similar. First observe that acting on any ambient form field, we have

$$\not\Delta^\ell(2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not\Delta) = 2\ell\not\Delta^\ell\boldsymbol{d}+\not\Delta^\ell\varepsilon(\boldsymbol{X})\not\Delta = 2\ell\not\Delta^\ell\boldsymbol{d}+[\not\Delta^\ell,\varepsilon(\boldsymbol{X})]\not\Delta+\varepsilon(\boldsymbol{X})\not\Delta^{\ell+1}\,.$$

Now recall from the Tables 1 and 2 that $[\not\Delta,\varepsilon(\boldsymbol{X})] = -2\boldsymbol{d}$, and that $\not\Delta$ and $\boldsymbol{d}$ commute. Thus $[\not\Delta^\ell,\varepsilon(\boldsymbol{X})]\not\Delta = -2\ell\not\Delta^\ell\boldsymbol{d}$, giving

$$\not\Delta^\ell(2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not\Delta) = \varepsilon(\boldsymbol{X})\not\Delta^{\ell+1}\,.$$

On the other hand, from the definition of $\varepsilon(\not{\mathbb{D}})$, we have that $\varepsilon(\not{\mathbb{D}})V = (2\ell\boldsymbol{d}+\varepsilon(\boldsymbol{X})\not\Delta)V$ for $V \in \mathcal{T}^{k-1}(\ell-n/2+1)$. □

Interpreting the proposition down on the underlying conformal manifold M, the second display of the proposition gives $\iota(\not{\mathbb{D}})\not\Delta_\ell = \not\Delta_{\ell+1}\iota(X)$ on $\mathcal{T}^{k+1}[\ell-n/2]$. The formal adjoint of this is $\not\Delta_\ell^*\varepsilon(\not{\mathbb{D}}) = \varepsilon(X)\not\Delta_{\ell+1}^*$ on $\mathcal{T}^k[1+\ell-n/2]$, while the other display of the proposition gives $\not\Delta_\ell\varepsilon(\not{\mathbb{D}}) = \varepsilon(X)\not\Delta_{\ell+1}$ on the same space. Thus combining these gives (19).

2.4. **Naturality.** The construction of the ambient manifold does not depend on choosing a particular metric from the conformal class. So it is conformally invariant. However the manifold is not unique. For our purposes we have fixed some choice of ambient metric and we must check that operators finally obtained on M via the ambient construction do not depend on the choices made in arriving at our particular ambient manifold. It clearly suffices to show that the operators are natural for the underlying conformal structure on M i.e., given by universal formulae polynomial in the metric g and its inverse, the Levi connection, its curvature and covariant derivatives. In fact there is an algorithm [22, 6] for expressing the operators we require here (and in fact a significantly larger class [23]) in terms of such a formula. Here we sketch some of the ingredients.

One basic idea behind this algorithm is to understand how certain tractor fields and operators, the explicit formulae for which are already known, turn up on the ambient manifold. Closely related to $\varepsilon(\not{\mathbb{D}})$ and $\iota(\not{\mathbb{D}})$ is the operator $\boldsymbol{D} := \boldsymbol{\nabla}(n+2\boldsymbol{\nabla}_{\boldsymbol{X}}-2)+\boldsymbol{X}\boldsymbol{\Delta}$. This acts tangentially on any ambient tensor field and since it is homogeneous determines an operator between tractor bundles. It is straightforward to verify that this resulting operator on M is exactly the tractor-D operator which we met earlier in the proof of proposition 1.1. Recall that for a metric from the conformal class, is given by

$$D_AV := (n+2w-2)wY_AV + (n+2w-2)Z_A{}^a\nabla_aV - X_A(\nabla_p\nabla^pV + wJV)\,,$$

where V is a section of any tractor bundle of weight w. In these formulae ∇ means the coupled tractor–Levi-Civita connection.

Next consider the tractor curvature K, which is defined by $[\nabla_a, \nabla_b]V^A = K_{ab}{}^A{}_BV^B$ for $V \in \mathcal{T}$. In terms of a choice of metric from the conformal class, this has the formula

$$K_{abCE} = Z_C{}^cZ_E{}^eC_{abce} - 4X_{[C}Z_{E]}{}^e\nabla_{[a}\mathrm{P}_{b]e}\,.$$

This may be inserted invariantly into the space $\mathcal{T}^3 \otimes \mathcal{T}^2$ by $K \mapsto \mathbb{X}^3{\cdot}K = \varepsilon(X)\mathbb{Z}^2{\cdot}K$, where for taking the inner product implicit in the '$\cdot$' we view K as a 2-form (taking values in $\mathcal{T}^2$). Let us write Ω for $\mathbb{Z}^2{\cdot}K$. Then from the relationship of the tractor connection to the ambient connection, as we outlined earlier, it follows easily that the conformally invariant field $\varepsilon(X)\Omega$ is exactly the tractor field corresponding to $\varepsilon(\boldsymbol{X})\boldsymbol{R}$. Using this and the ambient Bianchi identity one finds that, in dimensions other 4, the tractor field on M exactly corresponding to to the ambient curvature $\boldsymbol{R}$ is $W := \frac{3}{(n-2)(n-4)}\iota(D)\varepsilon(X)\Omega$ which is readily expanded to give an explicit formula for $\boldsymbol{R}$ in terms of the underlying conformal structure.

The difference between the ambient Bochner and form Laplacians is given by

$$\not{\boldsymbol{\Delta}} = -\boldsymbol{\Delta} - \boldsymbol{R}\sharp\sharp\,,$$

(where $\boldsymbol{\Delta} := \boldsymbol{\nabla}^A\boldsymbol{\nabla}_A$.) It follows immediately that, as operators on k-forms,

$$\varepsilon(\not{\boldsymbol{D}}) = \varepsilon(\boldsymbol{D}) - \varepsilon(\boldsymbol{X})\boldsymbol{R}\sharp\sharp\,,\quad \iota(\not{\boldsymbol{D}}) = \iota(\boldsymbol{D}) - \iota(\boldsymbol{X})\boldsymbol{R}\sharp\sharp\,,$$

and so for the corresponding operators on M we have

$$\varepsilon(\not{D}) = \varepsilon(D) - \varepsilon(X)\Omega\sharp\sharp\,,\quad \iota(\not{D}) = \iota(D) - \iota(X)\Omega\sharp\sharp\,.$$

So from the formulae for D and K we have naturality and explicit formulae for these operators.

For higher order operators the first step is to express the relevant ambient operator as a composition of low order tangential operators that we already understand in terms of natural operators on M. Let us consider, for example, the operator $\not{\Box}_2 : \mathcal{T}^k[2-n/2] \to \mathcal{T}^k[-2-n/2]$. This arises from $\not{\boldsymbol{\Delta}}^2$ on the ambient manifold. Since $\not{\boldsymbol{\Delta}}^2$ acts tangentially on $\tilde{V} \in \boldsymbol{\mathcal{T}}^k(2-n/2)$, we are free to chose an extension of $\tilde{V}$ off $\mathcal{Q}$ that simplifies our calculations without affecting $\not{\boldsymbol{\Delta}}^2\tilde{V}|_{\mathcal{Q}}$. For example, given $\tilde{V}|_{\mathcal{Q}}$, it is easily verified that we can arrange that $\boldsymbol{\Delta}\tilde{V} = O(\boldsymbol{Q})$ (where $O(\boldsymbol{Q})$ in meant in the sense of formal power series). Then using that $\not{\boldsymbol{\Delta}} = -\boldsymbol{\Delta} - \boldsymbol{R}\sharp\sharp$ we have

$$\begin{aligned}\not{\boldsymbol{\Delta}}^2\tilde{V} &= (\boldsymbol{\Delta} + \boldsymbol{R}\sharp\sharp)(\boldsymbol{\Delta} + \boldsymbol{R}\sharp\sharp)\tilde{V}\\ &= \boldsymbol{\Delta}^2\tilde{V} + (\boldsymbol{\Delta}\boldsymbol{R})\sharp\sharp\tilde{V} + 2(\boldsymbol{\nabla}_{|A|}\boldsymbol{R})\sharp\sharp\boldsymbol{\nabla}^{|A|}\tilde{V} + \boldsymbol{R}\sharp\sharp(\boldsymbol{R}\sharp\sharp\tilde{V}) + O(\boldsymbol{Q})\,,\end{aligned}$$

where the bars around indices indicate that they are to be ignored for the purposes of expanding the $\sharp$'s. From the Bianchi identity and the Ricci flatness of the ambient metric (for simplicity we assume that $n \notin \{4,6\}$) it follows that $(\boldsymbol{\Delta}\boldsymbol{R})\sharp\sharp\tilde{V} = -\frac{1}{2}\boldsymbol{R}\sharp\sharp(\boldsymbol{R}\sharp\sharp\tilde{V})$ and so this combines with the last term. Similarly, since $\boldsymbol{X}_A\boldsymbol{\nabla}^AV = \boldsymbol{\nabla}_{\boldsymbol{X}}\tilde{V} = (2-n/2)\tilde{V}$ and $\boldsymbol{D}_A\boldsymbol{R} = (n-6)\boldsymbol{\nabla}_A\boldsymbol{R} - \boldsymbol{X}_A\boldsymbol{\Delta}\boldsymbol{R}$ we can replace $\boldsymbol{\nabla}_{|A|}\boldsymbol{R}$ with $\frac{1}{(n-6)}\boldsymbol{D}_A\boldsymbol{R}$

provided we further adjust the coefficient of the last term. On the other hand if we write $V \in \mathcal{T}^k[2-n/2]$ for the tractor field equivalent to $\tilde{V}|_{\mathcal{Q}}$ then it is clear that $\boldsymbol{R}\sharp\sharp(\boldsymbol{R}\sharp\sharp\tilde{V})|_{\mathcal{Q}}$ is equivalent to $W\sharp\sharp(W\sharp\sharp V)$. We have an explicit formula for W and so we can deal with these terms. Next note that since $\boldsymbol{\Delta}\tilde{V} = O(\mathcal{Q})$ it follows that $2\boldsymbol{\nabla}^A\tilde{V} = 2\boldsymbol{\nabla}^A\tilde{V} - \boldsymbol{X}^A\boldsymbol{\Delta}\tilde{V} + O(\mathcal{Q}) = \boldsymbol{D}^A\tilde{V} + O(\mathcal{Q})$. Thus, modulo terms of the form $\boldsymbol{R}\sharp\sharp(\boldsymbol{R}\sharp\sharp\tilde{V})$, and modulo $O(\mathcal{Q})$, the term $2(\boldsymbol{\nabla}_{|A|}\boldsymbol{R})\sharp\sharp\boldsymbol{\nabla}^{|A|}\tilde{V}$ is a multiple of $(\boldsymbol{D}_{|A|}\boldsymbol{R})\sharp\sharp\boldsymbol{D}^{|A|}\tilde{V}$. The restriction of this to $\mathcal{Q}$ is equivalent to $(D_{|A|}W)\sharp\sharp D^{|A|}V$ which is a combination of standard tractor objects for which we have explicit formulae.

Finally there is the term $\boldsymbol{\Delta}^2\tilde{V}$. From the relationship between $\boldsymbol{D}$ and D it follows easily that $\boldsymbol{\Delta}$ acts tangentially on ambient tensors homogeneous of weight $(1-n/2)$ and in this case descends to the conformally invariant $\Box$ on tractor fields of weight $(1-n/2)$ on M. The operator $\boldsymbol{D}$ lowers homogeneous weight by 1 and so $\boldsymbol{\Delta D}$ acts tangentially on $\tilde{V}$ and $\boldsymbol{\Delta D}\tilde{V}|_{\mathcal{Q}}$ is equivalent to $\Box DV$. Once again $\Box D$ is a composition of operators for which we have explicit formulae. This deals with $\boldsymbol{\Delta}^2\tilde{V}$ since $\boldsymbol{\Delta D}V = \boldsymbol{\Delta}(2\boldsymbol{\nabla} - \boldsymbol{X\Delta})\tilde{V} = -\boldsymbol{X}\boldsymbol{\Delta}^2\tilde{V} + 2[\boldsymbol{\Delta},\boldsymbol{\nabla}]\tilde{V}$ and expanding the commutator here yields ambient curvature terms each of which is either $O(\mathcal{Q})$ due to the conditions on the ambient Ricci curvature or can be dealt with by a minor variation of the ideas discussed above.

More generally for the powers $\not{\Delta}^m$ (as in (17)) that we require it is straightforward to show inductively [23] that there is an algorithm for re-expressing each of these, modulo $O(\mathcal{Q})$, as an operator polynomial $\boldsymbol{X}$,$\boldsymbol{D}$, $\boldsymbol{R}$, the ambient metric $\boldsymbol{h}$, and its inverse $\boldsymbol{h}^{-1}$. A formula for the corresponding operator on M is then given by a tractor expression which is the same formula except with $\boldsymbol{X}$,$\boldsymbol{D}$, $\boldsymbol{R}$, $\boldsymbol{h}$, and $\boldsymbol{h}^{-1}$ formally replaced by, respectively, X, D W, h and h^{-1}.

3. Lecture 3 – Proving the Theorem – a sketch

Henceforth we restrict ourselves to the setting where the underlying conformal manifold is even dimensional.

Recall that the conformal gauge operator that we constructed for the Maxwell operator arose from a single tractor operator, taking values in a subbundle of the tractor bundle $\mathbb{T}^{n/2-1}[-1-n/2]$ consisting of elements of the form

$$\begin{pmatrix} & 0 & \\ * & & 0 \\ & * & \end{pmatrix}.$$

Since the tractor-projectors X and Y are null and $\iota(X)Y = 1$ it follows at once from (8) that this subbundle may alternatively be characterised as the elements of the form $\iota(X)F$. Let us write $\mathbb{G}_k$ (with section space denoted $\mathcal{G}_k$) to denote the subbundle of $\mathbb{T}^k[k-n]$ consisting of elements of the form $\iota(X)F$ and write $\mathbb{G}^k$ (with section space $\mathcal{G}^k$) for the quotient bundle which pairs with this in conformal integrals. That is $\mathbb{G}^k := \mathbb{T}^k[-k]/\mathbb{V}^k[-k]$ where $\mathbb{V}^k[-k] = \mathrm{Ker}(\varepsilon(X) : \mathbb{T}^k[-k] \to \mathbb{T}^{k+1}[1-k])$ (equivalently $\mathbb{V}^k[-k]$ is the subbundle of $\mathbb{T}^k[-k]$ consisting of elements of the form $\varepsilon(X)U$). Note that from the composition series (10) for $\mathcal{T}^k$ we see that the section spaces have composition series

$$\mathcal{G}^k = \begin{array}{c}\mathcal{E}^{k-1}\\ \oplus \\ \mathcal{E}^k\end{array} \quad \text{and} \quad \mathcal{G}_k = \begin{array}{c}\mathcal{E}_k\\ \oplus \\ \mathcal{E}_{k-1}\end{array} .$$

We will write $q_k : \mathcal{E}^k \to \mathcal{G}^k$ for the canonical conformally invariant injection which, in a choice of conformal scale, is given by $\mathbb{Z}^k\cdot$. Similarly we write $q^k : \mathcal{G}_k \to \mathcal{E}_k$ for the conformal map to the quotient which, in a conformal scale, is given by $\mathbb{Z}\bullet$, where the bullet indicates the inner product on the form tractor indices. Note that the composition $\iota(X)q_k$ vanishes, and so also the dual operator $q^k\varepsilon(X)$.

In this language the (Maxwell operator, gauge) pair appearing as the first term in the sum (10) is really a single conformally invariant operator $\mathcal{E}^{n/2-1} \to \mathcal{G}_{n/2-1}$. To generalise this we should obviously find an operator, $\mathcal{E}^k \to \mathcal{G}_k$. However the symmetry of Maxwell detour complex suggests that there might be a more general operator taking $\mathcal{G}^k$ to $\mathcal{G}_k$. Given our observations above this is easily constructed. First observe that $\varepsilon(X)$ gives a conformally invariant bundle injection $\varepsilon(X) : \mathbb{G}^k \to \mathbb{T}^{k+1}[1-k]$. The conformally invariant adjoint operation is $\iota(X) : \mathbb{T}^{k+1}[k-1-n] \to \mathbb{G}_k$. Thus composing $\not{\Box}_{\ell+1}$ fore and aft with these gives the required operator,

$$\big(\iota(X)\not{\Box}_{\ell+1}\varepsilon(X) := \mathbb{K}_\ell\big) : \mathcal{G}^k \to \mathcal{G}_k \,,$$

where $\ell := n/2 - k$. Note that this is manifestly formally self-adjoint since $\not{\Box}_{\ell+1}$ is formally self-adjoint. The non-triviality is an easy consequence of the ellipticity of $\not{\Box}_{\ell+1}$ and the classification of operators on the conformal sphere (as discussed in lecture 2). Finally by the algorithm for constructing a formula for these operators one finds they are natural for $k \in \{0, 1, \dots, n/2\}$.

This at once gives candidates for the required long operators, viz. the full composition $L_k = q^k\mathbb{K}_\ell q_k$ as in the diagram

$$\mathcal{E}^k \xrightarrow{q_k} \begin{array}{c}\mathcal{E}^{k-1}\\ \oplus \\ \mathcal{E}^k\end{array} \xrightarrow{\mathbb{K}_\ell} \begin{array}{c}\mathcal{E}_k\\ \oplus \\ \mathcal{E}_{k-1}\end{array} \xrightarrow{q^k} \mathcal{E}_k .$$

By construction the L_k are formally self-adjoint and once again the non-triviality follows from the non-triviality of the $\mathbb{K}_\ell$ and the classification of the operators between forms in the conformally flat setting. The candidates for extensions of the L_k to conformally invariant elliptic operators (as in part (*iv*) of the Theorem) are the operators obtained by simply omitting the final projection, that is $\mathbb{L}_\ell := \mathbb{K}_\ell q_k$.

Toward establishing that these operators have the desired properties it is useful to make an observation related to the geometry of the underlying constructions. This is that, since $2\boldsymbol{X} = d\boldsymbol{Q}$, and $\boldsymbol{Q}$ is a defining function for $\mathcal{Q}$ in $\tilde{M}$, it follows that $\mathbb{G}^k$ may be naturally identified with $\wedge^k T^*\mathcal{Q}/\sim$ where $U_p \sim V_q$ if $U_p = \rho^s_*(V_q)$ for some $s \in \mathbb{R}_+$. This is an easy consequence of our recovery of the tractor bundles from a similar quotient of $T\tilde{M} \cong T^*\tilde{M}$. The exterior derivative on $\mathcal{Q}$ preserves the subspace of forms homogeneous with respect to the canonical $\mathbb{R}_+$ action $\rho(s)$ and so this determines an operator (for each k)

$$\tilde{d} : \mathcal{G}^k \to \mathcal{G}^{k+1} \quad \text{with formal adjoint} \quad \tilde{\delta} : \mathcal{G}_{k+1} \to \mathcal{G}_k .$$

Under the described geometric interpretation of $\mathcal{G}^k = \mathcal{E}^{k-1} \oplus \mathcal{E}^k$ the $\mathcal{E}^k$-part of the composition series arises from $\pi^* \mathcal{E}^k$. Since exterior differentiation commutes with the pull-back it follows that

$$\tilde{d} q_k = q_{k+1} d \quad \text{on } \mathcal{E}^k \quad \text{and} \quad q^k \tilde{\delta} = \delta q^{k+1} \quad \text{on } \mathcal{E}_{k+1} ,$$

where the second result follows by taking the formal adjoint of the first.

More generally we get an operator $\tilde{d} : \mathcal{G}^k[w] \to \mathcal{G}^{k+1}[w]$ given by

$$\mathbb{Y}^k \cdot \alpha + \mathbb{Z}^k \cdot \mu = \begin{pmatrix} \alpha \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} w\mu - \varepsilon(\nabla)\alpha \\ \varepsilon(\nabla)\mu \end{pmatrix} = \mathbb{Y}^{k+1} \cdot (w\mu - \varepsilon(\nabla)\alpha) + \mathbb{Z}^{k+1} \cdot \varepsilon(\nabla)\mu ,$$

and a formal adjoint $\tilde{\delta}$ for this. This generalisation of $\tilde{d}$ still arises from the exterior derivative on $\mathcal{Q}$, except now restricted to appropriately homogeneous sections of $\Lambda^k T^* \mathcal{Q}$. From these origins, or alternatively the explicit formula displayed, it is clear that $\tilde{d}^2 = 0$ (and hence also $\tilde{\delta}^2 = 0$). Also, the operator $\tilde{d}$ satisfies the anti-derivation rule $\tilde{d}(\varepsilon(U)V) = \varepsilon(\tilde{d}U)V + (-1)^k \varepsilon(U)\tilde{d}V$ for U in $\mathcal{G}^k[w]$ and V in any $\mathcal{G}^{k'}[w']$.

These operators turn up as factors in the components of $\mathbb{K}_\ell$. Consider $\varepsilon(X)\mathbb{K}_\ell$. As a map on $\mathcal{G}_k$ we have

$$\varepsilon(X) : \begin{matrix} \mathcal{E}_k \\ \oplus \\ \mathcal{E}_{k-1} \end{matrix} \to \begin{matrix} \mathcal{E}_{k+1} \\ \oplus \\ \mathcal{E}_k \end{matrix} \quad \text{by} \quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ u \end{pmatrix} . \tag{20}$$

Now $\varepsilon(X)\mathbb{K}_\ell = \varepsilon(X)\iota(X)\not{\square}_{\ell+1}\varepsilon(X)$. Since X is null, $\varepsilon(X)$ and $\iota(X)$ anticommute. Then by (19) we have $\varepsilon(X)\not{\square}_{\ell+1} = \not{\square}_\ell \varepsilon(\not{D})$. Thus $\varepsilon(X)\mathbb{K}_\ell = -\iota(X)\not{\square}_\ell \varepsilon(\not{D})\varepsilon(X)$. But recall $\varepsilon(\not{D})\varepsilon(X)$ arises from the ambient composition $\varepsilon(\not{\boldsymbol{D}})\varepsilon(\boldsymbol{X})$ and by (18) this is exactly $(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} + 2)\boldsymbol{d}\varepsilon(\boldsymbol{X}) = -(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} + 2)\varepsilon(\boldsymbol{X})\boldsymbol{d}$, where finally we have used the anti-commutativity of $\varepsilon(\boldsymbol{X})$ and $\boldsymbol{d}$. Thus on $\mathcal{G}^k$ (which is a quotient of $\mathcal{T}^k[-k]$) $\varepsilon(\not{D})\varepsilon(X) = -(n - 2k + 2)\varepsilon(X)\tilde{d} = -2(\ell + 1)\varepsilon(X)\tilde{d}$. Hence overall we have $\varepsilon(X)\mathbb{K}_\ell = 2(\ell + 1)\iota(X)\not{\square}_\ell \varepsilon(X)\tilde{d} = 2(\ell + 1)\mathbb{K}_{\ell-1}\tilde{d}$. By taking the formal adjoint there is a corresponding result for $\mathbb{K}_{\ell+1}\iota(X)$, and we summarise these surprising results in the following lemma which is central to the subsequent discussions.

Lemma 3.1. *As operators on $\mathcal{G}^k$ we have*

$$\varepsilon(X)\mathbb{K}_\ell = 2(\ell + 1)\mathbb{K}_{\ell-1}\tilde{d} \quad \text{and} \quad \mathbb{K}_{\ell+1}\iota(X) = 2(\ell + 2)\tilde{\delta}\mathbb{K}_\ell .$$

We are now ready to construct the operators of the Theorem. We start by looking at $\mathbb{L}_\ell = \mathbb{K}_\ell q_k$. Note that $\iota(X)$ and $\varepsilon(Y)$ are well defined on $\mathcal{G}^k$. Recall that on $\mathcal{T}^k$, and therefore also on $\mathcal{G}^k$,

$$\iota(X)\varepsilon(Y) + \varepsilon(Y)\iota(X) = h(X, Y) = 1 .$$

Using this, and since $\iota(X)q_k = 0$, we have $\mathbb{K}_\ell q_k = \mathbb{K}_\ell \iota(X)\varepsilon(Y)q_k$. Thus from the lemma we have

$$\mathbb{K}_\ell q_k = 2(\ell + 1)\tilde{\delta}\mathbb{K}_{\ell-1}\varepsilon(Y)q_k . \tag{21}$$

The claim is that this single conformally invariant operator gives the conformally invariant elliptic system $(\delta Q_{k+1}d,\ \delta Q_k)$ of part *(iv)* of the theorem.

$\mathbb{K}_\ell q_k$ takes values in $\mathcal{G}_k = \mathcal{E}_k \oplus \mathcal{E}_{k-1}$. As observed above, to obtain the component with range $\mathcal{E}_k$ we project with q^k. Applying this to both sides of the last display we have $L_k := q^k \mathbb{K}_\ell q_k = 2(\ell+1) q^k \tilde{\delta} \mathbb{K}_{\ell-1} \varepsilon(Y) q_k$. Then recall $q^k \tilde{\delta} = \delta q^{k+1}$ so

$$L_k = 2(\ell+1)\delta q^{k+1} \mathbb{K}_{\ell-1} \varepsilon(Y) q_k$$

which establishes that L_k has δ as a left factor. Continuing on to show that it also has d as a right factor involves similar arguments. First observe that $\iota(Y)\varepsilon(X) + \varepsilon(X)\iota(Y)$ is well-defined and acts as the identity on $\mathcal{G}_{k+1}$ and recall $q^{k+1}\varepsilon(X) = 0$. So we may insert $\iota(Y)\varepsilon(X)$ to obtain

$$L_k = 2(\ell+1)\delta q^{k+1}\iota(Y)\varepsilon(X)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k = 4\ell(\ell+1)\delta q^{k+1}\iota(Y)\mathbb{K}_{\ell-2}\tilde{d}\varepsilon(Y)q_k\,,$$

where to obtain the last expression we have used the Lemma to exchange $\varepsilon(X)\mathbb{K}_{\ell-1}$ with a multiple of $\mathbb{K}_{\ell-2}\tilde{d}$. On the other hand it is easily verified that, in a conformal scale $\sigma \in \mathcal{E}[1]$, the corresponding $\varepsilon(Y)$ agrees with $\varepsilon(\sigma^{-1}\tilde{d}\sigma)$. From the anti-derivation rule for $\tilde{d}$, $\tilde{d}^2 = 0$, and that the log-derivative $\sigma^{-1}\tilde{d}\sigma$ is annihilated by $\tilde{d}$, it follows that $\tilde{d}$ anti-commutes with $\varepsilon(Y)$. Finally we have already noted that $\tilde{d}q_k = q_{k+1}d$ and so by this general construction we obtain, for each k, long operators which factor through δ and d,

$$L_k = -4\ell(\ell+1)\delta q^{k+1}\iota(Y)\mathbb{K}_{\ell-2}\varepsilon(Y)q_{k+1}d\,.$$

This completes part *(iii)* since from their non-triviality it follows easily that each L_k has leading term $(\delta d)^{n/2-k}$ (at least up to a constant multiple).

This result for the L_k suggests

$$M_k := q^k \iota(Y)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k \qquad (22)$$

(or some multiple thereof) is a candidate for Q_k.

Next we examine the $\mathcal{E}_{k-1}$-component of (21). In terms of the projectors from lecture 1 this is the coefficient of $\mathbb{X}^k = \varepsilon(X)\mathbb{Z}^{k-1}$. Since q^{k-1} is given, in a choice of conformal scale, by $\mathbb{Z}^{k-1}\bullet$ it follows that $q^{k-1}\iota(Y)\mathbb{K}_\ell q_k$ is the $\mathcal{E}_{k-1}$-component of $\mathbb{K}_\ell q_k$. Composing $q^{k-1}\iota(Y)$ with the right-hand-side of (21) brings us to $2(\ell+1)q^{k-1}\iota(Y)\tilde{\delta}\mathbb{K}_{\ell-1}\varepsilon(Y)q_k$. That $\{\iota(Y), \tilde{\delta}\}$ vanishes on $\mathcal{G}_{k+1}$ is just the formal adjoint of the result for $\{\tilde{d}, \varepsilon(Y)\}$ on $\mathcal{G}^{k-1}$. We have already that $q^{k-1}\tilde{\delta} = \delta q^k$ and so

$$q^{k-1}\iota(Y)\mathbb{K}_\ell q_k = -2(\ell+1)\delta q^k\iota(Y)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k = -2(\ell+1)\delta M_k\,.$$

Summarising then, we have that in a choice of conformal scale

$$\mathbb{K}_\ell q_k : \mathcal{E}^k \to \begin{array}{c}\mathcal{E}_k\\ \oplus \\ \mathcal{E}_{k-1}\end{array} \text{ is given by } \quad u \mapsto \begin{pmatrix} \delta M_{k+1} du \\ \delta M_k u \end{pmatrix}, \quad k = 1, \ldots n/2-1\,,$$

where M_k is defined by (22) above and we have ignored the details of non-zero constants. Note that by construction $\mathbb{K}_\ell q_k$ is conformally invariant. Therefore δM_k is conformally invariant on the null space of $\delta M_{k+1}d$. This is more fundamental than the transformation formula. Nevertheless, observe that the transformation formula claimed in part *(iv)* is now immediate from the invariance of $\mathbb{K}_\ell q_k$ and the conformal transformation formula $\widehat{\mathbb{Z}} = \mathbb{Z} + \varepsilon(\Upsilon)\mathbb{X}$. Thus part *(iv)* is proved.

For Part *(ii)* of the Theorem we should examine the conformal transformation law of $M_k = q^k\iota(Y)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k$. This looks potentially complicated as neither $\iota(Y)$ nor $\varepsilon(Y)$ is conformally invariant. Let us simplify initially and consider just part of this viz. $\mathbb{K}_{\ell-1}\varepsilon(Y)q_k$. Although $\mathbb{K}_{\ell-1}$ maps $\mathcal{G}^{k+1} \to \mathcal{G}_{k+1}$ we will show that the image of $\mathbb{K}_{\ell-1}\varepsilon(Y)q_k : \mathcal{C}^k \to \mathcal{G}_{k+1}$ lies entirely in the $\mathcal{E}_k$ part of

$$\mathcal{G}_{k+1} = \begin{array}{c}\mathcal{E}_{k+1}\\ \oplus \\ \mathcal{E}_k\end{array} .$$

Suppose then that φ is a closed k-form and consider $\varepsilon(X)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi$. By the lemma, and the commutation rules observed above, this gives (a constant multiple of)

$$\mathbb{K}_{\ell-2}\tilde{d}\varepsilon(Y)q_k\varphi = -\mathbb{K}_{\ell-2}\varepsilon(Y)\tilde{d}q_k\varphi = -\mathbb{K}_{\ell-2}\varepsilon(Y)q_k d\varphi = 0 .$$

From (20) this exactly proves that in a choice of scale we have $\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi = \mathbb{X}^{k+1}\cdot(\tilde{M}_k)$, or in the matrix notation,

$$[\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi]_g = \begin{pmatrix} 0 \\ \tilde{M}_k\varphi \end{pmatrix}$$

for some operator $\tilde{M}_k : \mathcal{C}^k \to \mathcal{E}_k$. Recall one recovers the coefficient of $\mathbb{X}^{k+1}$ by left composing with $q^k\iota(Y)$ and so $\tilde{M}_k\varphi = q^k\iota(Y)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi = M_k\varphi$. How does this formula transform conformally? For the purposes of this calculation we may take Y to be the metric dependent section in $\mathcal{G}^1[1]$ given by $Y = \sigma^{-1}\tilde{d}\sigma$ where $\sigma \in \mathcal{E}[1]$ is the conformal scale, that is $g = \sigma^{-2}\boldsymbol{g}$. Then conformal rescaling

$$g \mapsto \hat{g} = e^{2\omega}g\,, \quad \text{corresponds to} \quad \sigma \mapsto \hat{\sigma} = e^{-\omega}\sigma\,,$$

whence

$$Y \mapsto \hat{Y} = Y - \tilde{d}\omega\,.$$

From this we have

$$\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi \mapsto \mathbb{K}_{\ell-1}\varepsilon(\hat{Y})q_k\varphi = \mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi - \mathbb{K}_{\ell-1}\varepsilon(\Upsilon)q_k\varphi$$

where, as usual $\Upsilon := d\omega$. Now observe that, by the anti-derivation rule, $\tilde{d}q_k\omega\varphi = \tilde{d}\omega q_k\varphi = \varepsilon(\Upsilon)q_k\varphi$, since $\tilde{d}q_k = q_{k+1}d$ and $d\varphi = 0$. Thus the conformal variational term in the display may be written as the composition $\mathbb{K}_{\ell-1}\tilde{d}q_k\omega\varphi$. Using the Lemma we have $2(\ell+1)\mathbb{K}_{\ell-1}\tilde{d} = \varepsilon(X)\mathbb{K}_\ell$ and so the conformal variation term is $\varepsilon(X)\mathbb{K}_\ell q_k\omega\varphi$ – at least if we ignore the division by $2(\ell+1)$. Clearly this is also annihilated by the left action of $\varepsilon(X)$ and so is of the form $\mathbb{X}^{k+1}\cdot\tilde{L}_k\omega\varphi$, or

$$\begin{pmatrix} 0 \\ \tilde{L}_k\omega\varphi \end{pmatrix}$$

for some operator $\tilde{L}_k : \mathcal{E}^k \to \mathcal{E}_k$. Once again we recover the coefficient by composing on the left with $q^k\iota(Y)$ to obtain $q^k\iota(Y)\varepsilon(X)\mathbb{K}_\ell q_k\omega\varphi = q^k\mathbb{K}_\ell q_k\omega\varphi$ (since $\{\iota(Y), \varepsilon(X)\} = 1$

and $q^k\varepsilon(X) = 0$) which we recognise as $L_k\omega\varphi$ (i.e. $\tilde{L}_k = L_k$). So in summary we have shown the conformal transformation

$$[\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\varphi]_g = \begin{pmatrix} 0 \\ M_k\varphi \end{pmatrix} \mapsto [\mathbb{K}_{\ell-1}\varepsilon(\widehat{Y})q_k\varphi]_g = \begin{pmatrix} 0 \\ M_k - \frac{1}{2(\ell+1)}L_k\omega\varphi \end{pmatrix}.$$

Thus we exactly recover the result claimed in part (*ii*) if we take

$$Q_k = (n+2)n\ldots(n-2k+2)M_k\,.$$

Recovering Branson's Q-curvature. So far in our construction of the operators $\mathbb{K}_\ell$, we have only used $\not\Delta^\ell$ on $\mathcal{T}^{n/2-\ell}(\ell-n/2)$ whereas as we have observed already this operator is tangential on $\mathcal{T}^k(\ell-n/2)$ for any k. The upshot of the latter observation is that for each ℓ, $\mathbb{K}_\ell$ generalises to give an operator

$$\mathbb{K}_\ell : \mathcal{G}^k[w] \to \mathcal{G}_k[-w]\,,$$

where $w = \ell - n/2 + k$. Thus we get order 2ℓ conformally invariant operators between weighted forms

$$(L^k_\ell := q^k\mathbb{K}_\ell q_k) : \mathcal{E}^k[w] \to \mathcal{E}_k[w]\,.$$

These are natural for $\ell \leq n/2-1$, and also for $\ell = n/2$ if $k=0$. In this generalised setting Lemma 3.1 still holds in the sense that for example on $\mathcal{G}^k[w]$ we have

$$\varepsilon(X)\mathbb{K}_\ell = 2(\ell+1)\mathbb{K}_{\ell-1}\tilde{d}\,.$$

Thus for $L^k_\ell = q^k\mathbb{K}_\ell q_k = q^k\iota(Y)\varepsilon(X)\mathbb{K}_\ell q_k$ we obtain the alternative formula

$$L^k_\ell = 2(\ell+1)q^k\iota(Y)\mathbb{K}_{\ell-1}\tilde{d}q_k\,.$$

Now for $\mu \in \mathcal{E}^k$, and $\sigma \in \mathcal{E}[1]$ a conformal scale, we have $\sigma^w\mu \in \mathcal{E}^k[w]$. We will apply L^k_ℓ to this. First note that in terms of the splittings for the conformal scale we have

$$q_k\sigma^w\mu = \mathbb{Z}^k\cdot(\sigma^w\mu)\,.$$

So, from the explicit formula for $\tilde{d}$, we have

$$\begin{aligned}\tilde{d}q_k\sigma^w\mu &= w\sigma^w\mathbb{Y}^{k+1}\cdot\mu + \sigma^w\mathbb{Z}^{k+1}\cdot\varepsilon(\nabla)\mu \\ &= w\sigma^w\varepsilon(Y)\mathbb{Z}^k\cdot\mu + \sigma^w\mathbb{Z}^{k+1}\cdot d\mu \\ &= w\sigma^w\varepsilon(Y)q_k\mu + \sigma^w q_{k+1}d\mu\,,\end{aligned}$$

where we used that the Levi Civita connection for σ annihilates σ. Using this again we obtain

$$L^k_\ell\sigma^w\mu = 2w(\ell+1)\sigma^w q^k\iota(Y)\mathbb{K}_{\ell-1}\varepsilon(Y)q_k\mu + 2(\ell+1)\sigma^w q^k\iota(Y)\mathbb{K}_{\ell-1}q_{k+1}d\mu\,.$$

If μ is closed then the second term vanishes and taking the coefficient of w and setting, in this, $\ell = n/2$ (i.e., $w=0$) yields, up to a multiple, the Q_k operator on μ. In particular if $k=0$ and we take $\mu = 1$ we obtain $Q_0 1$. But in this case this construction is exactly recovering the Q-curvature according to Branson's original definition by dimensional continuation, since by construction the operators L^0_ℓ agree with the GJMS operators [24] on densities.

We have not only shown that $Q_0 1$ is the usual Q-curvature but also that the Q_k operators, defined earlier without the use of dimensional continuation, also arise from the analogous dimensional continuation argument but now applied to the operators L_ℓ^k between form densities. Implicitly we are using that the operators $\mathbb{K}_\ell$ are given by universal formulae, polynomial in the dimension, and with coefficients in terms of a stable basis of Riemannian invariants. This is an immediate consequence of the tractor formulae for the operators $\not{\Box}_\ell$ since the basic tractor objects (such as the tractor-D operator and the W tractor are given by formulae in this form).

It is easily shown too [6] that defining the Q-curvature to be $Q_0 1$ with Q_0 defined by (22) is equivalent to the definitions of both [19] and [22].

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