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Q-CURVATURE AND SPECTRAL INVARIANTS

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The concept of Q-curvature was introduced in [8, 19, 9, 10], and has since been seen to be a central object in conformal geometry and geometric analysis; see for example [22, 21, 23, 24, 25, 31, 32, 39, 41, 42, 43]. The Q-curvature in large part governs the movement within a conformal class of the functional determinant of positively elliptic operators with reasonable conformal properties; in particular the conformal Laplacian. It also governs the movement of the Cheeger half-torsion and related detour torsion quantities which we shall examine in detail below, reporting on recent joint work with Rod Gover. Closely related to this is a natural, geometric expression of Beckner's higher dimensional Moser-Trudinger-Onofri inequality in terms of Q-curvature. Graham and Hirachi [39] have recently shown that the Q-curvature also provides a natural higher-dimensional generalization of Weyl relativity, in that its total metric variation is the Fefferman-Graham obstruction tensor, itself a higher-dimensional analogue of the Bach tensor. The original construction of the Q-curvature was based on the GJMS operator series constructed in [40], but in the other direction, the Q-curvature implicitly contains within it enough information to construct the critical GJMS operator P via conformal variation.

This material was the subject of 3 lectures presented at the 24$^{\text{th}}$ Czech Winter School on Geometry and Physics held in Srní in January 2004. These notes draw on recent work with Rod Gover and with Mike Eastwood. The author would like to thank Pierre Albin, Alice Chang, Mike Eastwood, Rod Gover, Robin Graham, and Paul Yang for enlightening discussions on this material.

1. Introduction to Q-curvature

We begin by describing the vacua that Q-curvature is meant to fill.

The *Einstein tensor*, or divergence-free Ricci tensor $\mathsf{E} := \mathsf{r} - \frac{1}{2}\mathsf{K}g$ (where g is a pseudo-Riemannian metric, r is its Ricci tensor, and K is its scalar curvature) is the total metric variation of the scalar curvature in dimensions $n > 2$. This means that if we take a smooth curve of metrics $g(\varepsilon)$, denote $(d/d\varepsilon)|_{\varepsilon=0}$ by a $\bullet$, and suppose that

$$g(0) = g\,, \qquad g^\bullet = h\,,$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

then

$$\left(\int \mathsf{K}\, dv_g\right)^{\bullet} = \int h^{ab}\mathsf{E}_{ab} dv_g \tag{1}$$

for compactly supported h, where dv_g is the pseudo-Riemannian measure. The indices on the right in (1), and whenever indices are used below, are abstract ones; in particular no choice of frame is implied. An index appearing twice, once up and once down, indicates a contraction. (1) shows how the Einstein-Hilbert action $\int \mathsf{K}$ leads to the Einstein equation.

In the first half of the last century, some mathematicians and physicists considered alternatives to the Einstein-Hilbert action in dimension 4 that are invariant under uniform scaling $\bar{g} = \alpha^2 g$, where α is a positive constant. The most famous of these is the action of *Weyl relativity*,

$$\mathcal{W}(g) := \int |\mathsf{C}|^2 dv_g ,$$

where C is the *Weyl conformal curvature tensor* of g (see (45) below for a formula). Since uniform scaling by α induces the response $\mathcal{W}(\bar{g}) = \alpha^{n-4}\mathcal{W}(g)$ in this functional, it is scale-invariant in dimension 4. Moreover, the Weyl integrand is *conformally invariant* in dimension 4: if $\widehat{g} = \Omega^2 g$ for Ω a smooth positive function, then $(|\mathsf{C}|^2 dv)_{\widehat{g}} = (|\mathsf{C}|^2 dv)_g$. Thus instead of critical metrics for the Weyl functional, one has critical *conformal classes*

$$[g] := \{\Omega^2 g \mid 0 < \Omega \in C^\infty(M)\} ,$$

where M is the underlying manifold. To further specify a metric within a conformal class, one might try to use the *scalar curvature prescription equation*

$$(\Delta + \tfrac{1}{6}\mathsf{K})\,\Omega = \tfrac{1}{6}\widehat{\mathsf{K}}\Omega^3 \qquad (n = 4) \tag{2}$$

and demand, for example, that $\widehat{\mathsf{K}}$ be one's favorite constant (usually ± 1 or 0).

The total metric variation of the functional $\mathcal{W}(g)$ is called the *Bach tensor*:

$$\left(\int |\mathsf{C}|^2 dv_g\right)^{\bullet} = \int h^{ab}\mathcal{B}_{ab} dv_g \qquad (n = 4) .$$

Note that this defines $\mathcal{B}$ uniquely, and that as the total metric variation of a conformally invariant quantity, the Bach tensor must be conformally invariant. The calculation corresponding to this last-mentioned property goes as follows. Let $e^{\eta\omega}$ be a 1-parameter group of conformal factors; here the parameter η runs through $\mathbb{R}$, and ω is a smooth function. Let $g(\varepsilon)$ be the one-parameter family of metrics above, and compute

$$\left.\frac{\partial^2}{\partial\varepsilon\partial\eta}\right|_{(\varepsilon,\eta)=(0,0)} \mathcal{W}(e^{2\eta\omega} g(\varepsilon))$$

in two ways, using the fact that the mixed partials in different orders coincide. Differentiating first with respect to ε, we get

$$\left(\int (e^{-2\eta\omega}h^{ab})(\mathcal{B}_{e^{2\varepsilon\omega}g})_{ab} e^{4\eta\omega} dv_g\right)' = 2\int \omega h^{ab}\mathcal{B}(g)_{ab} dv_g + \int h^{ab}\mathcal{B}'_{ab} dv_g ,$$

where the prime denotes $(d/d\eta)|_{\eta=0}$. But differentiating first with respect to η, we get 0. The conclusion is that

$$\mathcal{B}'_{ab} = -2\omega\mathcal{B}_{ab} \qquad (n=4)\,,$$

since h was arbitrary. This shows that $\mathcal{B}_{ab}$ is a conformally invariant section of $\mathcal{E}_{(ab)}[-2]$, the symmetric 2-tensor densities of conformal weight -2. (See the lectures of Rod Gover at this conference, [36], for notational conventions on conformal tensor density bundles.) In addition, $\mathcal{B}$ must be trace-free, since if h is a function times g, the variation in the h direction is a conformal variation. This gives

$$\mathcal{B}_{ab} \in \mathcal{E}_{(ab)_0}[-2] \qquad (n=4)\,,$$

where the subscript 0 on the indices means "trace free".

In higher even dimensions, there is an established generalization of the Bach tensor, namely the *Fefferman-Graham obstruction tensor* $\mathcal{A}_{ab}$ [30]. The Fefferman-Graham *ambient metric construction* starts with a conformal structure of signature (p,q) on an n-dimensional manifold M, and attempts to construct, on a collar $\mathcal{M}$ of the conformal metric bundle $\mathcal{Q}$ over M, a Taylor series for a pseudo-Riemannian metric h of signature $(p+1,q+1)$ with vanishing Ricci tensor. In even dimensions, the recursion for this Taylor series is obstructed at finite order, and one cannot continue without the vanishing of $\mathcal{A}_{ab}$. The tensor $\mathcal{A}_{ab}$ is conformally invariant when viewed as an element of $\mathcal{E}_{(ab)_0}[2-n]$. (Note that since we start only with a conformal structure on M, the obstruction to the ambient metric construction must be a conformally invariant object of some kind.) In dimension 4, $\mathcal{A}$ coincides with the Bach tensor.

Robin Graham has pointed out that one way to predict several properties, in particular the conformal weight, of $\mathcal{A}$ is to look at the linearization D of the nonlinear operator carrying g to $\mathcal{A}(g)$. If $h_{ab} \in \mathcal{E}_{(ab)_0}[2]$, so that $h^{ab} \in \mathcal{E}^{(ab)_0}[-2]$, then

$$Dh = \mathcal{A}^\bullet\,, \quad \text{where} \quad g^\bullet = h\,.$$

Thus if we know that $\mathcal{A}_{ab}$ is conformally invariant, say as an element of $\mathcal{E}_{(ab)_0}[w]$, then

$$D : \mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(ab)_0}[w]$$

is conformally invariant. A qualitative inspection of the way in which the obstruction arises shows that D is nontrivial in the conformally flat case. Thus we may consult the classification of conformally invariant differential operators on the sphere in [5] to find that the only possibility is

$$D : \mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(ab)_0}[2-n]\,. \tag{3}$$

In fact, Weyl group considerations in the conformally flat case force the weight $2-n$ in the target even if D is not known to be differential.

Question. Is there a quantity which generalizes the Weyl action to higher even dimensions, in the sense that its total metric variation is $\mathcal{A}_{ab}$?

Answer. Yes, the Q-curvature, according to a recent result of Graham and Hirachi [39]. That is,

$$\left(\int Q dv_g\right)^{\bullet} = \int h^{ab}\mathcal{A}_{ab} dv_g\,.$$

How does this fit with the above 4-dimensional discussion? In dimension 4,

$$Q = \frac{1}{6}(\Delta \mathsf{K} - \mathsf{K}^2 - 3|\mathsf{r}|^2)$$

is a linear combination of the *Pfaffian* (or Euler integrand) Pff, $|\mathsf{C}|^2$, and the exact divergence $\Delta\mathsf{K}$. Since the integral of $\Delta\mathsf{K}$ is identically zero, and that of Pff is independent of the metric, the total metric variation of Q is (up to a constant factor) the same as that of $|\mathsf{C}|^2$.

This indicates that the way to generalize the phenomenon exhibited by the pair $(|\mathsf{C}|^2, \mathcal{B})$ is not, for example, to look at $|\mathsf{C}|^4$ in dimension 8; such things will produce zero in the conformally flat case, rather than an operator like (3). The 6-dimensional case is illuminating in this regard; see Sec. 4 below.

In addition to its interesting total metric variation, the Q-curvature has as its conformal variation the *critical GJMS operator* P (see Definition 2 and Remark 4 below).

Note that $\mathcal{A}$, like any total metric variation, is divergence free. Indeed, to take the variation in the direction of a one-parameter family of diffeomorphisms, one just takes

$$h^{ab} = \nabla^a X^b + \nabla^b X^a$$

for some vector field X. Integrating by parts in the diffeomorphism invariance condition for Q, one obtains

$$0 = \int (\nabla^a X^b + \nabla^b X^a)\mathcal{A}_{ab} = -2\int X^a \nabla^b \mathcal{A}_{ab}.$$

Since X was arbitrary, $\nabla^b \mathcal{A}_{ab} = 0$.

The Q-curvature turns up naturally from an approach in an apparently unrelated direction. The *Moser-Trudinger inequality* [44, 4] says that for a suitably differentiable function ω on the sphere S^2,

$$\log \int_{S^2} e^{2(\omega - \bar{\omega})} d\xi \le \int_{S^2} \omega(\Delta\omega) d\xi, \tag{4}$$

where $d\xi$ is normalized round measure, and $\bar{\omega} := \int_{S^2} \omega\, d\xi$ is the average value of ω. Furthermore, one has equality in (4) if and only if ω is the conformal factor of a conformal diffeomorphism h; that is,

$$(h^{-1})^* g_0 = e^{2\omega} g_0\,, \tag{5}$$

where g_0 is the round metric.

In [4], Beckner generalized this to higher dimensions (see also Carlen-Loss [20]). Looking at even dimensions for simplicity, Beckner's inequality says that

$$\log \int_{S^n} e^{n(\omega - \bar{\omega})}\, d\xi \le \frac{n}{2(n-1)!} \int_{S^n} \omega(P\omega)\, d\xi\,, \tag{6}$$

where

$$P = \Delta\{\Delta + n - 2\}\{\Delta + 2(n-3)\}\{\Delta + 3(n-4)\}\ldots\{\Delta + \tfrac{n}{2}(\tfrac{n}{2} - 1)\}\,, \tag{7}$$

with equality if and only if ω is a conformal factor; i.e. iff (5) holds.

Inequalities closely related to Beckner's play roles in recent work on important problems; for example, de Branges' resolution of the Bieberbach conjecture (via the Lebedev-Mihlin inequality), and Perelman's work on the Poincaré conjecture (via

Gross' logarithmic Sobolev inequality). All these inequalities are endpoint derivatives of sharp borderline Sobolev imbedding inequalities, or duals of such.

Question. Is there an expression of Beckner's inequality in terms of some local invariant?

Something of a template for such an expression may be taken from the *Yamabe problem* of prescribing constant scalar curvature. In dimension 4, this problem is governed by equation (2) above; in dimensions $n \geq 3$, the generalization is

$$(8) \qquad \underbrace{\left(\Delta + \frac{n-2}{4(n-1)}\mathsf{K}\right)}_{=:Y} u = \frac{n-2}{4(n-1)}\widehat{\mathsf{K}}u^{(n+2)/(n-2)}, \qquad u = \Omega^{(n-2)/2}.$$

The Yamabe problem is attacked [51, 50, 3, 48] by looking at the *Yamabe quotient*

$$\frac{(Yu,u)_{L^2}}{\|u\|^2_{L^{2n/(n-2)}}},$$

which encodes information about the borderline Sobolev embedding $L^2_1 \hookrightarrow L^{2n/(n-2)}$. According to the Yamabe equation (8), the Yamabe quotient is

$$\frac{n-2}{4(n-1)}\int \widehat{\mathsf{K}}\, dv_{\widehat{g}},$$

provided $\widehat{g} = \Omega^2 g$ has total volume 1. Thus the Yamabe problem is closely related to the search for critical points of a locally defined functional, namely $\int \widehat{\mathsf{K}}\, dv_{\widehat{g}}$.

Answer. The Q-curvature describes Beckner's inequality, though in a somewhat different way from that suggested by the Yamabe template above. Suppose $\widehat{g} = e^{2\omega}g$ is a metric conformal to the round metric g, and with the same volume. Let $\mathbf{Q}$ denote the $(-n)$-density version of the Q-curvature. This is akin to always considering $Q_g dv_g$ instead of Q_g. Then Beckner's inequality says exactly that

$$0 \leq \int_{S^n} \omega(\widehat{\mathbf{Q}} + \mathbf{Q}),$$

with equality if and only if (5).

This may seem a little unsatisfying, as it mentions the conformal factor, measured from the round metric, explicitly. A more invariant way to describe this is to look at *cocycles* on the conformal class $[g] := \{e^{2\omega}g \mid \omega \in C^\infty(M)\}$. The conformal factor

$$\omega = \omega(\widehat{g}, g) = \tfrac{1}{2}\log(\widehat{g}/g)$$

is one such cocycle, since it satisfies the condition

$$(9) \qquad \widehat{g} = e^{2\omega}g,\ \widehat{\widehat{g}} = e^{2\eta}\widehat{g} \ \Rightarrow\ \omega(\widehat{\widehat{g}}, g) = \omega(\widehat{\widehat{g}}, \widehat{g}) + \omega(\widehat{g}, g).$$

(Each side of (9) is an expression for $\omega + \eta$.)

A more subtle cocycle, valued in $\mathbb{R}$ rather than (as ω is) in $C^\infty(M)$, is

$$(10) \qquad \mathcal{H}(\widehat{g}, g) = \int_M \omega(\widehat{\mathbf{Q}} + \mathbf{Q}).$$

Because ω is alternating, i.e. $\omega(\widehat{g}, g) = -\omega(g, \widehat{g})$, so is $\mathcal{H}$. The cocycle condition on $\mathcal{H}$,

$$\mathcal{H}(\widehat{\widehat{g}}, g) = \mathcal{H}(\widehat{\widehat{g}}, \widehat{g}) + \mathcal{H}(\widehat{g}, g),$$

is not at all obvious, and is intertwined with the other properties that make $\mathbf{Q}$ what it is; see Remark 8 below.

2. Inequalities

Pending a detailed discussion below of the cocycle property of $\mathcal{H}(\widehat{g}, g)$, let us assume it, as well as the related conformal change law for $\mathbf{Q}$,

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega \,,$$

where $\mathbf{P}$ is the critical GJMS operator. There is some ambiguity, for general conformal classes, as to what should be considered the "best version" of $\mathbf{Q}$, but in the conformally flat case (the backdrop for all the sharp inequalities on round S^n that we shall discuss), $\mathbf{Q}$ is unique.

One aspect of the cocycle (10) that is quite relevant to its appearance in determinant and torsion quantities is its lack of scale invariance. If we change $\widehat{g}$ to $e^{2\alpha}\widehat{g}$, then $\widehat{\mathbf{Q}}$ and $\mathbf{Q}$ are unaffected, but ω changes to $\omega+\alpha$. This adds $2\alpha \int \mathbf{Q}$ to $\mathcal{H}(\widehat{g}, g)$, since $\int \mathbf{Q}$ is conformally invariant. (See Remark 6 below.) One cure for this is to add a volume penalty:

$$\tilde{\mathcal{H}}(\widehat{g}, g) := \int_M \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) - \frac{2\int \mathbf{Q}}{n} \log \frac{\operatorname{vol} \widehat{g}}{\operatorname{vol} g} \,, \tag{11}$$

or simply to restrict to a slice in the conformal class consisting of metrics of a fixed volume. Our volume penalty is clearly also a cocycle, so $\tilde{H}$ is a cocycle.

There is now some prospect of getting a *minimal metric* g for this cocycle; that is, a metric g in the conformal class satisfying

$$0 \le \tilde{\mathcal{H}}(\widehat{g}, g) \,, \qquad \text{all } \widehat{g} \in [g] \,.$$

Beckner's exponential class inequality identifies the minimal metrics for the round conformal class as the round metrics:

Theorem 1. [Restatement of [4], Theorem 1] *In the round conformal class on S^n for $n \ge 2$, the minimal metrics for $\tilde{\mathcal{H}}(\widehat{g}, g)$ are exactly the positive constant multiples of the $h^* g_0$, where h is a conformal transformation and g_0 is the standard round metric.*

Another way of writing this is as follows. Let $⨍ = (1/\mathrm{w}_n) \int$, where w_n is the volume of round S^n. Then

$$\begin{aligned} 0 \le \tilde{\mathcal{H}}(\widehat{g}, g) &= \int \omega(\widehat{\mathbf{Q}} + \mathbf{Q}) - \frac{2\int \mathbf{Q}}{n} \log \frac{\int e^{n\omega} dv_g}{\int dv_g} \\ &= \mathrm{w}_n ⨍ \omega \underbrace{(\widehat{\mathbf{Q}} + \mathbf{Q})}_{2\mathbf{Q}+\mathbf{P}\omega} - \frac{2\mathrm{w}_n}{n} \left(⨍ \mathbf{Q} \right) \log ⨍ e^{n\omega} \\ &= 2(n-1)!\mathrm{w}_n \bar{\omega} + \mathrm{w}_n ⨍ \omega \mathbf{P} \omega - \frac{2\mathrm{w}_n \mathbf{Q}}{n} \log ⨍ e^{n\omega} \\ &= \mathrm{w}_n \left\{ ⨍ \omega \mathbf{P} \omega - \frac{2(n-1)!}{n} \log ⨍ e^{n(\omega - \bar{\omega})} dv_g \right\} . \end{aligned}$$

where $\bar{\omega}$ is the average value of ω, since Q takes the constant value $(n-1)!$ at the round metric ([9], Theorem 2.8(f)). Since P has the form (7) on S^n, we have arrived at (6).

Prospective generalizations of the Beckner-Moser-Trudinger inequality to more general manifolds are *Adams-Fontana type* inequalities [1, 33]. These take the form (on a compact Riemannian manifold (M,g))

$$\frac{1}{\operatorname{vol} g}\int_M \exp\left(\frac{\beta(u-\bar{u})^2}{\|\nabla^{n/2}u\|_2^2}\right) \le c_\beta\,, \tag{12}$$

where β is a positive constant, and c_β is a positive constant depending on β. The form of this inequality that may be readily compared with a Beckner-Moser-Trudinger inequality is

$$\frac{1}{\operatorname{vol} g}\log\int_M e^{n(u-\bar{u})} \le \log c_\beta + \frac{n^2}{4\beta}\|\nabla^{n/2}u\|_2^2\,. \tag{13}$$

Indeed, by the Schwartz inequality at each $x\in M$,

$$2\cdot\sqrt{\beta}(u-\bar{u})\cdot\frac{n}{2\sqrt{\beta}}\|\nabla^{n/2}u\|_2^2 \le \beta(u-\bar{u})^2+\frac{n^2}{4\beta}\|\nabla^{n/2}u\|_2^4\,,$$

so

$$n(u-\bar{u}) \le \frac{\beta(u-\bar{u})^2}{\|\nabla^{n/2}u\|_2^2}+\underbrace{\frac{n^2}{4\beta}\|\nabla^{n/2}u\|_2^2}_{\text{const}}\,, \tag{14}$$

and

$$\begin{aligned}\log\left(\frac{1}{\operatorname{vol} g}\int e^{n(u-\bar{u})}\right) &\le \log\frac{1}{\operatorname{vol} g}\int \exp(\text{RHS(eqn. (14))})\\ &= \frac{n^2}{4\beta}\|\nabla^{n/2}u\|_2^2+\log\left(\frac{1}{\operatorname{vol} g}\int \exp\frac{\beta(u-\bar{u})^2}{\|\nabla^{n/2}u\|_2^2}\right)\,.\end{aligned}$$

Assuming the Adams-Fontana inequality (12), this is $\le$ RHS(eqn. (13)).

To assess the meaning of the Adams-Fontana inequality for the round conformal class on S^4, note that [1, 33] allow us to take $\beta = 32\pi^2+\varepsilon$, where $\varepsilon>0$, in dimension 4, so that

$$\frac{n^2}{4\beta}=\frac{1}{8\pi^2}-\varepsilon'\,,$$

where ε' is any small positive number. On round S^4,

$$\frac{1}{8\pi^2}=\underbrace{\frac{3}{8\pi^2}}_{\frac{1}{\operatorname{vol} S^4}}\cdot\underbrace{\frac{1}{3}}_{\frac{4}{2(4-1)!}}\,,$$

so qualitatively we have reached the Beckner-Moser-Trudinger form. The various sharpnesses (the best constant, extremals, and the operator P) still require substantial work to achieve.

Inequalities and cocycles involving *subcritical Q-curvatures* also turn up in the problem of estimating determinant and torsion quantities. The idea here (first taken up in [6]) is that a GJMS-type operator $P_m = \Delta^{m/2} + \mathrm{LOT}$ for $m < n$ (where here and below, "LOT" means "lower order terms"), since it takes the form

$$\underbrace{\delta(\Delta^{m/2-1} + \mathrm{LOT})d}_{P_m^0} + \frac{n-m}{2}Q_m \tag{15}$$

for some local invariant Q_m, and is invariant $\mathcal{E}[(m-n)/2] \to \mathcal{E}[(-m-n)/2]$, gives rise to a Yamabe-type prescription problem

$$\left(P_m^0 + \frac{n-m}{2}Q_m\right)u = \frac{n-m}{2}\widehat{Q}_m u^{(n+m)/(n-m)}, \qquad u := e^{(n-m)\omega/2}. \tag{16}$$

In fact, the Yamabe problem is the $m = 2$ special case of this, with $Q_2 = \mathsf{J}$. (J is a normalized scalar curvature; see (44) below for a formula.) Let

$$\mu = \inf_{u>0} \frac{\int u P_m u}{\|u\|_q^2}, \qquad q := \frac{2n}{n-m}.$$

The quotient under the infimum is the Q_m-*Yamabe quotient*, governing the borderline Sobolev imbedding $L^2_{m/2} \hookrightarrow L^q$. By Hölder's inequality,

$$\begin{aligned}\mu \le \frac{\int uP_m u}{\|u\|_q^2} &\le \frac{\int u^2 |P_m u/u|}{\|u\|_q^2} \le \frac{\|u^2\|_{n/(n-m)}\|P_m u/u\|_{n/m}}{\|u\|_q^2} \\ &= \|P_m u/u\|_{n/m} = \left\|\frac{n-m}{2}\widehat{Q}_m e^{m\omega}\right\|_{n/m} = \frac{n-m}{2}\left\{\int |\widehat{Q}_m|^{n/m} dv_{\hat{g}}\right\}^{m/n},\end{aligned}$$

using the prescription equation (16). Thus

$$\left(\frac{2\mu}{n-m}\right)^{n/m} \le \int |\widehat{Q}_m|^{n/m} dv_{\hat{g}}. \tag{17}$$

This looks particularly nice when n/m is an even integer; in this case we may remove the absolute value signs from (17).

On the other hand, Beckner's sharp form of the Sobolev imbedding inequality on S^n [4], says (when rewritten in the language above) that in normalized round measure on S^n,

$$\|f\|_q^2 \le \frac{2}{(n-m)Q_m^{\mathrm{round}}}(P_m^{\mathrm{round}} f, f)_2,$$

where $Q_m^{\mathrm{round}} = \Gamma(\frac{n+m}{2})/\Gamma(\frac{n-m+2}{2})$ is the value of Q_m and P_m^{round} is the (subcritical) order m GJMS operator on round S^n: by [7], Remark 2.23,

$$\begin{aligned}P_m^{\mathrm{round}} &= \left\{\Delta + \left(\tfrac{n+m}{2} - 1\right)\tfrac{n-m}{2}\right\}\left\{\Delta + \left(\tfrac{n+m}{2} - 2\right)\left(\tfrac{n-m}{2} + 1\right)\right\} \cdot \\ &\quad \ldots \left\{\Delta + \left(\tfrac{n}{2} + 1\right)\left(\tfrac{n}{2} - 2\right)\right\}\left\{\Delta + \tfrac{n}{2}\left(\tfrac{n}{2} - 1\right)\right\}.\end{aligned}$$

That is, we take the final $m/2$ factors in the expression (7) for the critical GJMS operators.

Furthermore, Beckner's result gives the case of equality: exactly if f is a nonzero constant multiple of $e^{(n-m)\omega/2}$, where $e^{2\omega}g_0$ is related to the round metric g_0 by a conformal diffeomorphism. This tells us that μ is attained, in particular, when f is the function 1. Because of the form (15),

$$\mu = \frac{n-m}{2} Q_m^{\text{round}} = \frac{((n+m-2)/2)!}{((n-m-2)/2)!}.$$

In particular, with $m = 2$,

$$0 \le \int_{S^n} \left\{ (|\mathsf{J}|^{n/2} dv)_{\widehat{g}} - (\mathsf{J}^{n/2} dv)_g \right\},$$

where g is the round metric, with equality iff $e^{2\omega}$ is a conformal diffeomorphism factor. Beyond the sphere, because of Schoen's solution of the Yamabe problem, we can say that if g minimizes the Yamabe functional in the conformal class $[g]$ on the compact manifold M, then J is constant, and

$$\begin{aligned} \mu &= \frac{\int_M \frac{n-2}{2} \mathsf{J}\, dv_g}{(\operatorname{vol} g)^{(n-2)/n}} = \frac{n-2}{2} \mathsf{J} (\operatorname{vol} g)^{2/n}, \\ (2\mu/(n-2)))^{n/2} &= \mathsf{J}^{n/2} \operatorname{vol} g = \textstyle\int \mathsf{J}^{n/2} dv_g, \\ 0 &\le \textstyle\int_M \left\{ (|\mathsf{J}|^{n/2} dv)_{\widehat{g}} - (\mathsf{J}^{n/2} dv)_g \right\}. \end{aligned}$$

More generally, if n/m is an even integer, we have an invariant problem about a cocycle,

$$\mathcal{H}_m(\widehat{g}, g) = \int_M \left\{ \left(Q_m^{n/m} dv\right)_{\widehat{g}} - \left(Q_m^{n/m} dv\right)_g \right\} = \int \left\{ \widehat{\mathbf{Q}}_m^{n/m} - \mathbf{Q}_m^{n/m} \right\},$$

where $\mathbf{Q}_m$ is the $\mathcal{E}[-m]$ version of Q_m. The problem is to find a metric g for which $\mathcal{H}_m(\widehat{g}, g)$ is always nonnegative. Because of Schoen's solution of the Yamabe problem, this is solved for $m = 2$; because of Beckner's inequalities, this is solved for all admissible m on the sphere S^n. This solves the higher-order Yamabe problem on the sphere, in the sense of finding the metrics that provide the infimum of the higher-order Yamabe functional. What it does not do is to rule out other metrics in the conformal class which might have a constant (but higher) Q_m.

Remarkably, for torsion and determinant quantities in dimension 4, the local term in the conformal change law for the scale-invariant functional is always described by a linear combination of the functional $\tilde{\mathcal{H}}$ of (11) above, and the functional $\mathcal{H}_2$ described just above. (The ambiguity in $\mathbf{Q}$ is just addition of a multiple of $|\mathsf{C}|^2$.) On the sphere S^4, this settles the extremal problem for this local term, provided the coefficients on $\tilde{\mathcal{H}}$ and $\mathcal{H}_2$ have the same sign (since they have the same extremals); see [12]. Moreover, many of these quantities have *only* a local term generically. A prospective global term arises when the conformal class admits a nontrivial null space for the relevant operator; for example the conformal Laplacian or Dirac operator. In particular, in the standard conformal class on S^4, the scale-invariant determinant quotients for the conformal Laplacian and the square of the Dirac operator are linear combinations of $\int \omega(\mathbf{Q} + \widehat{\mathbf{Q}})$ and $\int (\widehat{\mathsf{J}}^2 - \mathsf{J}^2)$. (Since this is a conformally flat situation, there is no ambiguity in $\mathbf{Q}$.) It is verified in [19] that the coefficients have the same sign for either the conformal Laplacian or the square of the Dirac operator. In [9, 10] it is shown that we may similarly extremize for these two operators in the standard conformal class on S^6, even though additional functionals appear. An invariant-theoretic result about general 6-dimensional manifolds, relevant to the representation of determinant and torsion quantities, appears below in Section 4.

Determinant quotients of a conformally invariant operator, or a power of such, are reasonably well covered in previous expositions. Thus we shall concentrate here on some recently discovered generalizations of Cheeger's half-torsion, which exhibit similar behavior. One interpretation of these quantities is that they are a kind of determinant for a non-elliptic operator like the one that gives Maxwell's equations. To complete the ellipticity picture, one needs other operators from an elliptic complex in which the non-elliptic operator lives; but to retain the delicate conformal change law, spectral quantities based on these operators must be added according to a very precise recipe. The global (null space) term for such quantities is important, as it encodes topological information via cohomology.

3. Detour torsion

This section describes joint work with Rod Gover.

Cheeger's half-torsion for the de Rham complex is a special case of a spectral invariant, the *detour torsion* [17], defined on the de Rham detour complexes introduced in [16]. The idea of detour complexes and detour torsions also makes sense for generalized Bernstein-Gelfand-Gelfand (BGG) diagrams. In this section, all manifolds are compact and Riemannian. They are also of even dimension n unless otherwise stated.

Though nonlocal, detour torsions have infinitesimal conformal variations in which the main term is local. These variational formulas may be "integrated up" to provide formulas for finite variations, in much the same way as one treats the functional determinants of conformally invariant operators. Both the infinitesimal and finite conformal variational formulas are sometimes called *Polyakov formulas.* In fact, these detour torsions are well-chosen products and quotients of functional determinants which individually behave badly under conformal change, but which behave well in the well-chosen aggregate. In a sense that will become apparent, the Cheeger half-torsion is a kind of determinant for the (non-elliptic) Maxwell operator, in which terms from earlier in the de Rham complex supply the needed ellipticity, but must be chosen with care to preserve as much good conformal behavior as possible. When we generalize to detour complexes, we do the same sort of thing with generalizations of the Maxwell operator; for example, the operators on differential forms introduced in [16] and described in [36]. The determinant of the critical GJMS operator is a half-torsion on its own – the half-torsion of a detour complex that "detours very early".

We first describe the Cheeger half-torsion. Working over a compact, Riemannian manifold, let d_k, δ_k, and

$$\Delta_k = \delta_{k+1} d_k + d_{k-1} \delta_k$$

be the usual Hodge-de Rham operators. The Hodge decomposition is

$$\mathcal{E}^k = \underbrace{\mathcal{R}(\delta) \oplus \mathcal{R}(d)}_{\mathcal{R}(\Delta)} \oplus \underbrace{(\mathcal{N}(d) \cap \mathcal{N}(\delta))}_{=:\mathcal{H}^k},$$

where $\mathcal{N}$ denotes the null space and $\mathcal{R}$ the range. The *zeta functions* of the complex are

$$\zeta(s, \Delta_k) := \mathrm{Tr}_{L^2}(\Delta_k|_{\mathcal{R}(\Delta_k)})^{-s}.$$

By standard theory (see [49] and references therein), the series implicit in the trace converges uniformly and absolutely on $\mathrm{Re}(s) > n/2$, so each zeta function is a holomorphic function of s in this half-plane. Each zeta function may then be continued analytically to a meromorphic function on $\mathbb{C}$. Any poles of such a zeta function are simple, and can occur only for $s \in \{1, 2, \ldots, n/2\}$. (See (29) below for a discussion of the nature of these poles.) In particular, $s = 0$ is a regular point, and we define the *functional determinant* of Δ_k to be

$$\det \Delta_k = \exp\left(-\zeta'(0, \Delta_k)\right).$$

Another view of the zeta functions is as follows. Let λ_j be the eigenvalues of Δ_k, listed as usual in increasing order. Then

$$\zeta(s, \Delta_k) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}$$

for sufficiently large $\mathrm{Re}(s)$. The nonzero form eigenvalues split into a list of δd eigenvalues, say μ_a, and a list of $d\delta$ eigenvalues, say ν_b. A key point in all discussions of index and torsion quantities is that much information is repeated in considering these lists for various k. Specifically, the nonzero δd eigenvalue list for k-forms is repeated as the nonzero $d\delta$ eigenvalue list for $(k+1)$-forms, since d and δ commute with Δ. This offers some scope for achieving interaction among the spectral invariants of the various Δ_k.

To set the stage for this, let us enrich our supply of zeta functions by defining such functions for the non-elliptic operators δd and $d\delta$. First, since d_0 and δ_1 are formal adjoints, the Hodge decomposition shows that

$$d_0 : \mathcal{R}(\delta_1) \leftrightarrow \mathcal{R}(d_0) : \delta_1 \quad \text{bijectively}.$$

Thus we have

$$\zeta(s, d_0\delta_1) := \mathrm{Tr}_{L^2}(d_0\delta_1|_{\mathcal{R}(d_0)})^{-s} = \mathrm{Tr}_{L^2}(\delta_1 d_0|_{\mathcal{R}(\delta_1)})^{-s} = \zeta(s, \Delta_0).$$

With this in place, we may take

$$\begin{aligned} \zeta(s, \delta_2 d_1) &= \mathrm{Tr}_{L^2}\left(\delta_2 d_1|_{\mathcal{R}(\delta_2)}\right)^{-s} \\ &= \zeta(s, \Delta_1) - \zeta(s, \Delta_0). \end{aligned}$$

Continuing in this way, we may define

$$\zeta(s, \delta_{k+1} d_k) \quad \text{and} \quad \zeta(s, d_{k-1}\delta_k),$$

regular at $s = 0$, with

$$\zeta(s, \delta_{k+1} d_k) = \zeta(s, d_k \delta_{k+1}).$$

A word of caution: differential operators without appropriate ellipticity or subellipticity properties will generally not have sensible zeta functions. In the case under consideration here, it is only the status of δd and $d\delta$ as partial Laplacians of an elliptic complex that allows us to define zeta functions for them.

A useful extension of the zeta function concept is obtained when we insert a multiplication operator just before tracing. If ω is a smooth function, let

$$\zeta(s, \Delta_k, \omega) := \mathrm{Tr}_{L^2}\left(\omega(\Delta_k|_{\mathcal{R}(\Delta_k)})^{-s}\right).$$

Like their $\omega = 1$ special cases above, these are meromorphic in s and regular at $s = 0$. In terms of kernel functions, these objects are related to their $\omega = 1$ special cases as follows:

operator	Δ^{-s}	$\omega\Delta^{-s}$
kernel function	$K(s,x,y)$	$\omega(x)K(s,x,y)$
Tr_{L^2}	$\int \mathrm{tr}_x K(s,x,x)$	$\int \omega(x)\mathrm{tr}_x K(s,x,x)$

Another word of caution: One needs to be careful in trying to use local partial zeta functions like $\zeta(s, \delta d, \omega)$. Let us introduce an abbrevation in which an underline stands for restriction to the correct range, as in

$$(\underline{\delta d})^{-s} = (\delta d|_{\mathcal{R}(\delta)})^{-s}.$$

Because this operator is of trace class for large $\mathrm{Re}(s)$, the operator $\omega(\underline{\delta d})^{-s}$ will be too. But there is no reason to expect regularity of this function at $s = 0$. In fact, this caution is closely related to the rarity of good torsion quantities. Such a torsion needs to be put together so that its conformal variation, which comes from some special combination of local partial zeta functions, is somehow guaranteed to be regular at $s = 0$.

As an operator from $\mathcal{E}^k$ to $\mathcal{E}^{k+1}$, the exterior derivative d_k is of course independent of the metric. The coderivative

$$\delta_k : \mathcal{E}^k[2k-n] \to \mathcal{E}^{k-1}[2k-2-n]$$

is conformally invariant. (Again, see [36] for notational conventions.) Thus when viewed as an operator from $\mathcal{E}^k$ to $\mathcal{E}^{k-1}$, the coderivative has the conformal deformation property

$$\widehat{g} = e^{2\omega} g \;\Rightarrow\; \widehat{\delta}_k\varphi = e^{(2k-2-n)\omega}\delta_k(e^{-(2k-n)\omega}\varphi)$$

for any $\varphi \in \mathcal{E}^k$. If we choose a scale g_0 within our conformal class and consider the conformal curve of metrics

$$g_\varepsilon := e^{2\varepsilon\omega} g_0,$$

then

$$\delta_k^\bullet\varphi = -(n-2k+2)\omega\delta_k\varphi + (n-2k)\delta_k(\omega\varphi),$$

where the $\bullet$ now denotes conformal variation.

Recall our underline notation from above; in particular

$$\begin{aligned} \underline{\Delta}_k &= \Delta_k|_{\mathcal{R}(\Delta_k)}, \\ \underline{\delta d}_k &= \delta_{k+1}d_k|_{\mathcal{R}(\delta_{k+1})}, \\ \underline{d\delta}_k &= d_{k-1}\delta_k|_{\mathcal{R}(d_{k-1})}. \end{aligned}$$

Simplifying the notation further by letting $\mathrm{Tr} = \mathrm{Tr}_{L^2}$, we have

$$\mathrm{Tr}\underline{\Delta}_k^{-s} = \mathrm{Tr}(\underline{\delta d}_k^{-s}) + \mathrm{Tr}(\underline{d\delta}_k^{-s}).$$

But, at least formally,

$$\mathrm{Tr}((\underline{\delta d})_k^{-s})^\bullet = -s\mathrm{Tr}\,\underbrace{(\delta_{k+1}d_k)^\bullet}_{\delta_{k+1}^\bullet d_k}(\underline{\delta d})_k^{-s-1}.$$

(The transition from formal calculations to rigorous ones involves the interchange of limiting operations, and thus hard-analytic estimates. We shall suppress such considerations here, but note that they are confronted in, for example, [18].)

Note that since there are now noncommuting operators involved (the variation of an operator like δd need not commute with the operator itself), this manipulation only works, even formally, with the L^2 trace out front. Continuing with the calculation, and using an informal notation in which multiplication by ω is denoted simply by ω, we have

$$\begin{aligned}\mathrm{Tr}((\underline{\delta d})_k^{-s})^\bullet &= -s\mathrm{Tr}\left(\{-(n-2k)\omega\delta_{k+1}+(n-2k-2)\delta_{k+1}\omega\}d_k(\underline{\delta d})_k^{-s-1}\right)\\ &= (n-2k)s\mathrm{Tr}\left(\omega(\underline{\delta d})_k^{-s}\right)-(n-2k-2)s\mathrm{Tr}\left(\omega(\underline{d\delta})_{k+1}^{-s}\right).\end{aligned}$$

Here, in rewriting the last term, we took advantage of the fact that $\delta_{k+1}:\mathcal{R}(d_k)\to\mathcal{R}(\delta_{k+1})$ is bijective.

This last step is a key point: the variation of δd on k-forms leads to terms in δd on k-forms, *and* in $d\delta$ on $(k+1)$-forms. It is from this that the interaction of different form orders will arise. Restating in terms of zetas and local zetas,

$$\zeta(s,(\delta d)_k)^\bullet = (n-2k)s\zeta(s,(\delta d)_k,\omega)-(n-2k-2)s\zeta(s,(d\delta)_{k+1},\omega)\,. \tag{18}$$

Similarly,

$$\zeta(s,(d\delta)_k)^\bullet = (n-2k+2)s\zeta(s,(\delta d)_{k-1},\omega)-(n-2k)s\zeta(s,(d\delta)_k,\omega)\,. \tag{19}$$

In each formula, local partial zeta functions at adjacent orders interact.

At first glance, it might seems as though the right sides of equations (18,19) vanish at $s=0$. In fact, these expressions make elementary sense only for large $\mathrm{Re}(s)$, but the s factors certainly influence how things look after analytic continuation. Recall the perils of local partial zetas: the individual terms in (18,19), without their s factors, generally have poles at $s=0$. Certain linear combinations

$$\kappa(s):=c_0\zeta(s,\Delta_0)+c_1\zeta(s,\Delta_1)+\cdots+c_n\zeta(s,\Delta_n)\,, \tag{20}$$

however, may be regular at $s=0$, and furthermore have conformally invariant $\kappa(0)$; that is, there may be *conformal indices* in the sense of [18]. The existence of such a conformal index is closely related to the presence of a Polyakov formula; that is, a determinant or torsion quantity having a variation whose main term is local.

Given $\kappa(s)$ defined by (20) and given k, the coefficient in $\kappa(s)^\bullet$ of $s\zeta(s,(\delta d)_k,\omega)$ is

$$(n-2k)(c_k+c_{k+1})\,,$$

while the coefficient of $s\zeta(s,(d\delta)_k,\omega)$ is

$$-(n-2k)(c_k+c_{k-1})\,.$$

One distinguished choice for the coefficient list will thus be $1,-1,1,-1,\ldots$. This is no surprise, as with this choice we are detecting the conformal invariance of the index of the de Rham complex (which has much more than just conformal invariance of course).

A slightly more subtle desideratum for a coefficient list is that it produce only full Laplacians in the variation; that is, that the coefficients of $s\zeta(s,(\delta d)_k,\omega)$ and $s\zeta(s,(d\delta)_k,\omega)$ agree. Via the above, this leads to

$$c_{k+1}=-c_{k-1}-2c_k\,,\qquad k\geq 1\,. \tag{21}$$

This doesn't produce a unique coupling, but in fact we need to demand more: that the $(s\zeta(s,(\delta d)_k,\omega), s\zeta(s,(d\delta)_k,\omega))$ coefficient pair in the variation be proportional to the $(\zeta(s,(\delta d)_k), \zeta(s,(d\delta)_k))$ coefficient pair in the original quantity,

$$(n-2k)(c_k+c_{k+1}, -c_k-c_{k-1}) = \lambda(c_k, c_k)$$

for some λ. Shifting k in the equality of second components, we get the system

$$\begin{aligned} (n-2k-\lambda)c_k + (n-2k)c_{k+1} &= 0\,, \\ (n-2k-2)c_k + (n-2k-2+\lambda)c_{k+1} &= 0\,, \end{aligned}$$

the determinant of which is $\lambda(2-\lambda)$. The choice $\lambda = 0$ gives us the coefficient list $1,-1,1,-1,\ldots$ associated with the index calculation. The choice $\lambda = 2$ gives the recursion

$$(n-2(k-1))c_k = -(n-2k)c_{k-1}\,. \tag{22}$$

The key point is that

for this choice, c_k may be taken to vanish for $k \geq n/2$.

If we set $c_{n/2} = \cdots = c_n = 0$, only the first half of the complex will be noticed by the calculation; this is the origin of the term *half-torsion.* A convenient normalization of the coefficient list is then $n, -(n-2), n-4, \ldots, \mp 4, \pm 2, 0$; that is,

$$c_k = \begin{cases} (-1)^k(n-2k)\,, & k < n/2\,, \\ 0\,, & k \geq n/2\,. \end{cases} \tag{23}$$

For this choice, if we define the local kappa function by

$$\kappa(s,\omega) := c_0\zeta(s,\Delta_0,\omega) + c_1\zeta(s,\Delta_1,\omega) + \cdots + c_n\zeta(s,\Delta_n,\omega)\,,$$

we get

$$\kappa(s)^\bullet = 2s\kappa(s,\omega)\,. \tag{24}$$

In hindsight, the ambiguity in the coupling that remained after (21) was the degree of freedom allowed in making linear combinations of the coefficient lists $1,-1,1,-1,\ldots$ and $(n, -(n-2), \cdots \pm 2, 0, \ldots, 0)$.

Let us fix the choice of coupling described above; that is,

$$\begin{aligned} \kappa(s,\omega) = n\zeta(s,\Delta_0,\omega) - (n-2)\zeta(s,\Delta_1,\omega) + (n-4)\zeta(s,\Delta_2,\omega) - \cdots \\ + (-1)^{n/2-1}\cdot 2\zeta(s,\Delta_{n/2-1},\omega)\,, \end{aligned} \tag{25}$$

with

$$\kappa(s) := \kappa(s,1)\,.$$

Since only local zeta functions of full Laplacians appear in the variation, $\kappa(s,\omega)$ is regular at $s=0$, so that by (24),

$\kappa(0)$ is a conformal invariant.

The *half-torsion* is $\kappa'(0)$. The Polyakov-type formula will now arise upon computation of the conformal variation of this.

Note that since the functional determinants of the Laplacians are defined as their $e^{-\zeta'(0)}$ quantities, the half-torsion is

$$\kappa'(0) = -\log\frac{(\det\Delta_0)^n(\det\Delta_2)^{n-4}\ldots}{(\det\Delta_1)^{n-2}(\det\Delta_3)^{n-6}\ldots}\,,$$

where the terms abbeviated by ... involve only the Δ_k for $k < n/2$.

The interesting task is now to find $\kappa'(0)^\bullet$, the conformal variation of the half torsion. By the foregoing, this will involve the $\zeta(s, \Delta_k, \omega) = \operatorname{Tr}(\omega \underline{\Delta}_k^{-s})$. The analytic continuation of zeta functions, as well as more involved procedures of commuting limit operations like our d/ds and $\bullet$, are accomplished by looking at things on the other side of the *Mellin transform*

$$(\mathcal{M}f)(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t)\, dt,$$

where $f(t)$ is a function on $[0, \infty)$. The Mellin transform performs the convenient trick of carrying $\exp(-t\lambda)$ to λ^{-s} for positive real λ. Thus it carries

$$\operatorname{Tr}(\omega \exp(-t\underline{\Delta}_k)) \mapsto \zeta(s, \Delta_k, \omega). \tag{26}$$

In more detail, the kernel functions for the operators being traced are

$$\omega(x) \sum_{\lambda_j \neq 0} e^{-\lambda_j t} \varphi_j(x) \otimes \varphi_j^*(y)$$

and

$$\omega(x) \sum_{\lambda_j \neq 0} \lambda_j^{-s} \varphi_j(x) \otimes \varphi_j^*(y)$$

respectively, where $\{\lambda_j\}$ is the eigenvalue list, the φ_j are the corresponding orthonormal basis of eigenforms, and the φ_j^* are the corresponding sections of the dual bundle. The Mellin transform acts only on the factors $e^{-\lambda_j t}$ to produce the factors λ_j^{-s}. To take the L^2 trace, we integrate over the diagonal $\{x = y\}$.

The L^2 trace on the left of (26) is closely related to the localized heat operator trace

$$Z(t, \Delta_k, \omega) := \operatorname{Tr}(\omega \exp(-t\Delta_k)),$$

in which the Δ is not underlined. This latter trace has the small-t asymptotic expansion

$$Z(t, \Delta_k, \omega) \sim \sum_{\text{even } i \geq 0} t^{(i-n)/2} \int \omega U_i \qquad \text{as} \quad t \downarrow 0. \tag{27}$$

Here the U_i are local scalar invariants of the metric; for forms, some of these for small i are computed in [46, 35]. The *leading terms* of all of the U_i have been computed; see [14]. The difference between the left sides of (27) and (26) is

$$\sum_{\lambda_j = 0} \int \omega |\varphi_j|^2. \tag{28}$$

Note that since the Hodge projection $\mathcal{P}_k$ onto the null space of Δ_k has kernel function

$$\sum_{\lambda_j = 0} \varphi_j(x) \otimes \varphi_j^*(y),$$

the quantity in (28) is actually

$$\operatorname{Tr} \omega \mathcal{P}_k,$$

so that

$$\operatorname{Tr} \omega \exp(-t\underline{\Delta}_k) \sim \sum_{\text{even } i \geq 0} t^{(i-n)/2} A_i(\Delta_k, \omega) \qquad \text{as} \quad t \downarrow 0,$$

where

$$A_i(\Delta_k,\omega) := \begin{cases} \int \omega U_n - \operatorname{Tr}\omega\mathcal{P}_k & \text{if } i=n\,, \\ \int \omega U_i & \text{otherwise}\,. \end{cases}$$

Now

$$\begin{aligned}\Gamma(s)\zeta(s,\Delta_k,\omega) &= \sum_{0\le \text{even } i\le m}\left(s-\frac{n-i}{2}\right)^{-1} A_i(\Delta_k,\omega) \\ &\quad + \int_0^1 t^{s-1}O(t^{(m-n+1)/2})\,dt + \int_1^\infty t^{s-1}(\operatorname{Tr}\omega\exp(-t\underline{\Delta}_k))\,dt\,,\end{aligned} \tag{29}$$

so at $s=0$,

$$\zeta(0,\Delta_k,\omega) = A_n(\Delta_k,\omega)\,,$$

since $\Gamma(s)$ has a simple pole at $s=0$.

By (24), we now have

$$\begin{aligned}\kappa'(0)^\bullet &= 2\kappa(0,\omega) \\ &= 2\sum_k c_k A_n(\Delta_k,\omega) \\ &= \underbrace{-2\sum_k c_k \operatorname{Tr}\omega\mathcal{P}_k}_{=:2\tau_{\text{glob}}(g,\omega)} + \underbrace{2\int\omega\sum_k c_k U_n[\Delta_k]}_{=:2\tau_{\text{loc}}(g,\omega)} \\ &=: 2\tau(g,\omega)\,.\end{aligned} \tag{30}$$

In naming the τ quantities, we make explicit the dependence on the metric g which was suppressed in earlier manipulations; this will be useful just below when we think in terms of functionals on the conformal class. If we put

$$\mathcal{U}_n := \sum_k c_k U_n[\Delta_k]\,, \tag{31}$$

we may re-express $\tau_{\text{loc}}(g,\omega)$ as $\int\omega\mathcal{U}_n$. Recall also that $\operatorname{Tr}\omega\mathcal{P}_k$ may be expressed in terms of any choice of L^2-orthonormal bases $\{\psi^k_m\}_{m=1}^{b_k}$ of the harmonic space $\mathcal{H}^k$ (denoting the k^{th} Betti number by b_k), as

$$\tau_{\text{glob}}(\omega) = -\sum_k c_k \sum_{m=1}^{b_k}\int\omega|\psi^k_m|^2\,.$$

The goal is now to "integrate up" the variation to find an expression for $\kappa'(0)^\wedge - \kappa'(0)$; that is, the difference between half-torsions at conformally related metrics $\widehat{g}=e^{2\omega}g$ and g. That is, we want a *conformal primitive* for the variation $\kappa'(0)^\bullet$. The meaning of this concept is as follows. Suppose we have a suitably smooth functional $\mathcal{V}(g,\omega)$, where g runs over a conformal class of Riemannian metrics on a manifold M, and ω runs over $C^\infty(M)$. A conformal primitive for $\mathcal{V}(g,\omega)$ (if such exists) is an alternating 2-metric functional $\mathcal{H}(\widehat{g},g)$ on the conformal class with the property that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathcal{H}(e^{2\varepsilon\omega}g,g) = \mathcal{V}(g,\omega)$$

for all (g,ω). Given a conformal primitive for $\mathcal{V}(g,\omega)$, we may artificially choose a background metric g_0, and get a one-metric functional $\mathcal{G}_{g_0}(g) := \mathcal{H}(g,g_0)$ with conformal variation $\mathcal{V}(g,\omega)$:

$$\mathcal{G}_{g_0}(g)^\bullet = \mathcal{V}(g,\omega)\,.$$

Having the same variation, these one-metric functionals for different g_0 must differ by a constant. That is,

$$\mathcal{H}(g,g_1) - \mathcal{H}(g,g_0) = c\,.$$

Since $\mathcal{H}$ is alternating, substitution of g_0 for g yields $c = \mathcal{H}(g_1,g_0)$, so we have the *cocycle condition*

$$\mathcal{H}(g,g_0) = \mathcal{H}(g,g_1) + \mathcal{H}(g_1,g_0)\,.$$

Conversely, given a one-metric functional $\mathcal{G}(g)$ on the conformal class with variation $\mathcal{V}(g,\omega)$, we may form a conformal primitive by taking $\mathcal{H}(\widehat{g},g) = \mathcal{G}(\widehat{g}) - \mathcal{G}(g)$. An advantage of the two-metric functional is that it is unique.

The claim is that $\tau_{\rm loc}(g,\omega)$ (which implicitly depends on g) has a conformal primitive; this statement is sensitive to the precise coefficient list c_k that defines the half-torsion. In addition, we claim that the contribution of the k-form harmonics to $\tau_{\rm glob}(g,\omega)$ for each k has a conformal primitive. The result of combining these ingredients will be a conformal primitive for $\tau(g,\omega)$.

To handle the global claim first, we adapt an argument of Ray and Singer [47]. Fix k, and fix an arbitrary basis $\mathbf{h} = \mathbf{h}^k$ of the $k^{\underline{\rm th}}$ *real* cohomology H^k. (Note that the $0,\dots,n/2-1$ form bundles admit distinguished real forms.) The *de Rham map* at a metric g is a natural isomorphism taking the g-harmonics $\mathcal{H}^k_g$ to H^k:

$$\mathcal{D}_g : \mathcal{H}^k_g \mapsto H^k.$$

If $\Psi = \{\psi_m\}$ is an orthonormal basis of $\mathcal{H}^k_g$, let $[\Psi/\mathbf{h}]$ be the determinant of the basis change from $\mathcal{D}\Psi$ to $\mathbf{h}$; that is, det $\mathcal{B}$, where $\mathbf{h} = \{h_m\}$ and

$$\mathcal{D}\psi_m = \sum_p \mathcal{B}_{mp} h_p\,.$$

Since the basis change in $\mathcal{H}^k_g$ between any two orthonormal bases is an orthogonal transformation, the quantity $[\Psi/\mathbf{h}]$ does not depend on the choice of the particular orthonormal basis, so we may also give it the name $[g:\mathbf{h}]$.

We would like to compute the conformal variation of $[g:\mathbf{h}]$. To this end, let $f_m(g) = \mathcal{D}_g^{-1}(h_m)$: Let $\omega \in C^\infty(M)$, and let

$$g_\varepsilon = e^{2\varepsilon\omega} g_0\,, \qquad \varepsilon \in \mathbb{R}$$

be a one-parameter conformal family of metrics. Consider the change of basis matrix $B_{mp}(\varepsilon)$ from $\{f_m(g_\varepsilon)\}$ to a g_ε-orthonormal basis $\psi_m(g_\varepsilon)$:

$$\psi_m = \sum_p B_{mp} f_p\,;$$

we have det $B = [g : \mathbf{h}]$. Since $\{\psi_m\}$ is orthonormal,

$$\delta_{uv} = (\psi_u, \psi_v) = \left(\sum_p B_{up} f_p, \sum_q B_{vq} f_q\right)$$
$$= \sum_{p,q} B_{up} B_{vq} \underbrace{(f_p, f_q)}_{=:C_{pq}} = (BCB^{\mathrm{T}})_{uv}\,,$$

so that

$$I = BCB^{\mathrm{T}}\,, \qquad (\det B)^2 = (\det C)^{-1}\,. \tag{32}$$

Clearly C varies smoothly with g, so that det B will also. So it will be enough to compute $(\det C)^{\bullet}$.

Since $f_m(\varepsilon)$ and $f_m(0)$ are cohomologous, we have forms $\varphi_m(\varepsilon)$ with $f_m(\varepsilon) = f_m(0) + d\varphi_m(\varepsilon)$. Thus

$$\mathrm{g}(\varepsilon)(f_m(\varepsilon), f_p(\varepsilon)) = \mathrm{g}(\varepsilon)(f_m(\varepsilon), f_p(0) + d\varphi_p(\varepsilon)) = \mathrm{g}(\varepsilon)(f_m(\varepsilon), f_p(0))\,,$$

where g is the form metric. In the last step, we have integrated by parts, using the fact that $f_m(\varepsilon)$ is harmonic (and thus annihilated by the coderivative δ) in the ε-metric. Similarly

$$\mathrm{g}(0)(f_m(0), f_p(0)) = \mathrm{g}(0)(f_m(\varepsilon) - d\varphi_m(\varepsilon), f_p(0)) = \mathrm{g}(0)(f_m(\varepsilon), f_p(0))\,.$$

Subtracting these equations, we get

$$\mathrm{g}(\varepsilon)(f_m(\varepsilon), f_p(\varepsilon)) - \mathrm{g}(0)(f_m(0), f_p(0)) = (\mathrm{g}(\varepsilon) - \mathrm{g}(0))(f_m(\varepsilon), f_p(0))\,.$$

Differentiating with respect to ε and then setting $\varepsilon = 0$, we have

$$C^{\bullet}_{mp} = \mathrm{g}^{\bullet}(f_m, f_p)\,.$$

Since $(g^k)^{\bullet} = -2k\omega g^k$ and $(dv_g)^{\bullet} = n\omega dv_g$, we have

$$C^{\bullet}_{mp} = (n - 2k)\mathrm{g}(f_m, \omega f_p)\,. \tag{33}$$

Left multiplying the first equation in (32) by B^{T} and then right multiplying by B, we get $C^{-1} = B^{\mathrm{T}}B$. Using this and the formula just obtained for the variation of C, we get

$$\begin{aligned}
-(\log\,[g : \mathrm{h}]^2)^{\bullet} &= -(\log(\det B)^2)^{\bullet} \\
&= (\log \det C)^{\bullet} \\
&= \mathrm{tr}(C^{-1}C^{\bullet}) = \mathrm{tr}(B^{\mathrm{T}}BC^{\bullet}) \\
&= \mathrm{tr}(BC^{\bullet}B^{\mathrm{T}}) \\
&= \sum_{q,m,p} B_{qm}\{(n-2k)\mathrm{g}(f_m, \omega f_p)\}B_{qp} \\
&= (n-2k)\sum_q \mathrm{g}(\psi_q, \omega\psi_q) \\
&= (n-2k)\mathrm{Tr}\,\omega\mathcal{P}_k\,.
\end{aligned}$$

But recall from (30) that

$$\tau_{\rm glob}(\omega) = -\sum_{k=0}^{n/2-1}(-1)^k(n-2k)\operatorname{Tr}\omega\mathcal{P}_k\,.$$

Combined with the formula just above, this gives

$$\left(\sum_{k=0}^{n/2-1}(-1)^k\log[g:\mathbf{h}^k]^2\right)^{\bullet} = \tau_{\rm glob}(\omega)\,.$$

This gives us a conformal primitive in the one-metric functional sense for $\tau_{\rm glob}(\omega)$, depending on our choices $\mathbf{h}^k$ of cohomology bases:

$$\mathcal{G}(g,\{\mathbf{h}^k\}) := \sum_{k=0}^{n/2-1}(-1)^k\log[g:\mathbf{h}^k]^2\,.$$

In the corresponding two-metric conformal primitive, the dependence on the cohomology bases washes out:

$$\mathcal{H}(\widehat{g},g) = \mathcal{G}(\widehat{g},\{\mathbf{h}^k\}) - \mathcal{G}(g,\{\mathbf{h}^k\}) = \sum_{k=0}^{n/2-1}(-1)^k\log\frac{[\widehat{g}:\mathbf{h}^k]^2}{[g:\mathbf{h}^k]^2}\,.$$

The quotient here is the square of the determinant of the basis change from $\mathcal{D}\Psi(g)$ to $\mathcal{D}\Psi(\widehat{g})$, so we are entitled to denote it $[\widehat{g}:g]_k^2$. Summarizing,

$$\mathcal{H}(\widehat{g},g) = \sum_{k=1}^{n/2-1}(-1)^k\log[\widehat{g}:g]_k^2 \tag{34}$$

is the (unique two-metric) conformal primitive of $\tau_{\rm glob}(\omega)$.

We can be more specific about what is happening with the $k=0$ term, where the lone harmonic is (up to a nonzero constant factor) the function 1. An L^2-orthonormal basis of the harmonic space at the metric g is given by the constant $\operatorname{vol}(g)^{-1/2}$. Thus in any eventual treatment of the extremal problem for the half-torsion, the global term contributed at $k=0$ combines with the volume penalty term in the functional (11). In fact, this is exactly what happens for the functional determinant of the Laplacian in dimension 2 [44, 45], which is of course the same as the half-torsion in dimension 2.

To work toward a conformal primitive of $\tau_{\rm loc}(\omega)$, first note that when we scale the metric uniformly,

$$\bar{g} = \alpha^2 g, \qquad 0<\alpha\in\mathbb{R},$$

the Laplacians scale by $\bar{\Delta}_k = \alpha^{-2}\Delta_k$. As a result, $\exp(-(\alpha^2 t)\bar{\Delta}_k) = \exp(-t\Delta_k)$, so the heat expansion (27) gives $U_n[\bar{\Delta}_k]dv_{\bar{g}} = U_n[\Delta_k]dv_g$. Taking the linear combination of U_n quantities under consideration here,

$$\bar{\mathcal{U}}_n dv_{\bar{g}} = \mathcal{U}_n dv_g\,.$$

Taking the $(-n)$-density version of $\mathcal{U}_n$ by using the *conformal metric* $\boldsymbol{g}$, a section of $\mathcal{E}_{(ab)}[2]$, and its inverse $\boldsymbol{g}^{-1}$, a section of $\mathcal{E}^{(ab)}[-2]$, to make metric contractions, we get a quantity $\mathbf{U}_n$ which is insensitive to uniform scaling:

$$\bar{\mathbf{U}}_n = \mathbf{U}_n\,.$$

4. Some invariant theory

Some of the invariant theory described in this section is joint work in progress with Mike Eastwood.

Counting the number of g and g^{-1} that must be used to contract to a scalar density, each monomial term in any expression for a natural Riemannian $(-n)$-density F as a universal polynomial in ∇ and the Riemann tensor R satisfies

$$N_\nabla + 2N_R = n\,, \tag{35}$$

where N_∇ (resp. N_R) is the number of occurrences of ∇ (resp. R) in the monomial. Looking at the conformal variations of ∇ and of R and arguing inductively (see, e.g., [6]), the conformal deformation law for such an F must take the form

$$\widehat{g} = e^{2\omega}g \;\Rightarrow\; \widehat{F} = F + \mathbf{X}_1[F](\Upsilon, g, g^{-1}, \nabla, R) + \cdots + \mathbf{X}_n[F](\Upsilon, g, g^{-1}, \nabla, R)\,, \tag{36}$$

where $\Upsilon := d\omega$, the connection ∇ and curvature R are computed in the metric g, and $\mathbf{X}_s$ is $(-n)$-density valued and universal, with homogeneity s in ω (or Υ):

$$\mathbf{X}_s[F](\varepsilon\Upsilon, g, g^{-1}, \nabla, R) = \varepsilon^s \mathbf{X}_s[F](\Upsilon, g, g^{-1}, \nabla, R)$$

for $\varepsilon \in \mathbb{R}$.

For ease of notation, we suppress the dependence of the $\mathbf{X}$ quantities on g and g^{-1}. As a consequence of (35) and the conformal deformations of ∇ and R,

$$n = N_\Upsilon + N_\nabla + 2N_R = s + N_\nabla + 2N_R \qquad \text{in} \quad \mathbf{X}_s(\Upsilon, \nabla, R)\,, \tag{37}$$

using the obvious extension of the notation of (35). As a result, the highest homogeneity term $\mathbf{X}_n(\Upsilon, \nabla, R)$ must take the form $c[F] \cdot g^{-1}(\Upsilon, \Upsilon)^{n/2}$ for some constant $c[F]$, as these are the only $(-n)$-density invariants satisfying (37).

All of the considerations of the last paragraph are valid when we set F equal to, for example, any linear combination of the $U_n[\Delta_k]$; we have not yet used the conformal index property, i.e. the fact that $\int \mathbf{U}_n$ is a conformal invariant. If $\int F$ is a conformal invariant, the top homogeneity term $c \cdot g^{-1}(\Upsilon, \Upsilon)^{n/2}$ must vanish, since for any real number ε,

$$\begin{aligned} 0 &= \int \{\mathbf{X}_1[F](\varepsilon\Upsilon, \nabla, R) + \cdots + \mathbf{X}_{n-1}[F](\varepsilon\Upsilon, \nabla, R) + c \cdot g^{-1}(\varepsilon\Upsilon, \varepsilon\Upsilon)^{n/2}\} \\ &= c[F]\varepsilon^n \int g^{-1}(\Upsilon, \Upsilon)^{n/2} + O(\varepsilon^{n-1}) \end{aligned}$$

as $\varepsilon \to \infty$. Taking the resulting slightly simplified version of (36) for $F = \mathbf{U}_n$ and with $\varepsilon\omega$ in place of ω, we have

$$(\mathbf{U}_n)(e^{2\varepsilon\omega}g) = \mathbf{U}_n(g) + \sum_{s=1}^{n-1} \varepsilon^s \mathbf{X}_s(\Upsilon, \nabla, R)\,.$$

Since $\tau(\omega)$ and $\tau_{\mathrm{glob}}(\omega)$ have conformal primitives, so does τ_{loc}; that is, there is a two-metric functional $\mathcal{H}_{\mathrm{loc}}(\widehat{g}, g)$ with

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{H}_{\mathrm{loc}}(e^{2\varepsilon\omega}g, g) = \tau_{\mathrm{loc}}(\omega) = \int \omega \mathbf{U}_n(g)\,.$$

Since we could replace g in this equation by $e^{2\varepsilon_0\omega}g$ for some ε_0, we have

$$\frac{d}{d\varepsilon}\mathcal{H}_{\mathrm{loc}}(e^{2\varepsilon\omega}g,g)=\int\omega\mathbf{U}_n(e^{2\varepsilon\omega}g)\,.$$

Integrating this in ε from 0 to 1, we get

$$\mathcal{H}_{\mathrm{loc}}(\widehat{g},g)=\int\omega\sum_{s=1}^{n-1}\frac{\mathbf{X}_s(\Upsilon,\nabla,R)}{s+1}\,, \tag{38}$$

where $\omega=\frac{1}{2}\log(\widehat{g}/g)$ and $\Upsilon=d\omega$. This somewhat brutal expression at least establishes that $\tau_{\mathrm{loc}}(g,\omega)$ has a locally computable conformal primitive. Note also that because of the universal nature of the calculation,

$$\mathcal{H}(\widehat{g},g)=-\mathcal{H}(g,\widehat{g})=-\int(-\omega)\sum_{s=1}^{n-1}\frac{\mathbf{X}_s(-\Upsilon,\widehat{\nabla},\widehat{R})}{s+1}=\int\omega\sum_{s=1}^{n-1}(-1)^s\frac{\mathbf{X}_s(\Upsilon,\widehat{\nabla},\widehat{R})}{s+1}\,,$$

so

$$\int_\omega\mathbf{X}_s(\Upsilon,\nabla,R)=(-1)^s\int_\omega\mathbf{X}_s(\Upsilon,\widehat{\nabla},\widehat{R})\,.$$

However, expressions like (38) as differential polynomials in the conformal factor separating two conformal scales are somewhat unsatisfying. For example, if we compute several such quantities (say, one from the half-torsion together with the determinants of the GJMS operators), it becomes clear that there are many constraints on the family $\{\mathbf{X}_s\}$; all the apparent moving parts are in reality not free to move independently, though the number of degrees of freedom does go up with the dimension. One would like to regroup all the terms to form invariants of the metrics g and $\widehat{g}$, with ω appearing explicitly only without derivatives attached. If some invariant theoretic conjectures that are currently being studied turn out to be correct, this can indeed be done. In low dimensions, where the invariants can be listed easily, there is no problem.

What we would like to assert is that

$$\mathcal{H}_{\mathrm{loc}}(\widehat{g},g)=\int\omega(\mathbf{Q}+\widehat{\mathbf{Q}})+\int(\widehat{\mathbf{F}}-\mathbf{F})\,, \tag{39}$$

where $\mathbf{Q}$ is some version of the Q-curvature, $\mathbf{F}$ is a local $(-n)$-density valued invariant of the metric, and as usual, ω is an abbreviation for the cocycle $\frac{1}{2}\log(\widehat{g}/g)$. Though the plus sign in the first term on the right in (39) seems odd at first glance, note that ω is alternating in g and $\widehat{g}$, so the integrand is also alternating. The idea of the first functional on the right in (39) is that it is a conformal primitive for the Q-curvature: besides being alternating, Remark 8 below shows that it is cocyclic and has conformal variaton

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\frac{1}{2}\int\log\frac{e^{2\varepsilon\eta}\widehat{g}}{g}(\mathbf{Q}[e^{2\varepsilon\eta}\widehat{g}]+\mathbf{Q}[g])\right)=\int\eta\mathbf{Q}[\widehat{g}]$$

for all $\eta\in C^\infty(M)$.

Let us immediately begin to explain the phrase "some version of the Q-curvature" used directly above.

Definition 2. Let $\mathcal{I}$ be the space of Riemannian invariant $(-n)$-densities in even dimension n. Given $F \in \mathcal{I}$, let $\mathbf{b}F$ be the universal linear operator defined by the linear term in the analogue of (36) (replacing $\mathbf{U}_n$ by F):

$$(\mathbf{b}F)\omega = \mathbf{X}_1[F](d\omega, \nabla, R).$$

Let $\mathcal{I}^{\mathrm{FSA}}$ be the subspace of $\mathcal{I}$ consisting of invariants F for which $\mathbf{b}F$ is formally self-adjoint. Let $\mathcal{I}^s$ be the subspace of $\mathcal{I}$ consisting of invariants F for which $\mathbf{X}_t[F]$ vanishes universally for $t > s$. Let

$$\mathcal{I}^{\mathrm{Q}} := \mathcal{I}^1 \cap \mathcal{I}^{\mathrm{FSA}}.$$

A *Q-curvature* is an element of $\mathcal{I}^{\mathrm{Q}}$ for which $\mathbf{b}Q$ has the form $\Delta^{n/2} + \mathrm{LOT}$. A P-operator (or *critical GJMS operator*) is an operator that appears as $\mathbf{b}Q$ for some Q-curvature Q.

Theorem 3. *There exists a Q-curvature.*

Remark 4. A Q-curvature in general even dimensions was first constructed in [9] (see also [10]); this construction uses properties of the GJMS operators which were completely verified in published form only later. More recently, Graham has shown that the original construction of the GJMS operators produces operators that can be written purely in terms of the ingredients $g, g^{-1}, \nabla, \mathrm{r}$. (That is, the Weyl tensor need not be used, if one writes things in just the right way.) As a result, the original construction of the Q-curvature also produces something built out of just these ingredients. It may be reasonable to conjecture that having a formula omitting the Weyl tensor pins down unique GJMS operators, and a unique curvature, but at present this is an open question.

Remark 5. Given $F \in \mathcal{I}$, if $\mathbf{X}_s[F]$ vanishes universally, then so do the $\mathbf{X}_t[F]$ for $t \geq s$. Thus to check for membership of F in $\mathcal{I}^0$, one only needs to know that $\mathbf{X}_1[F]$ vanishes universally; $\mathcal{I}^0$ is the space of *local conformal invariants.* To check for membership in $\mathcal{I}^1$, we just need $\mathbf{X}_2[F]$ to vanish universally. Indeed, the vanishing of $\mathbf{X}_s[F]$ is equivalent to the vanishing of $(d/d\varepsilon)^s|_{\varepsilon=0}F(e^{2\varepsilon\omega g})$ for all g and ω. Taking advantage of this universality to change g to $e^{2\varepsilon_0\omega}g$, we find that $(d/d\varepsilon)^s F(e^{2\varepsilon\omega g})$ vanishes for all (g, ω, ε), so all higher derivatives vanish also.

Remark 6. For any $F \in \mathcal{I}$, the operator $\mathbf{b}F$ has the form Td, since ω appears in $\mathbf{X}_1[F]$ only through $d\omega$. If $F \in \mathcal{I}^{\mathrm{FSA}}$, then $\mathbf{b}F$ must also take the form $\delta S d$. (This is not immediate, but follows from the canonical form of [26] and [37], Sec. 2.2.) If Q is a Q-curvature, the principal part of $P := \mathbf{b}Q$ is $\Delta^{n/2}$, so

$$P \text{ has the form } \delta((d\delta)^{n/2-1} + \mathrm{LOT})d.$$

Since under the usual conformal change $\widehat{Q} = Q + P\omega$, this shows in particular that Q is conformally invariant modulo exact divergences, so that the integral of Q is conformally invariant.

Remark 7. If $Q \in \mathcal{I}^{\mathrm{Q}}$, then the operator $P := \mathbf{b}Q$ is necessarily conformally invariant:

$$Q + P(\omega + \eta) = \widehat{\widehat{Q}} = \widehat{Q} + \widehat{P}\eta = Q + P\omega + \widehat{P}\eta,$$

in the notation of (9). Thus $\widehat{P}\eta = P\eta$ for all η.

Remark 8. Given $Q \in \mathcal{I}^{\mathrm{Q}}$ and $P = \mathbf{b}Q$, the quantity

$$\mathcal{Q}(\widehat{g}, g) := \tfrac{1}{2}\int \omega \underbrace{(Q + \widehat{Q})}_{2Q+P\omega} \tag{40}$$

is a cocycle on the conformal class $[g]$. As noted in connection with (10), it is alternating. For the cocycle condition, put $P = \mathbf{b}Q$; we compute that

$$\begin{aligned} \mathcal{Q}(\widehat{\widehat{g}}, \widehat{g}) + \mathcal{Q}(\widehat{g}, g) &= \tfrac{1}{2}\int \eta \underbrace{(2\widehat{Q} + \widehat{P}\eta)}_{2(Q+P\omega)+P\eta} + \tfrac{1}{2}\int \omega(2Q + P\omega) \\ &= \tfrac{1}{2}\int (\omega + \eta)(2Q + P(\omega + \eta)) = \mathcal{Q}(\widehat{\widehat{g}}, g)\,. \end{aligned} \tag{41}$$

Note that in the last step, the formal self-adjointness of P is used in equating $2\int \eta P\omega$ to $\int \eta P\omega + \int \omega P\eta$. The quantity $\mathcal{Q}(\widehat{g}, g)$ above has conformal variation $\int \omega Q$. In fact, the variation of $\mathcal{Q}(\widehat{g}, g)$ at the metric $\widehat{g}$ in the direction $\eta \in C^\infty$ is the first-order (in η) term in (41), namely $\int \eta\widehat{Q}$, as desired.

Definition 9. The *total conformal variation* of $F \in \mathcal{I}$ is

$$\partial F := (\mathbf{b}F)^* 1\,.$$

The null space of $\partial : \mathcal{I} \to \mathcal{I}$ is $\mathcal{I}^{\mathrm{ix}}$, the space of *conformal index densities.* $\mathcal{I}^{\mathrm{div}}$ is the subspace of $\mathcal{I}$ consisting of univeral exact divergences; that is, invariants of the form $\delta\varphi$ for some univeral element of $\mathcal{E}_a[2-n]$.

Remark 10. The idea of the total conformal variation is as follows. Suppose we take the conformal variation of $\int F$, and integrate by parts in the result until only undifferentiated occurrences of the conformal factor ω remain:

$$\left(\int F\right)^\bullet = \int F^\bullet = \int (\mathbf{b}F)\omega =: \int \omega G\,.$$

In the last step, the integration by parts is

$$\int 1 \cdot (\mathbf{b}F)\omega = \int (\mathbf{b}F)^* 1 \cdot \omega\,,$$

so that $G = (\mathbf{b}F)^* 1$.

Remark 11. If $F \in \mathcal{I}$, then $\mathbf{b}F$ has the form Td, so $\partial F = (\mathbf{b}F)^* 1 = \delta T^* 1$ is a universal exact divergence. Thus $\partial\mathcal{I} \subset \mathcal{I}^{\mathrm{div}}$. Since exact divergences integrate to 0 universally, they are annihilated by ∂; in particular, $\partial\partial = 0$.

Remark 12. $\mathcal{I}^{\mathrm{ix}}$ consists of the $F \in \mathcal{I}$ for which $\int F$ is conformally invariant; i.e. is a *conformal index* [18]. $\mathcal{I}^{\mathrm{ix}}$ is strictly larger than $\mathcal{I}^{\mathrm{div}}$, since the Pfaffian Pff is a conformal index density, but not a universal exact divergence (there are compact manifolds with nonzero Euler characteristic in even dimensions). $\mathcal{I}^{\mathrm{FSA}}$ is contained in $\mathcal{I}^{\mathrm{ix}}$, since for $F \in \mathcal{I}^{\mathrm{FSA}}$, the operator $\mathbf{b}F$ is of the form $\delta S d$, so that $(\int F)^\bullet = \int F^\bullet = \int \delta S d\omega = 0$.

The formal self-adjointness requirement on the P-operator associated to a Q-curvature is suggested by functional determinant and torsion problems, by way of the following considerations. Suppose $U \in \mathcal{I}^{\mathrm{ix}}$ arises as the conformal variation of some (not necessarily locally determined) quantity $\mathcal{D}$; that is, $\mathcal{D}^\bullet = \int \omega U$. The second variation of $\mathcal{D}$ in the direction pair (ω, η) then needs to be symmetric in ω and η. But this second variation is

$$\int \omega(\mathbf{b}U)\eta.$$

Thus the linear operator $\mathbf{b}U$ needs to be formally self-adjoint.

If $G = \partial F$, then a conformal primitive for G is

$$\mathcal{H}(\widehat{g}, g) = \int (\widehat{F} - F). \tag{42}$$

This is clearly antisymmetric and cocyclic. G arises as a conformal variation, so by the argument just above, $\mathbf{b}G$ is formally self-adjoint, and $G \in \mathcal{I}^{\mathrm{FSA}}$.

Summarizing, we have:

Proposition 13.

$$\partial\mathcal{I} \subset \mathcal{I}^{\mathrm{div}} \subset \mathcal{I}^{\mathrm{ix}}, \qquad \mathcal{I}^{\mathrm{Q}} + \partial\mathcal{I} \subset \mathcal{I}^{\mathrm{FSA}} \subset \mathcal{I}^{\mathrm{ix}}.$$

In particular, the natural $(-n)$-density $\mathbf{U}_n$ corresponding to the quantity in (31), *namely*

$$\mathbf{U}_n = \sum_{k=0}^{n/2-1} (-1)^k (n-2k) \mathbf{U}_n(\Delta_k),$$

lies in $\mathcal{I}^{\mathrm{FSA}}$.

The statement strong enough to guarantee the form (39) is thus:

Conjecture 14. $\mathcal{I}^{\mathrm{Q}} + \partial\mathcal{I} = \mathcal{I}^{\mathrm{FSA}}$.

We immediately have $\subset$; the question is whether we have $\supset$. To paraphrase, the conjecture says that each special conformal index density is the sum of something which is known to have a conformal primitive of the form (40), and something with a *local conformal primitive* (42).

Another invariant-theoretic statement, which S. Alexakis reports will be proved in his PhD dissertation [2], has something of the same flavor:

Conjecture 15. $\mathcal{I}^{\mathrm{ix}} = \mathbb{R} \cdot \mathsf{Pff} + \mathcal{I}^{\mathrm{div}} + \mathcal{I}^0$.

The impact of the truth or falsity of this statement on the status of Conjecture 14 is not immediately clear. One thing that it would imply is an analogue of the 4-dimensional statement that the total metric variation of $|\mathsf{C}|^2$ is the Bach tensor, by insuring that in even dimensions, some local conformal invariant has the Fefferman-Graham tensor as its total metric variation. Indeed, if we can write $\mathbf{Q}$ in the form

$$\mathbf{Q} = a\mathsf{Pff} + \delta\eta + \mathbf{S},$$

where $\mathbf{S}$ is a conformally invariant $(-n)$-density, then the metric variation of $\int \mathbf{S}$ must be $\mathcal{A}$, since the metric variations of $\int \mathsf{Pff}$ and $\int \delta\eta = 0$ vanish. But, as remarked earlier, the presence of high-order derivatives in $\mathcal{A}$ insures that for $n \geq 6$, such a $\mathbf{S}$ is not just a polynomial in C. In fact, Remark 17 below is a quantitative statement about just how far from a curvature polynomial $\mathbf{S}$ would have to be.

Perhaps more importantly, the verification of Conjecture 15 would provide a different route to (39), via a dimensional continuation argument, for quantities like the functional determinant of the Yamabe operator. Here one has a $\mathbf{U}_n$ analogous to that in Proposition 13 which extends to higher dimensions N and satisfies

$$\left(\int \mathbf{U}_n\right)^\bullet = (N-n)\int \omega \mathbf{U}_n\,, \qquad N \geq n\,.$$

The Q-curvature also has such an extension, since the subcritical Q-prescription equation in dimensions $N > n$, together with the critical prescription equation, imply that (for $\mathbf{Q} = \mathbf{Q}_n$)

$$\left(\int \mathbf{Q}\right)^\bullet = (N-n)\int \omega \mathbf{Q}\,, \qquad N \geq n\,.$$

By [9], Corollary 1.6 and the discussion preceding it,

$$\left.\frac{\int(\widehat{\mathbf{Q}}-\mathbf{Q})}{N-n}\right|_{N=n} = \frac{1}{2}\int \omega(\widehat{\mathbf{Q}}+\mathbf{Q})\,. \tag{43}$$

Given Conjecture 15, we may write

$$\int \mathbf{U}_n = b\int(\mathbf{Q}+\mathbf{L}) \qquad \text{in dimension } n\,,$$

where $\mathbf{L}$ is a local conformal invariant. Letting $\underline{\mathbf{Q}} := \mathbf{Q}+\mathbf{L}$ be our alternative Q-curvature, we extend to dimension $N \geq n$, and have

$$\left.\begin{aligned} \int(\mathbf{U}_n - b\underline{\mathbf{Q}}_n) &= (N-n)\int \mathbf{F}, \\ \left(\int(\mathbf{U}_n - b\underline{\mathbf{Q}}_n)\right)^\bullet &= (N-n)\int \omega(\mathbf{U}_n - b\underline{\mathbf{Q}}_n)\,, \end{aligned}\right\} \quad N \geq n.$$

where we use the subscript n on $\underline{\mathbf{Q}}$ to emphasize the fact that it is a subcritical Q-curvature for $N > n$. Implicit in this is a rational-in-N extension of $\mathbf{L}$, as well as the natural extension of $\mathbf{Q}_n$. Thus

$$\operatorname{Prim}\int \omega(\mathbf{U}_n - b\underline{\mathbf{Q}}_n) = \frac{1}{N-n}\left\{\int(\widehat{\mathbf{U}}_n - \mathbf{U}_n) - b\int(\widehat{\underline{\mathbf{Q}}_n} - \underline{\mathbf{Q}}_n)\right\} + \int(\widehat{\mathbf{F}} - \mathbf{F})$$

in dimension $N \geq n$. Going to dimension n and using (43), we get

$$\operatorname{Prim}\int \omega \mathbf{U}_n = \frac{1}{2}b\int \omega(\widehat{\underline{\mathbf{Q}}} + \underline{\mathbf{Q}}) + \int(\widehat{\mathbf{F}} - \mathbf{F})$$

in dimension n, as desired for (39). The dimensional continuation is justified by taking the product of the original manifold M with flat tori, using conformal factors that depend only on the M parameter, and deriving identities (one for each N) on M itself. These identities are then rationally continued in the parameter N.

In preparation for some explicit calculations, let

$$\mathsf{J} := \frac{\mathsf{K}}{2(n-1)}, \qquad \mathsf{P} := \frac{\mathsf{r} - \mathsf{J}g}{n-2}\,. \tag{44}$$

P is the *Schouten tensor*, and

$$\mathsf{J} = \mathsf{P}^a{}_a, \qquad \mathsf{J}_{|a} = \mathsf{P}^b{}_{a|b}\,.$$

The Weyl conformal curvature tensor is

$$\mathsf{C}^a{}_{bcd} = R^a{}_{bcd} + 2\mathsf{P}_{b[c}\delta^a{}_{d]} - 2\mathsf{P}^a{}_{[c}g_{d]b}\,. \tag{45}$$

All computations of conformal change laws for local $O(n)$ invariants maybe done using just the conformal change laws for $\mathsf{J}, \mathsf{P}, \mathsf{C}$,

$$\begin{aligned} \widehat{\mathsf{C}}^a{}_{bcd} &= \mathsf{C}^a{}_{bcd}\,, \\ \widehat{\mathsf{P}}_{ab} &= \mathsf{P}_{ab} - \omega_{|ab} + \omega_{|a}\omega_{|b} - \frac{1}{2}\omega_{|c}\omega_{|}{}^c g_{ab}\,, \\ \widehat{\mathsf{J}} &= e^{-2\omega}\left(\mathsf{J} - \omega_{|a}{}^a - \tfrac{n-2}{2}\omega_{|a}\omega_{|}{}^a\right)\,, \end{aligned} \tag{46}$$

together with the conformal change of the Levi-Civita connection on vector fields,

$$(\widehat{\nabla} - \nabla)_a X^d = (2\omega_{|(a}\delta_{b)}{}^d - \omega_{|}{}^d g_{ab})X^b\,,$$

and the fact that $(\widehat{\nabla} - \nabla)_a$ is a derivation over tensor product which commutes with contractions.

In dimension 4, a basis for $\mathcal{I}$ is J^2, $|\mathsf{P}|^2$, $|\mathsf{C}|^2$, $\Delta\mathsf{J}$. The Q-curvature used in [19] to study the determinant quotient is

$$Q = \Delta\mathsf{J} + 2(\mathsf{J}^2 - |\mathsf{P}|^2)\,,$$

and the associated P-operator is

$$P = \delta(d\delta + 2\mathsf{J} - 4\mathsf{P}\cdot)d\,,$$

where $\mathsf{P}\cdot$ is the natural action of a symmetric 2-tensor on 1-forms. This is manifestly formally self-adjoint. In addition,

$$\begin{aligned} 32\pi^2\mathsf{Pff} &= |\mathsf{C}|^2 + 8(\mathsf{J}^2 - |\mathsf{P}|^2)\,, \\ \mathcal{I}^{\text{ix}} &= \operatorname{span}\{Q, \Delta\mathsf{J}, |\mathsf{C}|^2\} = \operatorname{span}\{\mathsf{Pff}, \Delta\mathsf{J}, |\mathsf{C}|^2\}\,, \\ \mathcal{I}^0 &= \mathbb{R}\cdot|\mathsf{C}|^2\,, \\ \mathcal{I}^{\text{div}} &= \mathbb{R}\cdot\Delta\mathsf{J}\,, \\ \mathcal{I}^{Q} &= \operatorname{span}\{Q, |\mathsf{C}|^2\}\,, \\ \partial\mathcal{I} &= \mathbb{R}\cdot\partial|\mathsf{J}|^2 = \mathbb{R}\cdot\Delta\mathsf{J}\,. \end{aligned}$$

Since $\mathcal{I}^Q + \partial\mathcal{I}$ agrees with $\mathcal{I}^{\text{ix}}$, the space $\mathcal{I}^{\text{FSA}}$ wedged between them by Proposition 13 must also agree, verifying Conjecture 14 in dimension 4. It is also immediate to check Conjecture 15.

In dimension 6, the space $\mathcal{I}$ of local invariants of the correct homogeneity is generated by the following 17 quantities:

$$\begin{aligned} \mathrm{A} &:= \mathsf{J}_a{}^a{}_b{}^b = \Delta^2\mathsf{J}\,, \\ \mathrm{B}_1 &:= \mathsf{J}\mathsf{J}_{|a}{}^a = -\mathsf{J}\Delta\mathsf{J}\,, \\ \mathrm{B}_2 &:= \mathsf{P}_{ab}\mathsf{J}_{|}{}^{ab} = (\mathsf{P}, \mathrm{Hess}\mathsf{J})\,, \\ \mathrm{B}_3 &:= \mathsf{P}_{ab}\mathsf{P}^{ab}{}_{|c}{}^c = -(\mathsf{P}, \nabla^*\nabla\mathsf{P})\,, \\ \mathrm{B}_4 &:= \mathsf{P}_{ab|cd}\mathsf{C}^{acbd} = (\nabla\nabla\mathsf{P}, \underline{\mathsf{C}})\,, \\ \mathrm{C}_1 &:= \mathsf{J}_{|a}\mathsf{J}_{|}{}^a = |d\mathsf{J}|^2\,, \\ \mathrm{C}_2 &:= \mathsf{P}_{ab|c}\mathsf{P}^{ab}{}_{|}{}^c = |\nabla\mathsf{P}|^2\,, \\ \mathrm{C}_3 &:= \mathsf{P}_{ab|c}\mathsf{P}^{ac}{}_{|}{}^b = (\nabla\mathsf{P}, \underline{\nabla\mathsf{P}})\,, \\ \mathrm{C}_4 &:= \mathsf{C}_{abcd|e}\mathsf{C}^{abcd}{}_{|}{}^e = |\nabla\mathsf{C}|^2\,, \end{aligned} \qquad \begin{aligned} \mathrm{D}_1 &:= \mathsf{J}^3\,, \\ \mathrm{D}_2 &:= \mathsf{P}_{ab}\mathsf{P}^{ab}\mathsf{J} = |\mathsf{P}|^2\mathsf{J}\,, \\ \mathrm{D}_3 &:= \mathsf{P}_{ab}\mathsf{P}^a{}_c\mathsf{P}^{bc} = \operatorname{tr}(\mathsf{P}^3)\,, \\ \mathrm{D}_4 &:= \mathsf{P}_{ab}\mathsf{P}_{cd}\mathsf{C}^{acbd} = (\mathsf{P}\otimes\mathsf{P}, \underline{\mathsf{C}})\,, \\ \mathrm{D}_5 &:= \mathsf{C}_{abcd}\mathsf{C}^{abcd}\mathsf{J} = |\mathsf{C}|^2\mathsf{J}\,, \\ \mathrm{D}_6 &:= \mathsf{P}_{ab}\mathsf{C}^a{}_{cde}\mathsf{C}^{bcde}\,, \\ \mathrm{D}_7 &:= \mathsf{C}_{abcd}\mathsf{C}^{ab}{}_{ef}\mathsf{C}^{cdef} = \operatorname{tr}(\mathsf{C}^3)\,, \\ \mathrm{D}_8 &:= \mathsf{C}_{abcd}\mathsf{C}^a{}_e{}^c{}_f\mathsf{C}^{bedf}\,. \end{aligned} \tag{47}$$

Here, given any tensor $\varphi_{abc\star}$ with three or more arguments, $\varphi_{\underline{abc}\star} := \varphi_{acb\star}$. In $\mathrm{tr}(\mathsf{C}^3)$, the trace is as an operator on the two-forms $\mathcal{E}^2$.

D_7 and D_8 are manifestly conformally invariant (when viewed as (-6)-densities). A more subtle local conformal invariant is

$$\mathrm{I} = |W|^2 - 16(\mathsf{C}, U) + 16|\mathsf{A}|^2 ,$$

where A is the *Cotton tensor*, $\mathsf{A}_{abc} := 2\mathsf{P}_{a[b|c]}$,

$$W_{abcde} := \mathsf{C}_{abcd|e} + 2g_{e[a}\mathsf{A}_{b]cd} + 2g_{e[c}\mathsf{A}_{d]ab} ,$$

and

$$U_{abcd} := \mathsf{A}_{bcd|a} - \mathsf{P}_a{}^e\mathsf{C}_{ebcd} .$$

In terms of our basis, this expands as

$$\mathrm{I} = 32\mathrm{B}_4 + 32\mathrm{C}_2 - 32\mathrm{C}_3 + \mathrm{C}_4 + 16\mathrm{D}_6 .$$

This invariant is described in [30]; see [38], Sec. 3 for a detailed formula. The invariants D_7, D_8, I form a basis of the local conformal invariants $\mathcal{I}^0$.

The question now arises of which invariants in the quotient by the local conformal invariants have linear conformal change laws; that is, of identifying $\mathcal{I}^1/\mathcal{I}^0$ within $\mathcal{I}/\mathcal{I}^0$. One such is the Q-curvature computed by Gover and Peterson [37]:

$$\begin{aligned}\mathbf{Q} &:= 8|\nabla\mathsf{P}|^2 + 16\mathsf{P}_{ab}\mathsf{P}^{ab}{}_{|c}{}^c - 32\mathsf{P}_{ab}\mathsf{P}^a{}_c\mathsf{P}^{bc} - 16\mathsf{J}|\mathsf{P}|^2 + 8\mathsf{J}^3 - 8\mathsf{J}\mathsf{J}_{|c}{}^c \\ &\quad + \Delta^2\mathsf{J} + 16\mathsf{P}_{ab}\mathsf{P}_{cd}\mathsf{C}^{acbd} \\ &= \mathrm{A} - 8\mathrm{B}_1 + 16\mathrm{B}_3 + 8\mathrm{C}_2 + 8\mathrm{D}_1 - 16\mathrm{D}_2 - 32\mathrm{D}_3 + 16\mathrm{D}_4 .\end{aligned}$$

The conformal deformation of $\mathbf{Q}$ takes the form

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{P}\omega ,$$

where $\mathbf{P}$ is a formally self-adjoint operator of the form $\delta(d\delta + \mathrm{LOT})d$. A formula for $\mathbf{P}$ is given in [37], Sec. 2.2.

Additional terms with a linear conformal change law were found by Gover-Peterson [37] and Fefferman-Hirachi [32]. Gover and Peterson note that $\mathbf{G} := \Delta|\mathsf{C}|^2$ admits a linear law; this is apparent from the facts that $|\mathsf{C}|^2$ is an invariant (-4)-density, that

$$d : \mathcal{E}[-4] \to \mathcal{E}_a[-4] \qquad \text{changes by} \qquad \widehat{d} = d + 4\varepsilon(d\omega) ,$$

and that $\delta : \mathcal{E}_a[-4] \to \mathcal{E}[-6]$ is conformally invariant; thus

$$\widehat{\mathbf{G}} = \mathbf{G} + 4\delta\varepsilon(d\omega)|\mathsf{C}|^2 = \mathbf{G} + 4\delta(|\mathsf{C}|^2 d\omega) .$$

That is, $\mathbf{G}$ changes by a linear, formally self-adjoint operator. Expanded in the basis above,

$$\mathbf{G} = -32\mathrm{B}_4 - 2\mathrm{C}_4 - 4\mathrm{D}_5 - 16\mathrm{D}_6 + 2\mathrm{D}_7 + 8\mathrm{D}_8 .$$

Gover and Peterson also give a general machine for manufacturing additional terms with a formally self-adjoint linear change law in [37], Proposition 2.8, and remarks following this proposition. Fefferman and Hirachi [32] used an ambient space construction to produce another Q-curvature modification in dimension 6. They consider

$$\mathbf{H} := -\mathsf{C}_{abcd}\mathsf{C}^{abce}\mathsf{P}^d{}_e + |\mathsf{A}|^2 + \tfrac{1}{4}|\mathsf{C}|^2\mathsf{J} ,$$

which in the basis above is

$$\mathrm{H} = 2\mathrm{C}_2 - 2\mathrm{C}_3 + \tfrac{1}{4}\mathrm{D}_5 - \mathrm{D}_6 .$$

The conformal change of H is by

$$\widehat{\mathrm{H}} = \mathrm{H} + \underbrace{\left(\tfrac{1}{4}|\mathrm{C}|^2\Delta + 4\mathrm{P}_{ab|c}\mathrm{C}^{acbd}\nabla_d + \mathrm{C}^a{}_{cde}\mathrm{C}^{bcde}\nabla_a\nabla_b\right)}_{\mathbf{bH}}\omega .$$

Direct computation shows that the operator **bH** is formally self-adjoint; in particular, it can be written in the form $\delta S d$.

The matrix giving the ordered list

$$\begin{array}{l} \mathrm{I},\ \mathbf{Q},\ \mathbf{H}' := \mathbf{H} + \tfrac{1}{16}\mathrm{I},\ \mathbf{G}' := \mathbf{G} + \mathrm{I}, \\ \mathrm{D}_8,\ \mathrm{D}_7,\ \mathrm{D}_5,\ \mathrm{D}_4,\ \mathrm{D}_3,\ \mathrm{D}_2,\ \mathrm{D}_1,\ \mathrm{C}_3,\ \mathrm{C}_2, \mathrm{C}_1,\ \mathrm{B}_3,\ \mathrm{B}_2, \mathrm{B}_1 \end{array} \tag{48}$$

in terms of the ordered basis

$$\begin{array}{l} \mathrm{D}_6,\ \mathrm{A},\ \mathrm{B}_4,\ \mathrm{C}_4, \\ \mathrm{D}_8,\ \mathrm{D}_7,\ \mathrm{D}_5,\ \mathrm{D}_4,\ \mathrm{D}_3,\ \mathrm{D}_2,\ \mathrm{D}_1,\ \mathrm{C}_3,\ \mathrm{C}_2, \mathrm{C}_1,\ \mathrm{B}_3,\ \mathrm{B}_2, \mathrm{B}_1 \end{array}$$

is triangular (independent of the ordering of the invariants on the second line of each list). Thus (48) is a basis of the space of invariants $\mathcal{I}$, with $\mathrm{D}_7, \mathrm{D}_8, \mathrm{I}$ forming a basis of $\mathcal{I}^0$, and the classes of $\mathbf{Q}, \mathbf{H}', \mathbf{G}'$ spanning a 3-dimensional subspace of $\mathcal{I}^1/\mathcal{I}^0$. To find out whether $\mathcal{I}^1/\mathcal{I}^0$ has dimension exactly equal to 3, it is sufficient to test an indeterminate linear combination of

$$\mathrm{B}_1,\ \mathrm{B}_2,\ \mathrm{B}_3,\ \mathrm{C}_1,\ \mathrm{C}_2,\ \mathrm{C}_3,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3,\ \mathrm{D}_4,\ \mathrm{D}_5 . \tag{49}$$

Straightforward calculation shows that no nontrivial combination of these has a linear conformal change law.

To summarize,

Proposition 16. *In dimension 6, $\mathcal{I}^0$ has $\{\mathrm{I}, \mathrm{D}_8, \mathrm{D}_7\}$ as a basis, and $\mathcal{I}^{\mathrm{Q}}$ has $\{\mathbf{Q}, \mathbf{H}', \mathbf{G}', \mathrm{I}, \mathrm{D}_8, \mathrm{D}_7\}$ as a basis. The affine space of Q-curvatures is*

$$\mathbf{Q} + \mathrm{span}\{\mathbf{H}', \mathbf{G}', \mathrm{I}, \mathrm{D}_8, \mathrm{D}_7\} .$$

Let us now compute $\partial\mathcal{I}$. It is sufficient to consider the list (47) modulo $\mathcal{I}^{\mathrm{ix}}$, so *a fortiori* (by Proposition 13), it is enough to consider the list modulo $\mathcal{I}^{\mathrm{div}}$. It is not hard to see the classes of C_1, C_2, and the D_i form a basis of $\mathcal{I}/\mathcal{I}^{\mathrm{div}}$, so that

$$\dim \mathcal{I}^{\mathrm{div}} = 7 .$$

Since ∂ annihilates D_7 and D_8, we can find all integrated conformal variations by processing

$$\mathrm{C}_1,\ \mathrm{C}_2,\ \mathrm{D}_1, \ldots \mathrm{D}_6 .$$

We have:

$$\begin{aligned}
\partial C_1 &= 2A - 4B_1 - 4C_1,\\
\partial C_2 &= 2A + 2B_1 + 40B_2 - 8B_3 - 4B_4 + 18C_1 - 20C_2 + 36C_3\\
&\quad - 24D_2 + 144D_3 - 24D_4,\\
\partial D_1 &= -6B_1 - 6C_1,\\
\partial D_2 &= -2B_1 - 2B_2 - 2B_3 - 4C_1 - 2C_2,\\
\partial D_3 &= -6B_2 - 3C_1 - 3C_3 + 3D_2 - 18D_3 + 3D_4,\\
\partial D_4 &= 6B_2 - 6B_3 - 2B_4 - 12C_2 + 12C_3 - 6D_2 + 36D_3 - 6D_4,\\
\partial D_5 &= \mathbf{G} = -32B_4 - 2C_4 - 4D_5 - 16D_6 + 2D_7 + 8D_8,\\
\partial D_6 &= -12B_4 - 12C_2 + 12C_3 - \frac{1}{2}C_4 - D_5 - 4D_6 + \frac{1}{2}D_7 + 2D_8.
\end{aligned}$$

These total variations span a 6-dimensional space, one basis of which is

$$\partial C_1,\ \partial D_1,\ \partial D_3,\ \partial D_5 = \mathbf{G},\ \partial\underbrace{\left(C_2 - C_1 + D_1 + \tfrac{20}{3}D_3 - \tfrac{1}{8}D_5\right)}_{:=C_2'},\ \partial\underbrace{\left(D_6 - \tfrac{3}{8}D_5\right)}_{:=D_6'}. \tag{50}$$

Among the linear combinations of C_1, C_2, and D_1 through D_6, those annihilated by ∂ are spanned by

$$\begin{aligned}
&D_1 - 3D_2 + 2D_3 + D_4 + \tfrac{1}{8}D_5 - \tfrac{1}{2}D_6,\\
&C_1 - C_2 - D_1 - \tfrac{16}{3}D_3 + \tfrac{4}{3}D_4 - \tfrac{1}{12}D_5 + \tfrac{1}{3}D_6.
\end{aligned}$$

In particular, the Pfaffian must agree, up to a nonzero constant factor and a linear combination of D_7 and D_8, with the first of these.

The list (50) may be continued to a linearly independent list by appending

$$\mathbf{H}'' := \mathbf{H} + \frac{1}{6}\partial D_6',\ \mathbf{Q}' := \mathbf{Q} + \partial(-\tfrac{1}{2}C_1 - D_1 + 2C_2' - \tfrac{8}{3}D_6') + 4\mathbf{H}'',\ D_7,\ D_8. \tag{51}$$

Each expression in (50,51) is in $\mathcal{I}^{\mathrm{FSA}}$. The invariant I is linearly dependent on these:

$$I + \partial(D_5 + \tfrac{8}{3}D_6') + 8\mathbf{H}'' = D_7 + 4D_8. \tag{52}$$

Another basis of $\partial\mathcal{I}$, of course, is ∂C_1, ∂D_1, ∂D_3, ∂D_5, ∂C_2, ∂D_6. We may continue this to a linearly independent list in $\mathcal{I}^{\mathrm{FSA}}$ by appending $\mathbf{H}$, $\mathbf{Q}$, D_7, D_8. Though this is simpler to write down, the previous list (50,51) has triangularity properties with respect to the original list (47) that make it convenient for computation.

Collecting some information, in dimension 6 we have:

$$\begin{aligned}
&\dim\mathcal{I}^{\mathrm{ix}} = \dim\mathcal{I} - \dim\partial\mathcal{I} = 17 - 6 = 11,\\
&\mathcal{I}^0 = \mathrm{span}\{D_7, D_8, I\},\\
&\mathcal{I}^{\mathrm{Q}} = \mathrm{span}\{\mathbf{Q}, \mathbf{H}', \mathbf{G}'\} + \mathcal{I}^0,\quad \dim\mathcal{I}^{\mathrm{Q}} = 6,\\
&\dim\mathcal{I}^{\mathrm{div}} = 7,\\
&\dim(\mathcal{I}^{\mathrm{div}} \cap \mathcal{I}^{\mathrm{Q}}) \geq 1.
\end{aligned}$$

By the validity of Conjecture 15 in dimension 6 (which may be checked directly) and a dimension count (in $[\cdot]$ in the underbraces below), the sum

$$\underbrace{\mathcal{I}^{\mathrm{ix}}}_{[11]} = \underbrace{\mathbb{R}\cdot\mathsf{Pff}}_{[1]} + \underbrace{\mathcal{I}^{\mathrm{div}}}_{[7]} + \underbrace{\mathcal{I}^0}_{[3]}$$

must be direct:

$$\mathcal{I}^{\rm ix} = \mathbb{R} \cdot \mathsf{Pff} \oplus \mathcal{I}^{\rm div} \oplus \mathcal{I}^0 .$$

The dimension count for our conjecture goes as follows:

$$\overbrace{\mathcal{I}^{\rm ix}}^{[11]} \supset \mathcal{I}^{\rm FSA} \stackrel{?}{=} \overbrace{\underbrace{\mathcal{I}^{\rm Q}}_{[6]} + \underbrace{\partial\mathcal{I}}_{[6]}}^{[10]} .$$

The reason for the 10 in the overbrace on the right is that $\mathbf{G} \in \mathcal{I}^{\rm Q} \cap \partial\mathcal{I}$, and there is the additional linear dependence (52), so the number in the overbrace is ≤ 10. On the other hand, the linear indepencence of the list given by appending (51) to (50) establishes ≥ 10. Since $\mathcal{I}^{\rm FSA} \supset \mathcal{I}^{\rm Q} + \partial\mathcal{I}$, the conjecture will survive through dimension 6 if and only if $\dim \mathcal{I}^{\rm FSA} = 10$, if and only if $\mathcal{I}^{\rm ix} \setminus \mathcal{I}^{\rm FSA}$ is nonempty. For this, note that $\mathrm{B}_2 + \mathrm{C}_1 \in \mathcal{I}^{\rm div} \subset \mathcal{I}^{\rm ix}$. If $\mathbf{b}(\mathrm{B}_2 + \mathrm{C}_1) =: T$, then

$$\begin{aligned}(T - T^*)\omega = {} & \mathsf{J}_{|a}{}^a \Delta\omega + 2\mathsf{J}_{|a}(\Delta\omega)_|{}^a + 2|\mathsf{P}|^2\Delta\omega - \mathsf{J}\mathsf{J}_{|a}\omega_|{}^a + 2\mathsf{J}_{|a}\mathsf{P}^a{}_b\omega_|{}^b \\ & + 8\mathsf{P}_{ab}\mathsf{P}^a{}_{c|}{}^b\omega_|{}^c - 4\mathsf{P}_{ab}\mathsf{P}^{ab}{}_{|c}\omega_|{}^c + \mathsf{P}_{ab|c}{}^c\omega_|{}^{ab} \\ & + 12\mathsf{P}_{ab}\mathsf{P}^a{}_c\omega_|{}^{bc} + 2\mathsf{P}_{ab|c}\omega_|{}^{abc} - 2\mathsf{P}_{ab}\mathsf{C}^{acbd}\omega_{|cd} .\end{aligned} \tag{53}$$

In particular, if ω is chosen at a point x to have $(\nabla\omega)_x = 0$, $(\nabla\nabla\omega)_x = 0$, then the above becomes

$$2\mathsf{J}_{|a}(\Delta\omega)_|{}^a + 2\mathsf{P}_{ab|c}\omega_|{}^{abc} . \tag{54}$$

Note that this shows that even if we pursue this invariant theory in the conformally flat case only, there will be a nontrivial difference between $\mathcal{I}^{\rm ix}$ and $\mathcal{I}^{\rm FSA}$.

Since $\mathcal{I}^{\rm FSA}$ has codimension 1 in $\mathcal{I}^{\rm ix}$, the $\mathbf{b}F - (\mathbf{b}F)^*$ computed from any $F \in \mathcal{I}$ must be a constant multiple of the one in (53). Indeed, doing the same calculation for

$$\mathrm{B}_3 + \mathrm{C}_2 = \nabla^c(\mathsf{P}_{ab|c}\mathsf{P}^{ab}) \in \mathcal{I}^{\rm div} \subset \mathcal{I}^{\rm ix}$$

yields -1 times the expression in (53). The same calculation on the exact divergences A and $\mathrm{B}_1 + \mathrm{C}_1$ yields 0, so

$$\mathrm{A},\ \mathrm{B}_1 + \mathrm{C}_1 \in \mathcal{I}^{\rm FSA} .$$

It follows from (46) that any of the cubic curvature polynomials $\mathrm{D}_1, \ldots, \mathrm{D}_8$ have $\mathbf{b}\mathrm{D}_i$ of order at most 2 as a differential operator; thus $(\mathbf{b}\mathrm{D}_i)^*$ and $\mathbf{b}\mathrm{D}_i - (\mathbf{b}\mathrm{D}_i)^*$ also have order ≤ 2. Thus by (54), for any linear combination D of the D_i lying in $\mathcal{I}^{\rm ix}$, the constant multiple of (53) given by $\mathbf{b}\mathrm{D} - (\mathbf{b}\mathrm{D})^*$ is 0, so that

$$\mathcal{I}^{\rm ix} \cap \operatorname{span}\{\mathrm{D}_i\}_{i=1}^8 \subset \mathcal{I}^{\rm FSA} .$$

In particular,

$$\mathsf{Pff} \in \mathcal{I}^{\rm FSA} .$$

Remark 17. One concrete way in which Q-curvatures are very different from Pff is in the filtration of $\mathcal{I}$ by homogeneity degree in ∇ (the quantity N_∇ of (35)). Let $\mathcal{I}_k$ be the subspace of polynomials in $\mathcal{I}$ which are writable as a linear combinations of monomials with $N_\nabla \leq k$; then

$$\mathcal{I}_0 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_{n-2} \subset \mathcal{I}_{n-2} .$$

In this filtration, the Pfaffian lies in the most elite space $\mathcal{I}_0$, while a Q-curvature (in order to get the leading $\Delta^{n/2}\omega$ term in its conformal variation) lies in the same class in $\mathcal{I}_{n-2}/\mathcal{I}_{n-4}$ as does $\Delta^{n/2-1}\mathsf{J}$. Further conditions on the behavior of $\mathbf{Q}$ with respect to this filtration are implied by the fact that the total metric variation of $\mathbf{Q}$ is the Fefferman-Graham tensor. Let $\mathcal{J}_k$ denote the natural one-form $(2-n)$-densities with expressions having $N_\nabla \leq k$; then the exact divergences within $\mathcal{I}_k$ are $\delta\mathcal{J}_{k-1}$. Since $\delta\mathcal{J}_{n-3} = \mathcal{I}_{n-2}$, the leading term when we work modulo divergences is in

$$\mathcal{I}_{n-4}/(\mathcal{I}_{n-6} + \delta\mathcal{J}_{n-5}) \,. \tag{55}$$

If $n \geq 6$, this is one-dimensional, and generated by the class of

$$|\underbrace{\nabla \dots \nabla}_{(n-4)/2} \mathsf{J}|^2 \,. \tag{56}$$

In order to produce the Fefferman-Graham tensor $\mathcal{A}$ as its total metric variation, the class of $\mathbf{Q}$ in (55) must be nonzero, so a nonzero multiple of the class of (56). (This may be computed by looking at the corresponding leading term of $\mathcal{A}$; see [39], equation (2.2).) The same must be true of the putative S discussed above. In dimension 6 (where we know Conjecture 15 holds), this means that in the linear combination

$$\mathrm{S} = a\mathrm{D}_7 + b\mathrm{D}_8 + c\mathrm{I} \,,$$

we must have $c \neq 0$.

Remark 18. A smaller 6-dimensional invariant theory may be realized by restricting to flat conformal classes. Besides the vanishing of the Weyl tensor, such classes have $\mathsf{P}_{ab|c}$ symmetric in the last two indices, by the contracted Bianchi identity

$$\mathsf{C}_{abcd|}{}^{a} = 2(n-3)\mathsf{P}_{b[d|c]} \,.$$

As a result of this, the list (47) may be replaced by the list

$$\mathrm{A},\ \mathrm{B}_1,\ \mathrm{B}_2,\ \mathrm{C}_1,\ \mathrm{C}_2,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3 \,.$$

Specifically, the other 9 quantities are eliminated because

$$\mathrm{B}_3 = \mathrm{B}_2 + 6\mathrm{D}_3 - \mathrm{D}_2, \qquad \mathrm{C}_3 = \mathrm{C}_2 \,,$$

and B_4, C_4, and the D_i for $i \geq 4$ vanish. Modulo exact divergences, we have only C_1, D_1, D_2, and D_3. The exact divergences are spanned by

$$\begin{aligned} \mathrm{A} &= \nabla^b \mathsf{J}_{|a}{}^{a}{}_{b} \,, \\ \mathrm{B}_1 + \mathrm{C}_1 &= \nabla^a(\mathsf{J}\mathsf{J}_{|a}) \,, \\ \mathrm{B}_2 + \mathrm{C}_1 &= \nabla^b(\mathsf{P}_{ab}\mathsf{J}_{|}{}^{a}) \,, \\ \mathrm{B}_2 + \mathrm{C}_2 - \mathrm{D}_2 + 6\mathrm{D}_3 &= \nabla^c(\mathsf{P}_{ab|c}\mathsf{P}^{ab}) \,. \end{aligned}$$

The range of ∂ is 3-dimensional; *a priori* it is spanned by

$$\begin{aligned} \partial\mathrm{C}_1 &= 2(\mathrm{A} - 2\mathrm{B}_1 - 2\mathrm{C}_1) \,, \\ \partial\mathrm{D}_1 &= -6(\mathrm{B}_1 + \mathrm{C}_1) \,, \\ \partial\mathrm{D}_2 &= -2(\mathrm{B}_1 + 2\mathrm{B}_2 + 2\mathrm{C}_1 + \mathrm{C}_2 - \mathrm{D}_2 + 6\mathrm{D}_3) \,, \\ \partial\mathrm{D}_3 &= -3(2\mathrm{B}_2 + \mathrm{C}_1 + \mathrm{C}_2 - \mathrm{D}_2 + 6\mathrm{D}_3) \,. \end{aligned} \tag{57}$$

However the Pfaffian is a constant multiple of $D_1 - 3D_2 + 2D_3$, so there is one linear relation among the four quantities in (57). We have

$$\mathcal{I}^{\mathrm{ix}} = \mathcal{N}(\partial) = \mathcal{I}^{\mathrm{div}} + \mathbb{R} \cdot \mathsf{Pff}\,.$$

By the discussion around (53,54), $\mathcal{I}^{\mathrm{FSA}}$ will be 4-dimensional inside the 5-dimensional $\mathcal{I}^{\mathrm{ix}}$; in fact a basis is $\mathbf{Q}$, ∂C_1, ∂D_1, ∂D_2.

Remark 19. Another reasonably small invariant theory is that of 4-dimensional conformal structures with boundary; this is developed in [13] and used in [21].

A conjecture related to the above machinery, and suggested by the original construction of Q-curvature by polynomial continuation in the dimension is:

Conjecture 20. $\dim \mathcal{I}^{\mathrm{FSA}} + \dim \mathcal{I}^{\mathrm{div}} = \dim \mathcal{I}$.

Note that $\mathcal{I}^{\mathrm{div}} \subset \mathcal{I}^{\mathrm{FSA}}$, so this is more subtle than a conjecture about decompositions of $\mathcal{I}$.

5. Detour torsion (continued)

Again, this is joint work with Rod Gover. The list above in (50) gives us conformal primitives of the type we want for a 10-dimensional space of invariants, which must coincide with $\mathcal{I}^{\mathrm{FSA}}$. A combination

$$\underline{\mathbf{Q}} := q\mathbf{Q} + h\mathbf{H} + d_7 D_7 + d_8 D_8$$

has the conformal primitive $\int \omega(\widehat{\underline{\mathbf{Q}}} + \underline{\mathbf{Q}})$, while a combination

$$\partial \mathbf{F} := c_1 \partial C_1 + c_2 \partial C_2 + d_1 \partial D_1 + d_3 \partial D_3 + d_5 \partial D_5 + d_6 \partial D_6$$

has the conformal primitive $\int (\widehat{\mathbf{F}} - \mathbf{F})$.

What we have established is:

Theorem 21. *In dimension 6, $\mathbf{U}_n$ has the form $\underline{\mathbf{Q}} + \mathbf{F}$, where $\underline{\mathbf{Q}}$ and $\mathbf{F}$ are as above. Correspondingly, $\mathcal{H}_{\mathrm{loc}}$ has the form*

$$\mathcal{H}_{\mathrm{loc}}(g, \widehat{g}) = \int \omega(\underline{\mathbf{Q}} + \widehat{\underline{\mathbf{Q}}}) + \int (\widehat{\mathbf{F}} - \mathbf{F})\,.$$

Note that there are variations on this particular way of writing things which are still of the form (39). First, any conformal index density may be added to $\mathbf{F}$ without changing the quantity $\int (\widehat{\mathbf{F}} - \mathbf{F})$. The 11-dimensional space $\mathcal{I}^{\mathrm{ix}}$ of conformal index densities is spanned by the 7-dimensional space of exact divergences, together with D_7 and D_8, together with

$$\begin{gathered} D_1 - 3D_2 + 2D_2 + D_4 + \tfrac{1}{8} D_5 - \tfrac{1}{2} D_6\,, \\ C_1 - C_2 - D_1 - \tfrac{16}{3} D_3 + \tfrac{4}{3} D_4 - \tfrac{1}{12} D_5 + \tfrac{1}{3} D_6\,. \end{gathered}$$

The invariant on the first line just above is, modulo a linear combination of D_7 and D_8, the Pfaffian.

Second, we can change our way of writing things by viewing $\mathbf{G}$ as a Q-curvature modification rather than as something with a local conformal primitive, since it has both properties. Thus we can subtract aD_5, for any constant a, from $\mathbf{F}$, as long as we add $a\mathbf{G}$ to $\underline{\mathbf{Q}}$.

The half-torsion calculation is related to the *detour torsion* introduced in [17]; this detour torsion is a quantity attached to the *detour complexes* introduced in [16]. For $k < n/2$, let $\mathcal{E}_k := \mathcal{E}^k[2k-n]$. As a conformally invariant operator, the coderivative δ carries $\mathcal{E}_k$ to $\mathcal{E}_{k-1}$. If M is oriented, the Hodge star operator implements an isomorphism (as bundles for conformal structure) between $\mathcal{E}_k$ and $\mathcal{E}^{n-k}$, but we do not wish to assume orientability. In [16], it is shown that there are conformally invariant, formally self-adjoint differential operators $L_k : \mathcal{E}^k \to \mathcal{E}_k$ with the property that

$$L_k = \delta\left\{(\delta d)^{n/2-k-1} + \mathrm{LOT}\right\} d. \tag{58}$$

It follows that the sequence of operators formed by the beginning of the de Rham complex, followed by L_k, followed by the end of the de Rham co-complex (formed by taking the formal adjoint of the de Rham complex) is an elliptic complex:

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{L_k} \mathcal{E}_k \xrightarrow{\delta} \mathcal{E}_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_1 \xrightarrow{\delta} \mathcal{E}_0 . \tag{59}$$

We shall call (59) the k^{th} *de Rham detour complex.* It is worth emphasizing that this complex depends only on conformal structure, and not on the choice of a metric. The existence of this complex is not just a formal fact, but depends on the subtle construction in [16] of the L_k as operators with the factorization (58). In case the underlying manifold is orientable, a choice of orientation determines an isomorphism of (59) with

$$\mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{k-1} \xrightarrow{d} \mathcal{E}^k \xrightarrow{\star L_k} \mathcal{E}^{n-k} \xrightarrow{d} \mathcal{E}^{n-k+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n-1} \xrightarrow{d} \mathcal{E}^n .$$

The cohomology group of the k^{th} de Rham detour complex at $\mathcal{E}^p$ (resp. $\mathcal{E}_p$) for $p = 0, \cdots, k < n/2$ will be called $H^p_{L_k}$ (resp. $H^{n-p}_{L_k}$). If k has been fixed, we shall use the notations H^p_L and H^{n-p}_L. Let us fix k. Note that if $p < k$, then $H^p_L = H^p$, and H^{n-p}_L is the degree $n-p$ cohomology of the de Rham co-complex. By the factorization property $\delta(\bullet)d$ of L, the cohomology H^k naturally injects into H^k_L, and H^{n-k}_L naturally projects onto H^{n-k}. Since a choice of metric within the conformal class sets up a vector space isomorphism of H^p_L with the corresponding harmonic space (the joint null space of d and δ unless $p = k$ or $n-k$, in which case it is the joint null space of L and δ), all terms in the computation of the index by the alternating sum of cohomology dimensions cancel identically; that is, the index of a de Rham detour complex vanishes. One might conjecture that generically (in some sense of "generic" yet to be fully investigated), dim $H^k_L =$ dim H^k; this question generalizes one posed in [28]. A detailed discussion of the relative size of the detour and ordinary cohomologies, as well as some relevant estimates, are given in [16].

In introducing a torsion quantity for these complexes, one issue to be confronted immediately is that the coboundaries have different orders (1 for d and $n-2k$ for L). We can compensate for this, in the definition of the various zeta functions, by replacing $(\delta d)_p$ (for $0 \le p \le k-1$ and $n-k+1 \le p \le n$) with $(\delta d)_p^{n-2k}$, and similarly for $(d\delta)_p$ (for $1 \le p \le k$ and $n-k \le p \le n-1$). For any zeta function made purely from δ and d under this scheme (including local and partial ones),

$$\zeta^{\text{new}}(s) = \zeta^{\text{old}}((n-2k)s) .$$

In particular, the regularity (and in fact the value) at $s = 0$ is unchanged by this device, while the $\zeta'(0)$ quantity gets multiplied by $n-2k$. The zeta function being considered at the bundle $\mathcal{E}^k$ of (59) is that of

$$(d\delta)^{n-2k} + L^2 = \Delta^{n-2k} + \mathrm{LOT}\,,$$

since L is formally self-adjoint.

In analogy with (20), we consider linear combinations

$$\tilde{c}_0\zeta(s, \Delta_0^{n-2k}) + \cdots + \tilde{c}_{k-1}\zeta(s, \Delta_{k-1}^{n-2k}) + \tilde{c}_k\zeta(s, ((d\delta)_k)^{n-2k} + L^2) \\ + \tilde{c}_{n-k}\zeta(s, ((d\delta)_k)^{n-2k} + L^2) + \tilde{c}_{n-k+1}\zeta(s, \Delta_{k-1}^{n-2k}) + \cdots + \tilde{c}_n\zeta(s, \Delta_0^{n-2k})\,.$$

Because of the repetition of terms, we may condense this to

$$c_0\zeta(s, \Delta_0^{n-2k}) + \cdots + c_{k-1}\zeta(s, \Delta_{k-1}^{n-2k}) + c_k\zeta(s, (d\delta)_k^{n-2k} + L^2)\,,$$

then expand to

$$\begin{aligned} &c_0\zeta(s, (\delta d)_0^{n-2k}) + c_1\{\zeta(s, (d\delta)_1^{n-2k}) + \zeta(s, (\delta d)_1^{n-2k})\} \\ &\quad + \cdots + c_{k-1}\{\zeta(s, (d\delta)_{k-1}^{n-2k}) + \zeta(s, (\delta d)_{k-1}^{n-2k})\} + c_k\{\zeta(s, (d\delta)_k^{n-2k}) + \zeta(s, L^2)\} \\ &= c_k\zeta(2s, L) + \sum_{p=0}^{k-1} c_p\zeta((n-2k)s, (\delta d)_p) + \sum_{p=1}^{k} c_p\zeta((n-2k)s, (d\delta)_p) := \kappa_k(s)\,. \end{aligned}$$

Since L carries $\mathcal{E}^k$ to $\mathcal{E}^k[2k-n]$ in a conformally invariant manner, the conformal variation of L (viewed as an operator $\mathcal{E}^k \to \mathcal{E}^k$) in the direction ω is $(2k-n)\omega L$, so that

$$\mathrm{Tr}(L^{-s})^\bullet = (n-2k)s\mathrm{Tr}(\omega L^{-s})\,.$$

By this, (18), and (19), the conformal variation of $\kappa_k(s)$ in the direction ω is

$$\begin{aligned} &(n-2k)s\Bigg\{\sum_{p=0}^{k-1} c_p\{(n-2p)\zeta((n-2k)s, (\delta d)_p, \omega) \\ &\quad - (n-2p-2)s\zeta((n-2k)s, (d\delta)_{p+1}, \omega)\} \\ &\quad + \sum_{p=1}^{k} c_p\{(n-2p+2)\zeta((n-2k)s, (\delta d)_{p-1}, \omega) \\ &\quad - (n-2p)\zeta((n-2k)s, (d\delta)_p, \omega)\}\Bigg\} \\ &\quad + 2(n-2k)sc_k\zeta(2s, L, \omega) \\ &= (n-2k)s\Bigg\{\sum_{p=0}^{k-1}(c_p + c_{p+1})(n-2p)\zeta((n-2k)s, (\delta d)_p, \omega) \\ &\quad - \sum_{p=1}^{k}(c_p + c_{p-1})(n-2p)\zeta((n-2k)s, (d\delta)_p, \omega) + 2c_k\zeta(2s, L, \omega)\Bigg\}\,. \end{aligned}$$

If we choose $c_0, \ldots, c_k$ according to (22) (with p in place of k), then

$$(n-2p)(c_p + c_{p+1}) = 2\,, \qquad (n-2p)(c_p + c_{p-1}) = -2\,,$$

so the above becomes

$$2(n-2k)s\left\{c_k\zeta((n-2k)s,(d\delta)_k,\omega)+2c_k\zeta(2s,L,\omega)+\sum_{p=0}^{k-1}c_p\zeta((n-2k)s,\Delta_p,\omega)\right\}$$
$$=2(n-2k)s\left\{c_k\zeta((d\delta)_k^{n-2k}+L^2,\omega)+\sum_{p=0}^{k-1}c_p\zeta(s,\Delta_p^{n-2k},\omega)\right\}$$
$$=:2(n-2k)s\kappa_k(s,\omega)\,,$$

where $\kappa_k(s,\omega)$ is the local quantity corresponding to $\kappa_k(s)$. (In particular, $\kappa_k(s,1)=\kappa_k(s)$.)

Comparing with (24), the extra factor of $n-2k$ has appeared because the orders of all the Laplacians have been "pumped up" to match that of L^2. We could actually have used the $(n-2k)/2$ powers of the partial Laplacians, and L^1, but L^2 is what appears naturally as a partial Laplacian at $\mathcal{E}^k$.

In fact, the quantity $\kappa_{(n-2)/2}(s)$ (or its local generalization) is essentially Cheeger's $\kappa(s)$ quantity (or its local generalization). Since $L_{(n-2)/2}$ is the Maxwell operator $(\delta d)_{(n-2)/2}$, we have

$$\kappa_{(n-2)/2}(s,\omega)=\kappa(2s,\omega)\,.$$

For $k<(n-2)/2$ however, the L_k carry more subtle geometric information than just their principal parts $(\delta d)^{(n-2k)/2}$. At the other extreme, L_0 is the critical GJMS operator P. This means that $\kappa_0(s,\omega)$ is just $\zeta(2s,P,\omega)$.

Let

$$c_p:=(-1)^p(n-2p),\qquad p=0,\dots,k\,.$$

We now harvest the analogues of the conclusions we made above for the Cheeger $\kappa(s)$ quantity. First,

$$\kappa_k(0)\text{ is a conformal invariant.}$$

Second, the generalization of the heat expansion (27) to positively elliptic operators D of order 2ℓ is

$$Z(t,D,\omega)\sim\sum_{\text{even }t\geq 0}t^{(i-n)/2\ell}\int\omega U_i[D]\,. \tag{60}$$

Note that via the Mellin transform, these still correspond to zeta functions that are regular at $s=0$, and that furthermore the behavior at $s=0$ is still related to the t^0 coefficient, and through that to the U_n local coefficient. The harmonic spaces of the detour complexes still make a global contribution as in the calculation starting with (28). Following the calculation through, we get

$$\kappa_k'(0)^\bullet=2(n-2k)(\tau_k^{\rm loc}(g,\omega)+\tau_k^{\rm glob}(g,\omega))=:2(n-2k)\tau_k(g,\omega)\,,$$

where

$$\tau_k^{\rm loc}(g,\omega)=\int\omega\sum_{p=0}^{k}U_n[\Delta_{k,p}]\,,\qquad \tau_k^{\rm glob}(g,\omega)=-\sum_{p=0}^{k}c_k(-1)^k(n-2k){\rm Tr}\,\omega\mathcal{P}_{k,p}\,,$$

where $\Delta_{k,p}$ is the $p^{\underline{\text{th}}}$ Laplacian of the $k^{\underline{\text{th}}}$ detour complex, and and $\mathcal{P}_{k,p}$ is the projection onto the corresponding harmonic space $\mathcal{N}(\Delta_{k,p})$. (Note that the U_n quantity and this projection are insensitive to the powers used to level the orders of the Laplacians.

Recall that the harmonic space coincides with the de Rham harmonic space for $p \leq k-1$, and is the joint null space of δ and L_k for $p = k$.) All considerations on finding a conformal primitive of $\tau_k^{\mathrm{loc}}(g, \omega)$ are exactly as before. In particular, we may assert that this conformal primitive $\mathcal{H}_k^{\mathrm{loc}}(\widehat{g}, g)$ has the form (38). By our invariant theory above, we also have the right to assert that in dimensions 4 and 6, $\mathcal{H}_k^{\mathrm{loc}}(\widehat{g}, g)$ takes the form (39).

For the global calculation, we still have a de Rham bijection

$$\mathcal{D}_g : \mathcal{N}(\Delta_{k,p}) \to H^p_{L_k}$$

of harmonics with cohomology for the detour complexes. The calculation goes through as before, yielding a conformal primitive

$$\mathcal{H}_k^{\mathrm{glob}}(\widehat{g}, g) = \sum_{p=0}^{k} (-1)^p \log[\widehat{g} : g]^2_{k,p} \,.$$

Note that $[\widehat{g} : g]_{k,p}$ is the same as the $[\widehat{g} : g]_p$ of (34) as long as $p < k$, but that $[\widehat{g} : g]_{k,k}$ depends on the operator L_k.

We have proved:

Theorem 22. *The log of the detour torsion,*

$$\tau_k(g) := (-1)^k \zeta'(0, (d\delta)_k^{n-2k} + L_k^2) + \sum_{p=0}^{k-1} (-1)^p (n-2p) \zeta'(0, \Delta_p) \,,$$

has

$$\tau_k(\widehat{g}) - \tau_k(g) = \mathcal{H}_k^{\mathrm{loc}}(\widehat{g}, g) + \mathcal{H}_k^{\mathrm{glob}}(\widehat{g}, g) \,,$$

for $\mathcal{H}_{\mathrm{loc}}$ and $\mathcal{H}_{\mathrm{glob}}$ as above. The special case $\tau_{(n-2)/2}(g)$ is the Cheeger half-torsion.

Remark 23. As shown in [16], there is also an elliptic complex

$$(61) \quad \mathcal{E}_n \xrightarrow{\delta} \mathcal{E}_{n-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}_{n-k+1} \xrightarrow{\delta} \mathcal{E}_{n-k} \xrightarrow{L_{*,k}} \mathcal{E}^{n-k} \xrightarrow{d} \mathcal{E}^{n-k+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{n-1} \xrightarrow{d} \mathcal{E}^n \,,$$

the middle operator of which is conformally invariant, and is constructed in much the same way as L_k. This *co-detour complex* is *not* generally isomorphic to the detour complex (59), as is evident by taking the cohomology at the initial bundle: Take M to be compact and Riemannian conformal; then for the detour complex (with $k > 0$), dim H^0 is the number of connected components, while for the co-detour complex, dim H^0 is the number of *orientable* connected components. (See [16], Proposition 2.15.)

The fact that the main issues in the above are the Hodge decomposition in an elliptic complex, together with a specific form for the conformal variations of the operators involved, suggests that a version of the detour torsion might exist for generalized Bernstein-Gelfand-Gelfand (BGG) diagrams. In the even-dimensional Riemannian conformal case, these are diagrams of differential operators on S^n which are intertwining for representations of the conformal group $\mathrm{SO}_0(n+1,1)$, or its cover $\mathrm{Spin}_0(n+1,1)$. The representations involved are induced from representations of the maximal parabolic subgroup MAN for which the nilpotent part N acts trivially. The representations are parameterized by an M weight and an A weight. Since $\mathfrak{m} = \mathfrak{so}(n)$, the M parameter takes the form $[\lambda_1, \dots, \lambda_{n/2}]$, where all λ_a are integral, or all are properly half-integral, and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}| . \tag{62}$$

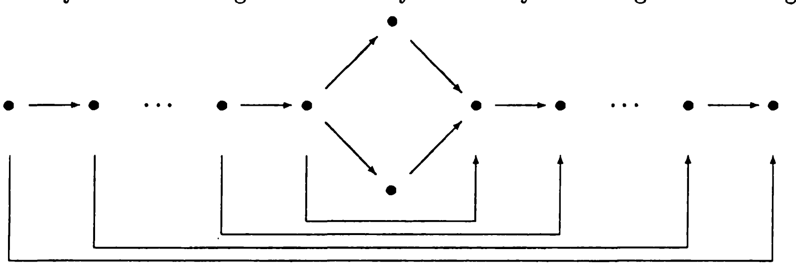
The tuple λ (which is said to be *dominant* if (62) holds) gives the coefficients in the expansion of the highest weight of the $\mathfrak{m}$ module in the basis consisting of the positive weights of the defining representation of $\mathfrak{so}(n)$. The $\mathfrak{a}$-weight can be any complex number; but according to the classification of invariant differential operators [5], only values in $\frac{1}{2}\mathbb{Z}$ can occur in the source or target for invariant differential operators. Furthermore, the $(\mathfrak{a}, \mathfrak{m})$ weight $[\lambda_0 | \lambda_1, \dots, \lambda_{n/2}]$ cannot occur for a differential operator unless $\lambda_0 - \lambda_1 \in \mathbb{Z}$.

The $(\mathfrak{a}, \mathfrak{m})$ weights arrange themselves into orbits under the *affine Weyl group* as follows. The *rho-shift* of $[\lambda_0 | \lambda_1, \dots, \lambda_{n/2}]$ is

$$\left(\lambda_0 + \frac{n}{2} \,\middle|\, \lambda_1 + \frac{n-2}{2}, \lambda_2 + \frac{n-4}{2}, \dots, \lambda_{n/2} + 1, \lambda_{n/2} \right) .$$

We shall use the difference between the round and square parentheses to indicate whether or not a weight has been rho-shifted. Two rho-shifted weights $(\mu_0 | \mu)$ and $(\nu_0 | \nu)$ are *affine Weyl equivalent* if the $(n/2+1)$-tuples involved differ by a permutation and an even number of sign changes. An *affine Weyl orbit* (equivalence class) is *regular* if the absolute values of the $n/2 + 1$ entries of the tuple are distinct.

It is easily seen that a regular affine Weyl orbit may be arranged into a diagram



in a unique way so that the dots ($n+2$ of them in all), representing rho-shifted weights $(\mu_0 | \mu)$, are in decreasing lexicographical order as we move to the right or down, and all tuples to the right of the bar are *strictly dominant* (the property of (62), but with $>$ signs).

By a theorem of Harish-Chandra, all intertwining operators (for principal series representations of $\mathrm{Spin}_0(n+1, 1)$) must pass between bundles in the same affine Weyl orbit. The Boe-Collingwood classification says that all *differential* intertwinors in a regular orbit pass between the bundles in the positions indicated by the picture above; and furthermore, there is a unique (up to constant multiples) nonzero differential intertwinor corresponding to each arrow. In addition, any composition of two arrows (with the exception of one linear combination of the arrows around the diamond, corresponding to the shortest *long operator*) vanishes, and the leading symbol complex at any such composition (including a the composition of a long and short operator) is exact.

These facts are readily generalized to general conformally flat metrics [29], to give conformally invariant differential operators in the positions indicated, between bundles induced by a $\mathfrak{so}(n)$-type (realized by a tensor-spinor bundle), and an $\mathfrak{a}$-weight (realized by a conformal density weight). The simplest example is the de Rham complex, which begins with

$$[0|0,\dots,0] = \left(\frac{n}{2}\,\middle|\,\frac{n-2}{2},\dots,1,0\right).$$

The long operators are the conformally flat special cases of the operators L_k of [16].

Beyond the conformally flat case, one knows that there are *curved generalizations* of each operator – natural, conformally invariant operators, except possibly in the case of the longest operator [27, 29]. However, with the exception of the de Rham complex, there is no longer any reason to expect that compositions of these operators vanish. Though there is some scope for constructing different curved generalizations, there is still no reason to expect that one can find versions for which compositions are identically zero. What one does know is that because leading symbols are determined by the conformally flat case, the composition of two adjacent operators has order lower than the sum of the orders of the two operators.

That the factorized form δQd can be asserted in the case of the long de Rham arrows, even in the conformally curved case, is quite unexpected, and is one of the major implications of [16].

To make our weight conventions completely clear, let us work out the weights of the de Rham complex in detail in dimension 6. In rho-shifted form, they are

$$(3|2,1,0) \to (2|3,1,0) \to (1|3,2,0) \to \begin{matrix}(0|3,2,1)\\ \oplus \\ (0|3,2,-1)\end{matrix} \to (-1|3,2,0) \to (-2|3,1,0) \to (-3|2,1,0).$$

In unshifted form, this is

$$[0|0,0,0] \to [-1|1,0,0] \to [-2|1,1,0] \to \begin{matrix}[-3|1,1,1]\\ \oplus \\ [-3|1,1,-1]\end{matrix} \to [-4,1,1,0] \to [-5|1,0,0] \to [-6|0,0,0].$$

When we realize these as tensor-density bundles, we encounter the fact that the tangent and cotangent bundles carry internal conformal weights. The effect of this is to raise the weight given just above by 1 for each down index, and to lower it by 1 for each up index. Since $\mathcal{E}^k$ carries k down indices, we get

$$\mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to \begin{matrix}\mathcal{E}^3_+\\ \oplus \\ \mathcal{E}^3_-\end{matrix} \to \mathcal{E}^4 \to \mathcal{E}^5 \to \mathcal{E}^6.$$

The bundles in the middle are the middle-forms of the two dualities; since $n = 6$ is of the form $4k+2$, these are the $\sqrt{-1}$-dual and $(-\sqrt{-1})$-dual 3-forms. A special feature of the de Rham BGG diagram is that it survives, as an elliptic complex of short operators, in the conformally curved case. The results of [16] show that (somewhat surprisingly) all the important properties of the long operators relevant to the current discussion also survive.

Another interesting example is the *deformation complex*, which we shall write out explicitly only in dimension 4. This is the complex whose initial short arrow is the conformal Killing operator S, which carries a vector field X to the trace-free part of

$\mathcal{L}_X g$, where $\mathcal{L}$ is the Lie derivative. The kernel of S consists of the conformal vector fields. This complex is constructed in the conformally flat case, and studied in detail, in [34]. Some of the deeper structure of this complex, involved with gauge companion operators, is explored in [15].

In terms of rho-shifted weights, the corresponding BGG diagram is

$$(3|2,0) \to (2|3,0) \to \begin{matrix} (0|3,2) \\ \oplus \\ (0|3,-2) \end{matrix} \to (-2|3,0) \to (-3|2,0)\,.$$

The unshifted weights are thus

$$[1|1,0] \to [0|2,0] \to \begin{matrix} [-2|2,2] \\ \oplus \\ [-2|2,-2] \end{matrix} \to [-4|1,0] \to [-5|1,0]\,.$$

A tensor-density realization is

$$\mathcal{E}^a \to \mathcal{E}_{(ab)_0}[2] \to \begin{matrix} \mathcal{W}^+ \\ \oplus \\ \mathcal{W}^- \end{matrix} \to \mathcal{E}_{(ab)_0}[-2] \to \mathcal{E}_a[-4]\,.$$

The bundles $\mathcal{W}^\pm$ are those of (self- and anti-self-dual) algebraic Weyl tensors; this is the totally trace-free subbundle of $\mathcal{E}^a{}_{bcd}$ with Riemann tensor symmetries. As mentioned, the first short arrow is the conformal Killing operator $X^a \mapsto \nabla_{(a}X_{b)_0}$. The short arrows directed at $\mathcal{W}^\pm$ give, in the conformally flat case, the linearizations of the maps carrying a conformal structure (represented by the conformal metric $\boldsymbol{g}$) to its Weyl tensors $\mathsf{C}^\pm$; note that a generic section of $\mathcal{E}_{(ab)_0}[2]$ may be viewed as a perturbation of conformal structure. The long arrow $\mathcal{E}_{(ab)_0}[2] \to \mathcal{E}_{(ab)_0}[-2]$ is the linearization of the (conformally invariant) Bach tensor, in the same sense. In higher dimensions, when we form the BGG beginning with the conformal Killing operator, the second-longest arrow gives the linearization of the Fefferman-Graham obstruction tensor (recall (3)).

Back in the 4-dimensional case, the deformation complex provides a good example of how some of the properties of a complex may persist beyond the conformally flat case, while not necessarily persisting for general conformal structures. Suppose $[g]$ is a Bach-flat conformal structure, and let B be the operator giving the Bach variation described above. We claim that $BS = 0$, where S is the conformal Killing operator. First, we extend the Lie derivative from functions to densities by requiring that

$$\tilde{\mathcal{L}}_X f = Xf - w(\operatorname{div} X)f/n\,, \qquad f \in \mathcal{E}[w]\,, \tag{63}$$

where $\operatorname{div} X := \nabla_a X^a$, and then extend on to tensor densities by requiring $\tilde{\mathcal{L}}_X$ to be a derivation. In these terms, the infinitesimal conformal diffeomorphism invariance of (for example) the conformal Laplacian $Y : \mathcal{E}[(2-n)/2] \to \mathcal{E}[(-2-n)/2]$ just reads $Y\tilde{\mathcal{L}}_X = \tilde{\mathcal{L}}_X Y$. In fact, a finite version of (63) may be used to *define* the concept of a w-density without reference to conformal structure, as a quantity which responds in a certain way (specifically, by acquiring a factor of a power of the Jacobian) to local diffeomorphisms; in particular, to coordinate changes. Now by a conformal analogue of the calculation that shows that the variation of the Riemann tensor R in the direction $\mathcal{L}_X g$ is $\mathcal{L}_X R$, we may compute that the variation of the Bach tensor $\mathcal{B}$ in the direction of the variation $\tilde{\mathcal{L}}_X \boldsymbol{g}$ of conformal structure is $\tilde{\mathcal{L}}_X \mathcal{B}$. But now note that $\tilde{\mathcal{L}}_X \boldsymbol{g}$ is automatically trace-free, and in fact is SX. This says exactly that $BS = 0$ at conformal structures where $\mathcal{B} = 0$.

For the same reason, at half-conformally flat structures (where $C^+ = 0$ or $C^- = 0$), the composition of S and the appropriate short arrow on the left half of the BGG diamond vanishes.

We now give the construction of detour torsions for BGG diagrams at conformal structures where the appropriate compositions vanish (in particular, for flat conformal structures).

Consider a regular BGG diagram, which, for convenience, we "flatten" by direct summing the bundles at the zenith and nadir of the diamond. Focusing the picture near the bundle E_k,

$$\cdots \longrightarrow E_{k-1} \xrightarrow{D_{k-1}} E_k \xrightarrow{D_k} E_{k+1} \longrightarrow \cdots ,$$

the assumption is that $D_k D_{k-1} = 0$, and the leading symbol complex is exact. (The additive normalization of k is such that the initial bundle of the diagram is E_0, and the final one E_n.) Here any one of the operators, or none, may be a long operator; that is, we may or may not be looking at a BGG *detour complex.* In the conformally flat case, classical theory guarantees only that the short arrow diagrams are locally exact. But one may compute by spectral methods like those in [11] that such detour diagrams are complexes when their coboundary compositions vanish. Mike Eastwood has pointed out that one may also get this from the fact that the local exactness property is preserved by Jantzen-Zuckerman translation. In any event, the exactness of the leading symbol complex may be observed, for arbitrary conformal structures, by looking at the conformally flat case.

There is a technical point that must be addressed before discussing the formal adjoints D_k^*. Given bundles E and F and a differential operator $D : E \to F$, the formal adjoint of D carries F^* to E^*. For our regular BGG diagrams, two things can happen with respect to dual bundles and formal adjoints: those made from the bundles and operators in the first half of the diagram lie in the second half of either (1) the same diagram, or (2) a different diagram. In fact, the dual of the bundle $(\lambda_0|\lambda)$ is $(-\lambda_0|\lambda^\dagger)$, where $\lambda^\dagger$ is the strictly dominant weight in the affine Weyl orbit of $-\lambda$. Thus $\lambda^\dagger = \lambda$ *unless*

$$n \equiv 0 \pmod 4 \qquad \text{and} \qquad \lambda_{n/2} \neq 0 . \tag{64}$$

For example, in the 4-dimensional diagram

$$(3|2,1) \to (2|3,1) \to \begin{array}{c}(1|3,2)\\ \oplus \\ (-1|3,-2)\end{array} \to (-2|3,-1) \to (-3|2,-1),$$

the $\mathfrak{m}$-bundle $(2,1)$ is its own dual, and similarly for the rest of the diagram; so the dual objects to those from the first half of the diagram live in the dual diagram

$$(3|2,-1) \to (2|3,-1) \to \begin{array}{c}(1|3,-2)\\ \oplus \\ (-1|3,2)\end{array} \to (-2|3,1) \to (-3|2,1).$$

On the other hand, the 6-dimensional diagram

$$(4|3,2,1) \to (3|4,2,1) \to (2|4,3,1) \to \begin{array}{c}(1|4,3,2)\\ \oplus \\ (-1|4,3,-2)\end{array} \to (-2|4,3,-1) \to (-3|4,2,-1) \to (-4|3,2,-1)$$

is self-dual, as (for example) the $\mathfrak{m}$ bundles $(3,2,1)$ and $(3,2,-1)$ are dual. The 4-dimensional deformation complex above is also self-dual, since (for example) the $\mathfrak{m}$-bundle $(2,0)$ is its own dual. The de Rham complex is self-dual in any dimension.

In the following discussion, we assume that we have a self-dual regular BGG; that is, we are *not* in the case (64). The coboundaries D_k have formal adjoints D_k^*, and (at each conformal scale) we have a Hodge decomposition

$$C^\infty(E_k) = \mathcal{R}(D_{k-1}) \oplus \mathcal{R}(D_k^*) \oplus \mathcal{H}^k,$$

where the *harmonic space* $\mathcal{H}^k$ is the joint null space of D_{k-1}^* and D_k. As for the de Rham case, we have the partial zeta functions

$$\begin{aligned} \zeta(s, D_k^* D_k) &= \mathrm{Tr}_{L^2}(D_k^* D_k|_{\mathcal{R}(D_k^*)})^{-s} = \mathrm{Tr}((\underline{D_k^* D_k})^{-s}), \\ \zeta(s, D_k D_k^*) &= \mathrm{Tr}_{L^2}(D_k D_k^*|_{\mathcal{R}(D_k)})^{-s} = \mathrm{Tr}((\underline{D_k D_k^*})^{-s}), \end{aligned} \tag{65}$$

and in fact these two are equal, because of the bijection

$$D_k : \mathcal{R}(D_k^*) \leftrightarrow \mathcal{R}(D_k) : D_k^*.$$

A difference between this general case and the de Rham case is that we must confront differing orders for the D_k even if long operators are not involved. In fact, orders may be computed by observing the drop in the weight to the left of the bar (in either the rho-shifted or unshifted regime). For example, the short operator orders in the 4-dimensional deformation complex above are 1,2,2,1, and the long operator giving the Bach tensor variation has order 4. We can handle this by letting P be a common integer multiple of all the orders p_k of D_k, and set the $k^{\underline{\text{th}}}$ Laplacian of the complex equal to

$$\Delta_k := (D_k^* D_k)^{P/p_k} + (D_{k-1} D_{k-1}^*)^{P/p_{k-1}}.$$

Each of the operators D_k has order $2P$. The generalized heat expansion (60) with $\ell = P$ implies that the partial zeta functions (65) and the local zeta functions $\zeta(s, \Delta_k, \omega)$ are regular at $s = 0$. Our levelling of the orders just above has the effect

$$\begin{aligned} \zeta(0, \Delta_k) &= \zeta(0, D_k^* D_k) + \zeta(0, D_{k-1}^* D_{k-1}), \\ \zeta'(0, \Delta_k) &= \frac{P}{p_k}\zeta'(0, D_k^* D_k) + \frac{P}{p_{k-1}}\zeta(0, D_{k-1} D_{k-1}^*) \end{aligned}$$

on the important spectral quantities.

Recall that the half-torsion and detour torsions of the de Rham complex are not conformally invariant, but are functionals on a conformal class that can be "tracked" via Polyakov-type formulas. This implies in particular that we are doing a calculation that cannot be made fully conformally invariant; in particular, $D_k^* D_k$ and $D_{k-1} D_{k-1}^*$ are not conformally invariant operators. Thus, just as in the de Rham case, we make a choice of conformal weights that is artificial for some of our operators; to harmonize with our treatment of that case, we may as well choose to have the D_k invariant in the chosen weights. Now if $D : (\mu_0|\mu) \to (\nu_0|\nu)$ is conformally invariant, then $D^* : (-\nu_0|\nu^\dagger) \to (-\mu_0|\mu^\dagger)$ is conformally invariant. However, in the compositions

D^*D and DD^*, performed at a conformal scale, we view D^* as carrying $(\nu_0|\nu)$ to $(\mu_0|\mu)$ conformally *covariantly*. Specifically, for this version of D^*,

$$(D^*)^\bullet\varphi = 2\nu_0 D^*(\omega\varphi) - 2\mu_0\omega D^*\varphi .$$

Applying this to our BGG diagram,

$$(D_k^*)^\bullet\varphi = 2\mu_0^{(k+1)} D_k^*(\omega\varphi) - 2\mu_0^{(k)}\omega D_k^*\varphi ,$$

where $\mu_0^{(k)}$ is the rho-shifted conformal weight of E_k. (Note that if no D_p for $p < k < n/2$ is a long operator, then $\mu_0^{(k)} = \mu_k^{(0)}$.)

The analogues of (18) and (19) are thus

$$\zeta(s,(D^*D)_k)^\bullet = 2\mu_0^{(k)} s\zeta(s,(D^*D)_k,\omega) - 2\mu_0^{(k+1)} s\zeta(s,(DD^*)_{k+1},\omega) ,$$
$$\zeta(s,(DD^*)_k)^\bullet = 2\mu_0^{(k-1)} s\zeta(s,(D^*D)_{k-1},\omega) - 2\mu_0^{(k)} s\zeta(s,(DD^*)_k,\omega) .$$

We now choose a linear combination of zeta functions,

$$\kappa_{(n-2)/2}(s) = c_0\zeta(s,\Delta_0) + c_1\zeta(s,\Delta_1) + \dots c_{(n-2)/2}\zeta(s,D_{(n-2)/2})$$

if no D_k is long, or

$$\kappa_k(s) = c_0\zeta(s,\Delta_0) + c_1\zeta(s,\Delta_1) + \dots c_k\zeta(s,D_k) \tag{66}$$

if D_k is long. The analogue of (23) is

$$c_p = \begin{cases} (-1)^p 2\mu_0^{(p)} , & p < k , \\ 0 , & p \geq k , \end{cases} \tag{67}$$

where we set $k = (n-2)/2$ if no long operators are used (so that (66) may be considered a unified formula for the two cases).

Note that this specializes to our previous choices for the de Rham and de Rham detour complexes. For the other example considered in detail above, the 4-dimensional deformation complex, it gives

$$6\zeta(s,(S^*S)^2) - 4\zeta(s,(SS^*)^2 + W^*W) ,$$

where W is the linearized Weyl tensor operator.

Most importantly, the conformal variation of $\kappa(s)$ is given by

$$\kappa_k(s)^\bullet = 2Ps\kappa_k(s,\omega) ,$$

where (putting $\sigma_p = \mu_0^{(p)}$)

$$\kappa_k(s) = 2\sigma_0\zeta(s,\Delta_0,\omega) - 2\sigma_1\zeta(s,\Delta_1,\omega) + \dots + (-1)^k \cdot 2\sigma_k\zeta(s,\Delta_k,\omega)$$

After application of the Mellin transform as in (29–31), we get

$$\begin{aligned}\kappa_k'(0)^\bullet &= 2P\left\{\mathcal{U}_n - \sum_{p=0}^{k}\sigma_p \operatorname{Tr}\omega\mathcal{P}_p\right\} \\ &=: 2P\{\tau_{\mathrm{loc}}(g,\omega) + \tau_{\mathrm{glob}}(g,\omega)\} = 2P\tau(g,w) ,\end{aligned}$$

where $\mathcal{U}_n$ is a natural $(-n)$-density in $\mathcal{I}^{\mathrm{FSA}}$, and $\mathcal{P}_p$ is the Hodge projection onto the harmonic sections of E_p (the joint null space of D_p and D_{p-1}^*, or alternatively, the null space of Δ_p). The local part has a conformal primitive of the form (38), and conjecturally, of the form (39); we are guaranteed the form (39) in dimensions 4 and 6. The global part has the conformal primitive

$$\mathcal{H}_{\text{glob}}(\widehat{g}, g) = \sum_{p=1}^{k}(-1)^p \log[\widehat{g} : g]_p^2,$$

where $[\widehat{g} : g]_p$ is the basis change determinant connecting the images in the BGG diagram's cohomology under the de Rham map of g- and $\widehat{g}$-orthonormal bases of Δ_p and $\widehat{\Delta}_p$.

Remarkably, once again the sequence of coefficients in this global term is $1, -1, 1, \ldots$ in the conformal primitive, despite the coefficients in its variation, the kappa quantity, which depend on the particular BGG. The reason for this is that the conformal weights σ_p are telling us about the conformally invariant global (L^2) inner products on the bundles in question. Going back to the half-torsion calculation, note the mechanism by which the coefficients $n - 2k$ were produced in the variation: we set the conformal weights to be the natural weights carried by $\mathcal{E}^k$. The pointwise inner product contracts two k-forms with

$$\underbrace{\boldsymbol{g}^{-1} \otimes \cdots \otimes \boldsymbol{g}^{-1}}_{k},$$

which carries a conformal weight of $-2k$ (as well as carrying $2k$ indices). But the conformal measure $dv_{\boldsymbol{g}}$ corresponding to $\boldsymbol{g}$ carries the conformal weight n. (This is just the Riemannian statement $\widehat{g} = e^{2\omega} g \Rightarrow dv_{\widehat{g}} = e^{n\omega} dv_g$ made intrinsically in terms of conformal structure.) The $(\mathfrak{a}, \mathfrak{m})$ weight of $\mathcal{E}^k$ is

$$[-k|\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0].$$

More generally, it is clear that for any bundle of the form $[\lambda_0|\ldots]$, the conformally invariant pointwise inner product will carry the weight $2\lambda_0$, so that the conformally invariant integrand for the global inner product will carry the weight $n+2\lambda_0 = 2(\lambda_0 + n/2)$, the leftmost entry in the rho-shifted weight $(\lambda_0 + n/2|\ldots)$ of the bundle. But the key step (33) in the argument giving the global part in the case of the half-torsion is clearly dependent exactly on identifying the conformally invariant global inner product.

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