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MODULAR SPACES OF LOW-DIMENSIONAL DRINFELD DOUBLES

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ABSTRACT. We construct the modular space of the semiabelian 4-dimensional Drinfeld double and present also the results for six-dimensional semisimple Drinfeld doubles. Implications for Poisson-Lie T-duality and especially Poisson-Lie T-plurality are mentioned.

1. INTRODUCTION

The interest in dualities of field theories, especially the string theory, in the middle 90s has led to investigation of duality of σ-models and in this context to the discovery of a generalization of abelian [1, 2] and non-abelian T-duality [3], the so-called Poisson–Lie T-duality [4, 5]. A crucial role in these considerations plays the notion of Manin triple and Drinfeld double.

Since then, several non-trivial examples of Poisson–Lie T-dual models, i.e. not contained in the class of non-abelian T-dual models, were constructed and considered both on the classical and quantum level, e.g. the pair of models on one of the Manin triples of the Drinfeld double $SO(3,1)$, see [6, 7].

Moreover also the knowledge of the modular spaces of Drinfeld doubles, i.e. the complete sets of their decompositions into different Manin triples, is of interest. Such modular spaces can be interpreted as spaces of σ-models mutually connected by Poisson–Lie T-duality. Consequently, if one is able to solve one of the σ-models, the solutions of all other models in the modular space follow. Up to recently only rather trivial examples of modular spaces were known. The author had presented a construction of modular spaces of semisimple 6-dimensional Drinfeld doubles in [12]. In the present paper we give a rather simpler example on 4-dimensional Drinfeld double and then briefly recall those 6-dimensional cases.

2. MANIN TRIPLES, DRINFELD DOUBLES AND THEIR MODULAR SPACES

The **Drinfeld double** D is defined as a connected Lie group such that its Lie algebra $\mathcal{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $\langle .,. \rangle$ can

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be decomposed into a pair of subalgebras $\mathcal{G}$, $\tilde{\mathcal{G}}$ maximal isotropic with respect to $\langle .,. \rangle$ and $\mathcal{D}$ as a vector space is the direct sum of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. This ordered triple of algebras $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ is called the **Manin triple**.

One can see that dimensions of the subalgebras are equal and that bases $(X_i), (\tilde{X}^i)$ in the subalgebras can be chosen so that

$$(1) \qquad \langle X_i, X_j \rangle = 0\,, \qquad \langle X_i, \tilde{X}^j \rangle = \langle \tilde{X}^j, X_i \rangle = \delta_i^j\,, \qquad \langle \tilde{X}^i, \tilde{X}^j \rangle = 0\,;$$

in the following we always assume that the bases of $\mathcal{D}$ are of this form. Due to the ad-invariance of $\langle .,. \rangle$ the algebraic structure of $\mathcal{D}$ is determined by the structure of maximal isotropic subalgebras; in the basis $(X_i), (\tilde{X}^i)$ the Lie product is given by

$$(2) \qquad [X_i, X_j] = f_{ij}{}^k X_k\,, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}{}_k \tilde{X}^k\,, \quad [X_i, \tilde{X}^j] = f_{ki}{}^j \tilde{X}^k + \tilde{f}^{jk}{}_i X_k\,.$$

It is clear that to any Manin triple $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ one can construct the dual one by interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$, i.e. interchanging $f_{ij}{}^k \leftrightarrow \tilde{f}^{ij}{}_k$ and such Manin triples give rise to the same Drinfeld double. On the other hand, it might be possible to decompose a given Drinfeld double into more than two Manin triples.

The set of all possible decompositions of the Lie algebra $\mathcal{D}$ of the Drinfeld double D into different Manin triples, i.e. all possible decompositions of $\mathcal{D}$ into two maximal isotropic subalgebras is called the **modular space** $\mathcal{M}(\mathcal{D})$ of the Drinfeld double. In general one can find the modular space if one knows the group of automorphisms of the Lie algebra $\mathrm{Aut}(\mathcal{D})$ and a complete list of non-isomorphic[1] Manin triples $(\mathcal{D},\mathcal{G}_i,\tilde{\mathcal{G}}_i)$ generating the Drinfeld double D. The part of modular space $\mathcal{M}(\mathcal{D})$ corresponding to Manin triples isomorphic to $(\mathcal{D},\mathcal{G}_i,\tilde{\mathcal{G}}_i)$ can then be written

$$(3) \qquad \mathcal{M}(\mathcal{D})_i = \frac{\mathrm{Aut}(\mathcal{D}) \bigcap O(\mathrm{n},\mathrm{n},\mathbf{R})}{\mathcal{H}_i}$$

where $O(\mathrm{n},\mathrm{n},\mathbf{R})$ consists of linear transformations leaving $\langle .,. \rangle$ invariant[2] and $\mathcal{H}_i$ is the subgroup of transformations leaving the isotropic subalgebras $\mathcal{G}_i, \tilde{\mathcal{G}}_i$ invariant. By coset space we mean for concreteness the right coset space $[a] = \mathcal{H}a$. The whole modular space $\mathcal{M}(\mathcal{D})$ is the union of $\mathcal{M}(\mathcal{D})_i$,

$$(4) \qquad \mathcal{M}(\mathcal{D}) = \bigcup_i \mathcal{M}(\mathcal{D})_i\,.$$

3. Poisson–Lie T-dual σ-models and Drinfeld doubles

Starting from a Drinfeld double one can construct the **Poisson-Lie T-dual σ-models** on it. The construction of the models is described in the papers [4, 5]. The models have target spaces[3] in the Lie groups G and $\tilde{G}$ corresponding to the Lie algebras $\mathcal{G}$, resp. $\tilde{\mathcal{G}}$ of a chosen Manin triple, and are defined on $(1+1)$-dimensional Minkowski spacetime M with light-cone coordinates $z, \bar{z}$ by the actions

$$(5) \quad S = \int dz d\bar{z} E_{ij}(g)(g^{-1}\partial_- g)^i (g^{-1}\partial_+ g)^j\,, \quad \tilde{S} = \int dz d\bar{z} \tilde{E}^{ij}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})_i (\tilde{g}^{-1}\partial_+ \tilde{g})_j\,,$$

[1] Manin triples $(\mathcal{D},\mathcal{G},\tilde{\mathcal{G}})$ and $(\mathcal{D},\mathcal{G}',\tilde{\mathcal{G}}')$ are isomorphic if and only if exists a $\langle .,. \rangle$-preserving automorphism ϕ of $\mathcal{D}$ s. t. $\phi(\mathcal{G}) = \mathcal{G}'$, $\phi(\tilde{\mathcal{G}}) = \tilde{\mathcal{G}}'$.

[2] Evidently the group of inner automorphisms $In(D) \in \mathrm{Aut}(\mathcal{D}) \bigcap O(\mathrm{n},\mathrm{n},\mathbf{R})$.

[3] Also a generalization to manifolds on which G, resp. $\tilde{G}$ act freely is possible.

where the coordinates of elements of $\mathcal{G}, \tilde{\mathcal{G}}$ are written in the bases (X_i), $(\tilde{X}^j)$, e.g.

$$g^{-1}\partial_\pm g = (g^{-1}\partial_\pm g)^i X_i$$

and the (non-symmetric) metrics E is

$$E(g) = (a(g) + E(e)b(g))^{-1}E(e)d(g)\,, \tag{6}$$

$E(e)$ is a constant matrix and $a(g), b(g), d(g)$ are submatrices of the adjoint representation of the group G on $\mathcal{D}$ in the basis $(X_i, \tilde{X}^j)$

$$Ad(g)^T = \begin{pmatrix} a(g) & 0 \\ b(g) & d(g) \end{pmatrix}. \tag{7}$$

The matrix $\tilde{E}(\tilde{g})$ is constructed analogously with

$$Ad(\tilde{g})^T = \begin{pmatrix} \tilde{d}(\tilde{g}) & \tilde{b}(\tilde{g}) \\ 0 & \tilde{a}(\tilde{g}) \end{pmatrix}, \qquad \tilde{E}(\tilde{e}) = E(e)^{-1}. \tag{8}$$

The duality can be most straightforwardly understood from the fact that both equations of motion of the above given lagrangian systems can be reduced from an equation of motion on the whole Drinfeld double, $l:\ M \to D$

$$\langle(\partial_\pm l)l^{-1}, \mathcal{E}^\pm\rangle = 0\,, \tag{9}$$

where subspaces $\mathcal{E}^+ = \mathrm{span}\,(X_i + E_{ij}(e)\tilde{X}^j)$, $\mathcal{E}^- = \mathrm{span}\,(X_i - E_{ji}(e)\tilde{X}^j)$ are orthogonal with respect to $\langle .,. \rangle$ and span the whole Lie algebra $\mathcal{D}$. One writes $l = g.\tilde{h}$, $g \in G$, $\tilde{h} \in \tilde{G}$ (such decomposition of group elements exists at least at the vicinity of the unit element, see [10]) and eliminates $\tilde{h}$ from (9), respectively $l = \tilde{g}.h$, $h \in G$, $\tilde{g} \in \tilde{G}$ and eliminates h from (9). The resulting equations of motion for g, resp. $\tilde{g}$ are the equations of motion of the corresponding lagrangian systems (see [4]).

Since the equations of motion are deduced[4] from equation (9) defined originally on D without any reference to $\mathcal{G}, \tilde{\mathcal{G}}$, the transitions between Manin triples of the same Drinfeld double D lead to other pairs of σ-models whose equations of motion are also derived from the same equation on D and in this sense all these models are equivalent. This generalization of the original T-duality was recently named **Poisson–Lie T-plurality** and some of its properties on quantum level were investigated using path integral methods in [11].

The pairs of σ-models whose Manin triples are connected by a transformation from the group $\mathcal{H}$ introduced in (3) describe the same pair of models in different coordinates since the maximal isotropic subalgebras are the same, only their bases have changed.

In the following we consider only the algebraic structure, the Drinfeld doubles as Lie groups can be obtained in principle by means of exponential map. All results can be transferred immediately to connected and simply connected Lie groups, the modular spaces of non-simply connected versions of Drinfeld doubles might be more or less different.

[4]In certain cases only locally.

4. Semiabelian Drinfeld double and its automorphisms

The semiabelian Drinfeld double in dimension 4 is the only Drinfeld double in this dimension possessing decompositions into two different pairs of isotropic subalgebras. It can be decomposed (see [8]) into either semiabelian Manin triple or type B non-abelian Manin triple (and their duals $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$):

- Semiabelian Manin triple (only nonzero brackets are displayed):

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [T_2, \tilde{T}^1] = T_2, \quad [T_2, \tilde{T}^2] = -T_1. \tag{10}$$

- Type B nonabelian Manin triple:

$$\begin{aligned} &[X_1, X_2] = X_2, \quad [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \\ &[X_1, \tilde{X}^1] = X_2, \quad [X_1, \tilde{X}^2] = -X_1 - \tilde{X}^2, \quad [X_2, \tilde{X}^2] = \tilde{X}^1, \end{aligned} \tag{11}$$

where $(X_i, \tilde{X}^j)$ denote the dual basis in the type B nonabelian Manin triple and $(T_i, \tilde{T}^j)$ is the basis in the semiabelian Manin triple.

The transformation of the dual basis between these Manin triples can be fixed as

$$\begin{aligned} X_1 &= -\tilde{T}^1 + \tilde{T}^2, \qquad & X_2 &= T_1 + T_2, \\ \tilde{X}^1 &= T_2, & \tilde{X}^2 &= \tilde{T}^1. \end{aligned} \tag{12}$$

In order to construct all possible automorphisms of $\mathcal{D}$ preserving the bilinear form $\langle .,. \rangle$ one starts by finding the derived subalgebras

$$\mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i], \qquad \mathcal{D}_0 \equiv \mathcal{D}.$$

It is easy to see that

$$\mathcal{D}_1 = \operatorname{span}[T_1, T_2, \tilde{T}^2], \qquad \mathcal{D}_2 = \operatorname{span}[T_1].$$

One easily proves that for any automorphism $\Phi : \mathcal{D} \to \mathcal{D}$ holds

$$\Phi(\mathcal{D}_i) = \mathcal{D}_i$$

by induction using $\Phi([\mathcal{D}_i, \mathcal{D}_i]) = [\Phi(\mathcal{D}_i), \Phi(\mathcal{D}_i)]$ and $\Phi(\mathcal{D}_0) = \Phi(\mathcal{D}) = \mathcal{D}$.

This invariance leads to restriction on the possible form of automorphisms Φ:

$$\begin{aligned} \Phi(T_1) &= \alpha^1 T_1, \qquad \alpha^1 \neq 0, \\ \Phi(T_2) &= \beta^1 T_1 + \beta^2 T_2 + \tilde{\beta}_2 \tilde{T}^2, \\ \Phi(\tilde{T}^2) &= \gamma^1 T_1 + \gamma^2 T_2 + \tilde{\gamma}_2 \tilde{T}^2. \end{aligned} \tag{13}$$

On $\mathcal{D}_1$ we have the following nontrivial condition

$$[\Phi(T_2), \Phi(\tilde{T}^2)] = -\Phi(T_1), \quad \text{i.e.} \quad \tilde{\beta}_2 \gamma^2 - \beta^2 \tilde{\gamma}_2 = -\alpha^1$$

together with conditions following from invariance of the bilinear form

$$\begin{aligned} \langle \Phi(T_2), \Phi(T_2) \rangle &= 2\beta^2 \tilde{\beta}_2 = 0, \\ \langle \Phi(\tilde{T}^2), \Phi(\tilde{T}^2) \rangle &= 2\gamma^2 \tilde{\gamma}_2 = 0 \\ \langle \Phi(T_2), \Phi(\tilde{T}_2) \rangle &= \tilde{\beta}_2 \gamma^2 + \beta^2 \tilde{\gamma}_2 = 1. \end{aligned}$$

Altogether we see that either

$$\alpha^1 = 1\,, \quad \tilde{\beta}_2 = \gamma^2 = 0\,, \quad \tilde{\gamma}_2 = \frac{1}{\beta^2}\,, \quad \beta^2 \neq 0$$

or

$$\alpha^1 = -1\,, \quad \beta^2 = \tilde{\gamma}_2 = 0\,, \quad \gamma^2 = \frac{1}{\tilde{\beta}_2}\,, \quad \tilde{\beta}_2 \neq 0\,.$$

To complete the construction of all possible $\langle .,. \rangle$-preserving automorphisms, we assume the general expression

$$\Phi(\tilde{T}^1) = \epsilon^1 T_1 + \epsilon^2 T_2 + \tilde{\epsilon}_1 \tilde{T}^1 + \tilde{\epsilon}_2 \tilde{T}^2$$

and impose

$$\langle \Phi(\tilde{T}^1), \Phi(T_j) \rangle = \delta^1_j\,, \quad \langle \Phi(\tilde{T}^1), \Phi(\tilde{T}^j) \rangle = 0\,.$$

After solving these equations and checking that the correct commutation relations follow we find the most general form of $\langle .,. \rangle$-preserving automorphisms – any $\langle .,. \rangle$-preserving automorphism is either of the form

$$\begin{aligned}
&\Phi_{(\beta^1,\beta^2,\gamma^1)}(T_1) = T_1 \\
&\Phi_{(\beta^1,\beta^2,\gamma^1)}(T_2) = \beta^1 T_1 + \beta^2 T_2 \\
&\Phi_{(\beta^1,\beta^2,\gamma^1)}(\tilde{T}^1) = -\gamma^1\beta^1 T_1 - \gamma^1\beta^2 T_2 + \tilde{T}^1 - \frac{\beta^1}{\beta^2}\tilde{T}^2 \\
&\Phi_{(\beta^1,\beta^2,\gamma^1)}(\tilde{T}^2) = \gamma^1 T_1 + \frac{1}{\beta^2}\tilde{T}^2
\end{aligned} \tag{14}$$

or

$$\begin{aligned}
&\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(T_1) = -T_1 \\
&\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(T_2) = \beta^1 T_1 + \tilde{\beta}_2 \tilde{T}^2 \\
&\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(\tilde{T}^1) = \gamma^1\beta^1 T_1 + \frac{\beta^1}{\tilde{\beta}_2} T_2 - \tilde{T}^1 + \gamma^1\tilde{\beta}_2\tilde{T}^2 \\
&\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(\tilde{T}^2) = \gamma^1 T_1 + \frac{1}{\tilde{\beta}_2} T_2
\end{aligned} \tag{15}$$

The transformations (15) can be obtained from (14) by composition with special transformation Φ_P:

$$\Phi_P(T_1) = -T_1\,, \quad \Phi_P(T_2) = \tilde{T}^2\,, \quad \Phi_P(\tilde{T}^1) = -\tilde{T}^1\,, \quad \Phi_P(\tilde{T}^2) = T_2\,.$$

In the basis $(X_j, \tilde{X}^j)$ defined by (12) these transformations read

$$\begin{aligned}
\Phi_{(\beta^1,\beta^2,\gamma^1)}(X_1) &= \frac{1}{\beta^2}(\beta^1+1)X_1 + \gamma^1(\beta^1+1)X_2 \\
&\quad + \gamma^1(\beta^2 - \beta^1 - 1)\tilde{X}^1 + \frac{\beta^1}{\beta^2}\tilde{X}^2 \\
\Phi_{(\beta^1,\beta^2,\gamma^1)}(X_2) &= (\beta^1+1)X_2 + (\beta^2 - \beta^1 - 1)\tilde{X}^1
\end{aligned}$$

$$\Phi_{(\beta^1,\beta^2,\gamma^1)}(\tilde{X}^1) = \beta^1 X_2 + (\beta^2 - \beta^1)\tilde{X}^1$$

$$\Phi_{(\beta^1,\beta^2,\gamma^1)}(\tilde{X}^2) = -\frac{\beta^1}{\beta^2}X_1 - \gamma^1\beta^1 X_2 + \gamma^1(\beta^1 - \beta^2)\tilde{X}^1 + (1 - \frac{\beta^1}{\beta^2})\tilde{X}^2 \tag{16}$$

respectively

$$\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(X_1) = -\gamma^1\tilde{\beta}_2 X_1 + \gamma^1(1-\beta^1)X_2 + (1-\beta^1)(\frac{1}{\tilde{\beta}_2} - \gamma^1)\tilde{X}^1 - (1+\gamma^1\tilde{\beta}_2)\tilde{X}^2$$

$$\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(X_2) = \tilde{\beta}_2 X_1 + (\beta^1 - 1)X_2 - (\beta^1 - 1)\tilde{X}^1 + \tilde{\beta}_2\tilde{X}^2$$

$$\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(\tilde{X}^1) = \tilde{\beta}_2 X_1 + \beta^1 X_2 - \beta^1\tilde{X}^1 + \tilde{\beta}_2\tilde{X}^2$$

$$\tilde{\Phi}_{(\beta^1,\tilde{\beta}_2,\gamma^1)}(\tilde{X}^2) = \gamma^1\tilde{\beta}^2 X_1 + \beta^1\gamma^1 X_2 + \beta^1(\frac{1}{\tilde{\beta}_2} - \gamma_1)\tilde{X}^1 + (\gamma^1\tilde{\beta}_2 - 1)\tilde{X}^2 \tag{17}$$

The structure of the constructed group of $\langle .,. \rangle$-preserving automorphisms $\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}) = \mathrm{Aut}(\mathcal{D}) \bigcap O(2,2)$ is the following:

- $\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D})$ is composed of 4 connected components ($\beta^2 > 0$, $\beta^2 < 0$ resp. $\tilde{\beta}_2 > 0$, $\tilde{\beta}_2 < 0$) and the whole group can be written as a 4-element subgroup $\{e, \Phi_P, \Phi_- \equiv \Phi_{(0,-1,0)}, \Phi_P\Phi_-\}$ acting on the connected component of unity $(\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e$
$$\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}) = \{e, \Phi_P, \Phi_-, \Phi_P\Phi_-\} \triangleright (\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e$$
- The connected component of unity $(\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e$ can be expressed as a semidirect product[5]
$$(\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e = F_1 \triangleright F_2$$
of the subgroup[6]
$$F_1 = \{\Phi \in (\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e | \gamma^1 = 0, \beta^2 > 0\} \simeq Af(1)$$
with the normal subgroup
$$F_2 = \{\Phi \in (\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e | \beta^1 = 0, \beta^2 = 1\} \simeq (\mathbf{R}, +)\,.$$
The decomposition of a general element of $(\mathrm{Aut}_{\langle .,. \rangle}(\mathcal{D}))_e$ is
$$\Phi_{(\beta^1,\beta^2,\frac{\gamma^1}{\beta^2})} = \Phi_{(0,1,\gamma^1)}\Phi_{(\beta^1,\beta^2,0)} \equiv ((\beta^1,\beta^2),\gamma^1)\,,$$
the action of F_1 on F_2 defining the semidirect product is
$$(\beta^1,\beta^2) \triangleright (\gamma^1) = (\beta^2\gamma^1)\,.$$

[5]If $\triangleright$ is an action of F_1 on F_2 then the semidirect product of the groups F_1, F_2 is defined $(f_1,f_2).(g_1,g_2) \equiv (f_1.g_1, f_2.(f_1 \triangleright g_2)), \forall f_1, g_1 \in F_1,\ f_2, g_2 \in F_2$.

[6]$Af(1) = \left\{ \begin{pmatrix} \beta^2 & \beta^1 \\ 0 & 1 \end{pmatrix} | \beta^2 > 0 \right\}$ is the group of affine transformations of the line.

5. Modular space of the semiabelian Drinfeld double

The construction of the modular space of the semiabelian Drinfeld double is now straightforward. We firstly find the subgroups $\mathcal{H}_{\text{semiabel}}, \mathcal{H}_{\text{type B}}$ of $\text{Aut}_{\langle.,.\rangle}(\mathcal{D})$ leaving the semiabelian Manin triple, resp. the type B nonabelian Manin triple invariant. The modular space can then be described as a union of the corresponding factor sets.

One can easily see that only elements of $\{e, \Phi_-\} \triangleright F_1$ leave invariant the isotropic subalgebras of the semiabelian Manin triple (i.e. the transformation matrix is block-diagonal) and similarly only elements of F_2 leave invariant the isotropic subalgebras of the type B nonabelian Manin triple. Also it is clear that no element of $\text{Aut}_{\langle.,.\rangle}(\mathcal{D})$ is block-antidiagonal in the bases either $(T_j, \tilde{T}^j)$ or $(X_j, \tilde{X}^j)$, i.e. there is no automorphism interchanging $\mathcal{G} \leftrightarrow \tilde{\mathcal{G}}$ of any of the Manin triples.

Therefore the modular space is composed of four parts, two of them isomorphic to

$$\mathcal{M}(\mathcal{D})_{\text{semiabel}} \simeq \frac{\text{Aut}_{\langle.,.\rangle}(\mathcal{D})}{\mathcal{H}_{\text{semiabel}}} = \{e, \Phi_P\} \triangleright F_2 \simeq \{e, \Phi_P\} \triangleright (\mathbf{R}, +) \tag{18}$$

and two isomorphic to

$$\begin{aligned} \mathcal{M}(\mathcal{D})_{\text{type B}} &\simeq \frac{\text{Aut}_{\langle.,.\rangle}(\mathcal{D})}{\mathcal{H}_{\text{type B}}} = \{e, \Phi_P, \Phi_-, \Phi_P\Phi_-\} \triangleright F_1 \\ &\simeq \{e, \Phi_P, \Phi_-, \Phi_P\Phi_-\} \triangleright Af(1) . \end{aligned} \tag{19}$$

Although the resulting parts of the modular space can be equipped by a group structure, their intrinsic meaning is only of manifolds.

It is rather suprising that (up to discrete transformation Φ_P) any semiabelian Manin triple can be obtained from a given one by composition of a fixed transformation, e.g. (12), to the type B nonabelian Manin triple, different choice of dual basis there and the inverse transformation to (12). Similarly the different type B nonabelian Manin triples correspond (up to discrete transformations Φ_P, Φ_-) to different choices of dual bases in a fixed semiabelian Manin triple.

6. Modular spaces of semisimple 6-dimensional real Drinfeld doubles

By a similar, albeit more complicated, approach one may find also the structure of 6-dimensional real Drinfeld doubles. Because the complete derivation was published elsewhere [12], we present only a list of results here.

6.1. Drinfeld doubles with the Lie algebra $sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$.

It is known (see [9])[7] that there are two classes of non-isomorphic Drinfeld doubles with the Lie algebra $sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$.

[7] Concerning the notation: the expression like $(\mathbf{6_a}|\mathbf{6_{1/a}}.\mathbf{i}|\mathbf{b})$ denotes the Manin triple with the first subalgebra $\mathcal{G}$ of the Bianchi class 6 and the value of parameter a, the second subalgebra $\tilde{\mathcal{G}}$ of Bianchi class 6 and the value of parameter $1/a$, the roman indices index the different possible pairings of bases in $\mathcal{G}$ and $\tilde{\mathcal{G}}$ and b corresponds to rescaling the form $\langle .,.\rangle$.

1. 2-parameter ($a > 1$, $b \in \mathbf{R} - \{0\}$) class of Drinfeld doubles that can be decomposed only into Manin triples isomorphic to

$$\begin{aligned}(\mathbf{6_a|6_{1/a}.i|b}) : &[X_1, X_2] = -aX_2 - X_3\,, \quad [X_2, X_3] = 0\,,\\ &[X_3, X_1] = X_2 + aX_3\,,\\ &[\tilde{X}^1, \tilde{X}^2] = -b(\frac{1}{a}\tilde{X}^2 + \tilde{X}^3)\,, \quad [\tilde{X}^2, \tilde{X}^3] = 0\,,\\ &[\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3)\end{aligned}$$

and its dual. The modular space consists of two components (corresponding to Manin triples with $\mathcal{G} = 6_a$ and $\tilde{\mathcal{G}} = 6_{1/a}$ resp. $\mathcal{G} = 6_{1/a}$ and $\tilde{\mathcal{G}} = 6_a$), each isomorphic to the homogeneous space

$$\mathcal{M}(\mathcal{D})_{(6_a|6_{1/a}.i|b)} \simeq \frac{SL(2,\mathbf{R})}{\mathbf{R} - \{0\}} \times \frac{SL(2,\mathbf{R})}{\mathbf{R} - \{0\}}\,.$$

2. 1-parameter ($b > 0$) class of Drinfeld doubles that possess decompositions into four non-isomorphic Manin triples, namely

$$\begin{aligned}(\mathbf{8|5.i|b}) : &[X_1, X_2] = -X_3\,, \quad [X_2, X_3] = X_1\,, \quad [X_3, X_1] = X_2\,,\\ &[\tilde{X}^1, \quad \tilde{X}^2] = -b\tilde{X}^2\,, \quad [\tilde{X}^2, \tilde{X}^3] = 0\,, \quad [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^3\end{aligned}$$

and

$$\begin{aligned}(\mathbf{6_0|5.iii|b}) : &[X_1, X_2] = 0\,, \quad [X_2, X_3] = X_1\,, \quad [X_3, X_1] = -X_2\,,\\ &[\tilde{X}^1, \tilde{X}^2] = 0\,, \quad [\tilde{X}^2, \tilde{X}^3] = -b\tilde{X}^2\,, \quad [\tilde{X}^3, \tilde{X}^1] = b\tilde{X}^1\end{aligned}$$

and their duals. The whole modular space then consists of four pieces, two isomorphic to

$$\mathcal{M}(\mathcal{D})_{(8|5.i|b)} \simeq \frac{(\{1,S\} \triangleright (SL(2,\mathbf{R})/\{1,-1\}) \times (\{1,S\} \triangleright (SL(2,\mathbf{R})/\{1,-1\})}{(\{1,S\} \times \mathbf{R}^+)}\,.$$

and two to

$$\mathcal{M}(\mathcal{D})_{(6_0|5.iii|b)} \simeq \frac{SL(2,\mathbf{R})}{\mathbf{R} - \{0\}} \times \frac{SL(2,\mathbf{R})}{\mathbf{R} - \{0\}}\,.$$

6.2. **Drinfeld doubles with the Lie algebra** $so(1,3)$**.** There are two classes of non-isomorphic Drinfeld doubles with the Lie algebra $so(1,3)$.

1. 2-parameter ($a \geq 1$, $b \in \mathbf{R} - \{0\}$) class of Drinfeld doubles that can be decomposed only into Manin triples isomorphic to

$$\begin{aligned}(\mathbf{7_a|7_{\frac{1}{a}}|b}) : &[X_1, X_2] = -aX_2 + X_3\,, \quad [X_2, X_3] = 0\,, \quad [X_3, X_1] = X_2 + aX_3\,,\\ &[\tilde{X}^1, \tilde{X}^2] = b(-\frac{1}{a}\tilde{X}^2 + \tilde{X}^3)\,, [\tilde{X}^2, \tilde{X}^3] = 0\,, \quad [\tilde{X}^3, \tilde{X}^1] = b(\tilde{X}^2 + \frac{1}{a}\tilde{X}^3)\,,\end{aligned}$$

and its dual. The modular space $\mathcal{M}(\mathcal{D})$ is

$$\mathcal{M}(\mathcal{D})_{(7_a|7_{\frac{1}{a}}|b)} \simeq \frac{SO(1,3)_+}{U(1) \times \mathbf{R}^+}$$

if $a = 1$ (the self-dual case $\mathcal{G} \simeq \tilde{\mathcal{G}}$) and consists of two such components if $a > 1$.

2. 1-parameter ($b \in \mathbf{R}^+$) class of Drinfeld doubles that possess decompositions into six non-isomorphic Manin triples, namely

$$
\begin{aligned}
(9|5|\mathrm{b}) : & [X_1,X_2] = X_3\,, \quad [X_2,X_3] = X_1\,, \quad [X_3,X_1] = X_2\,,\\
& [\tilde{X}^1,\tilde{X}^2] = -b\tilde{X}^2\,, \quad [\tilde{X}^2,\tilde{X}^3] = 0\,, \quad [\tilde{X}^3,\tilde{X}^1] = b\tilde{X}^3\,,\\
(8|5.\mathrm{ii}|\mathrm{b}) : & [X_1,X_2] = -X_3\,, \quad [X_2,X_3] = X_1\,, \quad [X_3,X_1] = X_2\,,\\
& [\tilde{X}^1,\tilde{X}^2] = 0\,, \quad [\tilde{X}^2,\tilde{X}^3] = b\tilde{X}^2\,, \quad [\tilde{X}^3,\tilde{X}^1] = -b\tilde{X}^1\,,
\end{aligned}
$$

and

$$
\begin{aligned}
(7_0|5.\mathrm{ii}|\mathrm{b}) : & [X_1,X_2] = 0\,, \quad [X_2,X_3] = X_1\,, \quad [X_3,X_1] = X_2\,,\\
& [\tilde{X}^1,\tilde{X}^2] = 0\,, \quad [\tilde{X}^2,\tilde{X}^3] = b\tilde{X}^2\,, \quad [\tilde{X}^3,\tilde{X}^1] = -b\tilde{X}^1\,,
\end{aligned}
$$

and their duals. The whole modular space then consists of six pieces, four (corresponding to Manin triples (9|5|b), (8|5.ii|b) and their duals) of them are isomorphic to

$$\mathcal{M}(\mathcal{D})_{(9|5|\mathrm{b})} \simeq \mathcal{M}(\mathcal{D})_{(8|5.\mathrm{ii}|\mathrm{b})} \simeq \frac{SO(1,3)_+}{U(1)}$$

and two (corresponding to Manin triple (7_0|5.ii|b) and its dual) are isomorphic to

$$\mathcal{M}(\mathcal{D})_{(7_0|5.\mathrm{ii}|\mathrm{b})} \simeq \frac{SO(1,3)_+}{U(1)\times\mathbf{R}^+}\,.$$

7. Conclusions

We have presented several simple but nontrivial examples of modular spaces of Drinfeld doubles. These are rather different from the known Abelian one, mainly $\mathrm{Aut}(\mathcal{D})\bigcap O(d,d,\mathbf{R})$ is in these cases (almost) the group of inner automorphisms $In(\mathcal{D})$, whereas in the Abelian case $In(\mathcal{D}) = \{\mathbf{1}\}$. In this sense they represent other extremal cases of modular spaces. Also we have encountered the fact that the modular spaces might be composed of parts of different dimensions. Consequently, after fixing one concrete T-plurality transformation and applying it to pairs of models on some set of isomorphic Manin triples one may obtain just one pair of models on another Manin triple written in different coordinates and vice versa.

We should also mention again that we have assumed that the Drinfeld double is simply connected. Therefore all automorphisms of Lie algebra could be raised to automorphisms of Lie group.

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