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HIGHER MONOGENICITY AND RESIDUE THEOREM FOR RARITA-SCHWINGER OPERATOR

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ABSTRACT. We define the notion of higher Spin-monogenic functions and prove Stokes and residue theorems for them.

1. INTRODUCTION

In this article we investigate various properties of higher spin analogy of Clifford analysis, see the monograph [1].

In particular, we employ representation theory (Littlewood-Richardson rule) to get decompositions on irreducibles of various tensor-spinor Spin-modules. Based on these results we define Rarita-Schwinger operator, which is higher spin analogy of Dirac operator in Clifford analysis. We then proceed to define higher spin (i.e. Rarita-Schwinger) monogenic functions. This notion is consequently used in the construction of Stokes and residue theorems for Rarita-Schwinger monogenic functions.

It is worth to emphasize that similar (Clifford like) analysis can be done for any other spinor like module, the only problem being connected with **explicit** construction of projectors on this module realized inside tensor-spinor tensor products of Spin-representations. This is just the place where Clifford analysis enters the game. Because we consider operators of the first order, the previous computations are in fact equivalent to the constructions of their symbols.

In this article we shall consider only the closest higher Spin-module beyond spinor module, i.e. the Rarita-Schwinger module. In fact, everything can be extended to any (spinor like) Spin-module. Likewise everything can be generalized towards higher (than one) degree forms valued in Rarita-Schwinger module (instead of zero and one forms as in this article). But note that potential complications coming from irreducibles of higher multiplicity could appear.

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2. Algebraic preliminaries – tensor product decompositions

The aim of this section is to decompose the tensor products of the $\mathrm{Spin}(n)$ representation with the highest weight $(1,0,\dots,0,1)$ (we call the corresponding representation higher spinor representation) with fundamental vector representation with highest weight $(1,0,\dots,0)$. To carry out the computation, we shall employ so called *Littlewood-Richardson rule* based on the notion of a "standard Young tableau" (see [3]).

2.1. Spin-standard Young tableaux.

Definition 2.1. Let $p=(p_1,\dots,p_m)$ be a partition of a natural number n, that is $p_1\geq\dots\geq p_m\geq 0$ are integers and $n=\sum_{i=1}^m p_i$. With p we associate its *Young diagram*, a figure consisting of m left justified rows of boxes, p_i boxes in the i-th row from the top. By a *Young tableau $\mathcal{T}$ of shape p* we mean a filling of the boxes with positive integers. A Young tableau is *standard* if the numbers in the boxes are non-decreasing in the rows and strictly increasing in the columns from the top to the bottom.

Remark 2.2. Note, that we use the convention introduced in [4], which is opposite to the one in [3] in the sense that the rôles of columns and rows are exchanged.

If we omit any column from a standard Young tableau $\mathcal{T}$, we get a standard Young tableau again. Let us number columns of Young tableau from the right hand side. It will be convenient to denote by $\mathcal{T}(k,l)$ the Young tableau consisting of the l-th up to the k-th columns of a Young tableau $\mathcal{T}$, and by $\mathcal{T}(k)$ the tableau $\mathcal{T}(k,1)$. If i is a positive integer, we define $c_{\mathcal{T}}(i)$ to be the number of boxes in $\mathcal{T}$ containing the number i.

To the dominant weight μ of a representation V_μ of a Lie algebra $\mathfrak{g}$ we associate a partition $p(\mu)=(p_1,\dots,p_m)$. The exact relation depends on the type of $\mathfrak{g}$, here we shall restrict ourselves only to the formulas for Lie algebras of type B_m and D_m. The formulas for other types of Lie algebras can be found in Appendix of [3].

Definition 2.3. Let ω_i be the i-th fundamental weight of a Lie algebra $\mathfrak{g}$, and let $\mu=\sum_{i=1}^m a_i\omega_i$ be the decomposition of the dominant weight μ of a representation V_μ of $\mathfrak{g}$. Then we associate to μ the partition $\mathbf{p}(\mu)=(p_1,\dots,p_m)$ with $p_i=\sum_{j=i}^{m-1}2a_j+a_m$ for $\mathfrak{g}$ of type B_m, and $p_i=\sum_{j=i}^{m-2}2a_j+a_{m-1}+a_m$ for $\mathfrak{g}$ of type D_m (the void sums give 0).

Definition 2.4. Let $\mathbf{h}$ be a column of a standard Young tableau such that it does not contain numbers i and $2m+1-i$ together. For $i=1,\dots,m$ we denote by $s_i(\mathbf{h})$ the columns defined in the following way:

> If $i<m$ and both $i+1$ and $2m+1-i$ are entries of the column $\mathbf{h}$, then $s_i(\mathbf{h})$ is the column obtained from $\mathbf{h}$ by replacing the entry $i+1$ by i and $2m+1-i$ by $2m-i$. If $i=m$, $\mathfrak{g}$ is of type B_m and $\mathbf{h}$ contains an entry with value $m+1$, then $s_i(\mathbf{h})$ is the column obtained from $\mathbf{h}$ by replacing $m+1$ by m. If $i=m$, $\mathfrak{g}$ is of type D_m and both $m+1$ and $m+2$ are entries of the column $\mathbf{h}$, then $s_i(\mathbf{h})$ is the column obtained

from $\mathbf{h}$ by replacing $m+1$ by $m-1$ and $m+2$ by m. In all other cases we set $s_i(\mathbf{h}) = \mathbf{h}$.

We say that a pair of columns $(\mathbf{h}, \mathbf{h}')$ is *admissible*, if there exists a sequence of different columns $(\mathbf{h}_0, \dots, \mathbf{h}_k)$, $k \geq 0$, such that

$$\mathbf{h} = \mathbf{h}_0, \quad \mathbf{h}' = \mathbf{h}_k,$$
$$s_{i_j}(\mathbf{h}_{j-1}) = \mathbf{h}_j \qquad \text{for } j = 1, \dots, k \text{ and some integers } 1 \leq i_j \leq m.$$

Definition 2.5. Let $\mathcal{T}$ be a Young tableau of shape $\mathbf{p}(\mu)$ that contains only positive integers smaller or equal to $2m$ and that does not contain integers i and $2m+1-i$ in the same column together. Denote $\bar{p_1} = p_1 - a_m$ for $\mathfrak{g}$ of type B_m, and $\bar{p_1} = p_1 - a_m - a_{m-1}$ for $\mathfrak{g}$ of type D_m. $\mathcal{T}$ is called Spin-*standard* if all of the following conditions hold:

- If $\mathfrak{g}$ is of type B_m then $\mathcal{T}$ is standard. If $\mathfrak{g}$ is of type D_m then we divide $\mathcal{T}$ into three tableaux: $\mathcal{T}_1 := \mathcal{T}(\bar{p_1})$, $\mathcal{T}_2 := \mathcal{T}(p_1 - a_m, \bar{p_1}+1)$, and $\mathcal{T}_3 := \mathcal{T}(p_1, p_1 - a_m + 1)$. Then each of the tableaux $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ is standard.
- Let $\mathbf{t}_i$ be the i-th column of $\mathcal{T}$. The pair of columns $(\mathbf{t}_{2i-1}, \mathbf{t}_{2i})$ is admissible for every $i = 1, \dots, \bar{p_1}/2$.

For $\mathfrak{g}$ of type D_m there must be two further conditions satisfied:

- In $\mathcal{T}_2$ (resp. $\mathcal{T}_3$) the number of integers in a column greater than m is odd (resp. even). The condition for $\mathcal{T}_1$ looks as follows: let $1 \leq i \leq \bar{p_1}/2 - 1$ and let the $(2i)$-th column $\mathbf{t}_{2i}$ of $\mathcal{T}_1$ consists of entries $(h_1, \dots, h_k)$ and the $(2i+1)$-st column $\mathbf{t}_{2i+1}$ of $(j_1, \dots, j_l)$, $k \leq l$. For any sequence of integers $1 \leq i_1 < \cdots < i_q \leq k$ such that
$$m + 1 - q \leq h_{i_1} < \cdots < h_{i_q} \leq m + q$$
$$m + 1 - q \leq j_{i_1} < \cdots < j_{i_q} \leq m + q$$
there holds $h_{i_1} + \cdots + h_{i_q} \equiv j_{i_1} + \cdots + j_{i_q} \mod 2$.
- This last condition is needed only if either $a_{m-1} > 0$ and $a_m > 0$, or $\sum_{i=1}^{m-2} a_i > 0$ and $a_{m-1} + a_m > 0$. Let $(k_1, \dots, k_s)$ be the entries of the most left column of $\mathcal{T}_1$. Denote by $\mathcal{R}$ the set $\{k_1, \dots, k_s, l_1, \dots, l_{m-s-1}, x\}$ with the following properties:
$2m \geq l_1 > \cdots > l_{m-s-1} > m$, $l_{m-s-1} > x$, $l_{m-s-1} > 2m + 1 - x$, $l_i \neq k_j$ and $x \neq k_j$ for all $1 \leq i \leq m - s - 1$, $1 \leq j \leq s$, and if $r \in \mathcal{R} \Longrightarrow 2m + 1 - r \notin \mathcal{R}$. If the number of integers strictly greater than m in $\mathcal{R} \setminus \{x\}$ is odd, then $x > m$ else $x \leq m$ ($\mathcal{R}$ is uniquely determined by these properties).

  Denote by $\mathcal{T}_2'$ the tableau obtained from $\mathcal{T}_2$ by adding one column of length m to the right of $\mathcal{T}_2$ and filling the boxes of this column with the elements of $\{2m + 1 - x\} \cup \mathcal{R} \setminus \{x\}$ in increasing order. Then the tableau $\mathcal{T}_2'$ defined above is standard.

  Denote by $\mathcal{T}_3'$ the tableau obtained from $\mathcal{T}_3$ by adding $(a_{m-1} + 1)$ columns of length m to the left of $\mathcal{T}_3$, with filling of the columns defined inductively as follows. The boxes of the most left added column of $\mathcal{T}_3'$ are filled with integers $\mathcal{R}$ in increasing order. Assume now that $(i-1)$-st column has already been filled, $2 \leq i \leq a_{m-1} + 1$. Let $(j_1, \dots, j_m)$ be the $(i-1)$-st row of $\mathcal{T}_2$. For $1 \leq l \leq m$ let $\mathcal{R}_l$ denote the m-tuple $(i_1, \dots, i_m)$ such that $i_1 < \cdots < i_m$ and $\{i_1, \dots, i_m\} = \{j_1, \dots, 2m+1-j_l, \dots, j_m\}$. Note that $\mathcal{R}_1, \dots, \mathcal{R}_m$ are lexicographically ordered;

the i-th row of $\mathcal{T}_3'$ is equal to $\mathcal{R}_{max}$, where $\mathcal{R}_{max}$ is maximal among those $\mathcal{R}_j'$s for which $\mathcal{T}_3'(i)$ with $\mathcal{R}_{max}$ the most left column is standard. Then the tableau $\mathcal{T}_3'$ defined above is standard.

2.2. The Littlewood-Richardson Rules.

Definition 2.6. Let μ be a dominant weight of a representation of $\mathfrak{g}$ and let $\mathcal{T}$ be a Spin-standard Young tableau of shape $\mathbf{p}(\mu) = (p_1, \dots, p_m)$. Define the *weight* of the tableau $\mathcal{T}$ as

$$\nu(\mathcal{T}) := \frac{1}{2}\left[(c_{\mathcal{T}}(1) - c_{\mathcal{T}}(2m))\varepsilon_1 + \cdots + (c_{\mathcal{T}}(m) - c_{\mathcal{T}}(m+1))\varepsilon_m\right] \tag{1}$$

where $(\varepsilon_1, \dots, \varepsilon_m)$ is the standard weight basis of $\mathfrak{g}$.

Remark 2.7. We suppose that the relation between the fundamental and the standard weight basis is the following. In the case of B_m we have $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}$ for $i = 2, \dots, m-1$ and $\varepsilon_m = 2\omega_m - \omega_{m-1}$. In the case of D_m we have $\varepsilon_1 = \omega_1$, $\varepsilon_i = \omega_i - \omega_{i-1}$ for $i = 2, \dots, m-2, m$ and $\varepsilon_{m-1} = \omega_m + \omega_{m-1} - \omega_{m-2}$.

For $1 \le l \le p_1$ denote by $\nu_l(\mathcal{T})$ the weight $2\nu(\mathcal{T}(l))$. If λ is a dominant weight for $\mathfrak{g}$, then a Spin-standard Young tableau $\mathcal{T}$ of shape $\mathbf{p}(\mu)$ is called λ-*dominant* if all the weights $2\lambda + \nu_1(\mathcal{T}), \dots, 2\lambda + \nu_{p_1}(\mathcal{T})$ are contained in the dominant Weyl chamber of $\mathfrak{g}$.

Theorem 2.8. *The decomposition of the tensor product $V_\lambda \otimes V_\mu$ into a sum of irreducible representations of $\mathfrak{g}$ is given by the formula*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathcal{T}} V_{\lambda + \nu(\mathcal{T})} \tag{2}$$

where $\mathcal{T}$ runs over all λ-dominant Spin-standard Young tableaux of shape $\mathbf{p}(\mu)$.

2.3. Tensor product $[1, 0 \dots, 0, 1] \otimes [1, 0 \dots, 0]$.

2.3.1. *The case of odd spin group B_m* = Spin$(2m+1)$. According to the notation and rules discussed before Th.2.8, we define $\lambda = [1, 0, \dots, 0], \mu = [1, 0, \dots, 0, 1]$. The conditions have in the case of Spin$(2m+1)$ $(m \ge 1)$ the following content:

- The *standard* Young tableau

$$\begin{array}{|c|c|c|}
\hline
i_1 & j_1 & j_2 \\
\hline
i_2 \\
\cline{1-1}
i_3 \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
i_{m-1} \\
\cline{1-1}
i_m \\
\cline{1-1}
\end{array}$$

is filled with integers from the set $\{1,\ldots,2m\}$, such that for all i, $1 \leq i \leq 2m$, the couple $\{i, 2m+1-i\}$ is not in the first column.

- The condition of admissibility of the second and third column (i.e. the second and third box from the left in the first row) implies, that there are two complementary possibilities (see the previous Young tableau):
  - $j_1 = j_2 \in \{1,\ldots,2m\}$,
  - $j_1 = m, j_2 = m+1$.

The last possibility, $j_1 = m, j_2 = m+1$, is excluded because the weight $2\lambda + 2\nu(\mathcal{T}(1)) = (2,0,\ldots,0,-1)$ is not dominant. From the same reason are excluded the possibilities $j_1 = j_2 \in \{3,\ldots,2m-1\}$. The only non-trivial contributions come from the cases $j_1 = j_2 \in \{1,2,2m\}$:

$$
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 \\
\cline{1-1}
3 \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
m-1 \\
\cline{1-1}
m \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2m & 2m \\
\hline
2 \\
\cline{1-1}
3 \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
m-1 \\
\cline{1-1}
m \\
\cline{1-1}
\end{array}
$$

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 2 \\
\hline
2 \\
\cline{1-1}
3 \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
m-1 \\
\cline{1-1}
m \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2 & 2 \\
\hline
3 \\
\cline{1-1}
4 \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} \\
\cline{1-1}
m \\
\cline{1-1}
2m-1 \\
\cline{1-1}
\end{array}
$$

They correspond to the weights $[2,0,\ldots,0,1]$ ($j_1 = j_2 = 1$), $[0,\ldots,0,1]$ ($j_1 = j_2 = 2m$), $[0,1,0,\ldots,0,1]$ ($j_1 = j_2 = 2$) and $[1,0,\ldots,0,1]$ ($j_1 = j_2 = 2$).

In the basis of fundamental weights the decomposition reads

$$
\begin{aligned}
[1,0,\ldots,0,1] \otimes [1,0,\ldots,0] \simeq {} & [2,0,\ldots,0,1] \oplus [1,0,\ldots,0,1] \\
& \oplus [0,1,0,\ldots,0,1] \oplus [0,\ldots,0,1]\,.
\end{aligned}
\tag{3}
$$

2.3.2. *The case of even spin group* $D_m = \mathrm{Spin}(2m)$. We shall start with Young tableaux attached to the weight $[1,0,\dots,0,1]$. Because $p_1 = 3, p_2 = 1, \dots, p_m = 1$, we have the following rules:

- (2.5) The *standard* Young tableaux

$$\begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline \end{array} \quad \mathcal{T}_1 = \mathcal{T}(1,2) = \mathcal{T}(2) \qquad \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline i_3 \\ \hline \vdots \\ \hline i_{m-1} \\ \hline i_m \\ \hline \end{array} \quad \mathcal{T}_3 = \mathcal{T}(3,3)$$

  where the tableau $\mathcal{T}(2)$=$\{0\}$ is trivial (=empty). Moreover, if i, $1 \leq i \leq 2m$, is the entry of the first column, the number $2m+1-i$ is not the entry of the first column. The number of integers greater than m in $\mathcal{T}_3$ is *even.*

- (2.5) The condition of admissibility of the second and third columns (i.e. the second and third boxes from the left in the first row) implies, that there is only one possibility (see the previous Young tableau) - $j_1 = j_2 \in \{1, 2, 2m\}$.

- (2.5) This condition is non-trivial because $\sum_{i=2}^{m-2} a_i > 0 \ \wedge \ a_{m-1} + a_m > 0$ is fulfilled. Let (j_1) be the value in the left column (=box) of $\mathcal{T}_1$ ($j_1 \in \{1, 2, 2m\}$). The set $\mathcal{R} = \{j_1, l_1, \dots, l_{m-2}, x\}, |\mathcal{R}| = m$, has properties:

  > $2m \geq l_1 > \cdots > l_{m-2} > m$, $l_{m-2} > x$, $l_{m-2} > 2m+1-x$, $l_i \neq j_1$ and $x \neq j_1$ for all $1 \leq i \leq m-2$, and if $r \in \mathcal{R} \Longrightarrow 2m+1-r \notin \mathcal{R}$. If the number of integers strictly greater than m in $\mathcal{R} \setminus \{x\}$ is odd, then $x > m$ else $x \leq m$ ($\mathcal{R}$ is uniquely determined by these properties).

  The tableau $\mathcal{T}_2$ is trivial, and so there is no condition on its modification. However the tableau $\mathcal{T}_3$ is restricted by the following condition: denote by $\mathcal{T}_3'$ the tableau obtained from $\mathcal{T}_3$ by adding one column to the right, i.e. the left column of $\mathcal{T}_3'$ contains integers $\{i_1, \dots, i_m\}$ and right columns contains integers (ordered in increasing order) $\mathcal{R}$. Then the tableau $\mathcal{T}_3'$ is *standard.*

1. $j_1 = j_2 = 1$; $1 \in \mathcal{R} \Longrightarrow 2m \notin \mathcal{R}$, and there is only one possibility: either m is odd, $x = m+1$ and $\mathcal{R} = \{1, 2m-1, 2m-2, \dots, m+2, m+1\}$, or m is even, $x = m$ and $\mathcal{R} = \{1, 2m-1, 2m-2, \dots, m+2, m\}$. The corresponding *standard* Young tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} m-1 \\ \cline{1-1} m \\ \cline{1-1}\end{array}$$

gives (irrespective of the parity of m) the highest weight $(\frac{5}{2},\frac{1}{2},\ldots,\frac{1}{2}) = [2,0,\ldots,0,1]$.

2. $j_1 = j_2 = 2$; $2 \in \mathcal{R} \Longrightarrow 2m-1 \notin \mathcal{R}$, and there is only one possibility (for x): either m is odd, $x = m+1$ and $\mathcal{R} = \{2, 2m, 2m-2, 2m-3\ldots, m+2, m+1\}$, or m is even, $x = m$ and $\mathcal{R} = \{2, 2m, 2m-2, 2m-3\ldots, m+2, m\}$. Taking into account the condition of even number of integers greater than m in $\mathcal{T}_1$, only two possibilities of *standard* Young tableaux survive:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} m-1 \\ \cline{1-1} m \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \multicolumn{1}{c}{\vdots} \\ \cline{1-1} m+1 \\ \cline{1-1} 2m-1 \\ \cline{1-1}\end{array}$$

They give (irrespective of the parity of m) the highest weights $(\frac{3}{2},\frac{3}{2},\frac{1}{2}\ldots,\frac{1}{2}) = [0,1,0\ldots,0,1]$ resp. $(\frac{3}{2},\frac{1}{2}\ldots,\frac{1}{2},-\frac{1}{2}) = [1,0\ldots,0,1,0]$.

3. $j_1 = j_2 = 2m$; $2m \in \mathcal{R} \Longrightarrow 1 \notin \mathcal{R}$, and there is only one possibility: either m is even, $x = m+1$ and $\mathcal{R} = \{2m, 2m-1, 2m-2, \ldots, m+2, m+1\}$, or m is odd, $x = m$ and $\mathcal{R} = \{2m, 2m-1, 2m-2, \ldots, m+2, m\}$. The corresponding *standard* Young tableau

$$\begin{array}{|c|c|c|}
\hline
1 & 2m & 2m \\
\hline
2 & \multicolumn{2}{c}{} \\
\cline{1-1}
3 & \multicolumn{2}{c}{} \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} & \multicolumn{2}{c}{} \\
\cline{1-1}
m-1 & \multicolumn{2}{c}{} \\
\cline{1-1}
m & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}$$

gives (irrespective of the parity of m) the highest weight $(\frac{1}{2}, \dots, \frac{1}{2}) = [0, \dots, 0, 1]$.

Summarizing together the case of D_m, in the basis of fundamental weights the decomposition reads

$$\begin{aligned}[1,0,\dots,0,1] \otimes [1,0,\dots,0] \simeq {} & [2,0,\dots,0,1] \oplus [1,0,\dots,0,1,0] \\ & \oplus [0,1,0,\dots,0,1] \oplus [0,\dots,0,1]\,. \end{aligned} \tag{4}$$

Note, that in the case of complementary spinor weight $[1,0,\dots,0,1,0]$ the decomposition differs from the last case by exchange of last two entries of the RHS:

$$\begin{aligned}[1,0,\dots,0,1,0] \otimes [1,0,\dots,0] \simeq {} & [2,0,\dots,0,1,0] \oplus [1,0,\dots,0,1] \\ & \oplus [0,1,0,\dots,0,1,0] \oplus [0,\dots,0,1,0]\,. \end{aligned} \tag{5}$$

Remark 2.9. In the context of Clifford analysis there is similar decomposition based on tensor products with spinor module, see [1],[5] instead of Rarita-Schwinger one. In this case one can use either representation theory in the spirit of previous paragraphs or (of course equivalent) intertwining operators (easily constructed because of single multiplicities of target Spin-modules). Within the context of representation theory, we talk about dual pair $(\mathrm{spin}(n), \mathrm{sl}(2))$.

3. Rarita-Schwinger operator

Let us consider compact Lie group G, i.e. in our case of interest it will be $G = \mathrm{Spin}(n)$, and its (complexified) representation on finite dimensional complex vector space with Hermitian scalar product $(V_{\mathbb{C}}, <,>)$, i.e. a homomorphism $\rho : G \longrightarrow GL_{\mathbb{C}}(V_{\mathbb{C}})$. Then $V_{\mathbb{C}}$ admits G-invariant Hermitian inner product

$$(v_1, v_2) := \int_G < \rho(g)v_1, \rho(g)v_2 > dg$$

such that the representation ρ on $(V_{\mathbb{C}}, <,>)$ is unitary w.r. to $(\,,)$. The symbol dg denotes (G-invariant) Haar measure on the group G.

If ρ is a representation of G on $(V_{\mathbb{C}}, (\,,))$, then ρ is the direct sum of irreducible representations, i.e. $V_{\mathbb{C}} \simeq V_1 \oplus \cdots \oplus V_m$, $m \in \mathbb{N}$ such that V_i $(i = 1, \dots, m)$ is irreducible (complex) finite dimensional G-module. In other words,

$$(v_i, v_j) = 0 \qquad \forall\, v_i \in V_i,\ v_j \in V_j,\ i \neq j\,.$$

Rarita-Schwinger operator $\mathcal{R}_S$ on a Spin-manifold M with conformal structure is conformally invariant differential operator acting between sections valued in Rarita-Schwinger $spin(n)$-modules:

$$\mathcal{R}_S : \mathcal{C}^\infty(M, S_{(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})}) \longrightarrow \mathcal{C}^\infty(M, S_{(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2})}) \tag{6}$$

One can consider this operator to be the second operator (behind Dirac operator) in the series of conformally invariant differential (elliptic) operators acting on spinor-like modules.

Let us denote by ∇ the covariant derivative of (the lift of) Levi-Civita connection on Spin-bundle of M with respect to a given Spin-structure. Let

$$i_{\frac{3}{2}} : S_{(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})} \hookrightarrow S_{(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})} \otimes \mathbb{V} \tag{7}$$

be a homomorphism (embedding) of Spin-modules, uniquely determined in its isomorphism class up to a multiple by element $\mathbb{C}^\star$; $\mathbb{V}$ is the fundamental vector representation.

Let us consider a local orthonormal frame $\{e_i\}_{i=1}^n$ and its (dual) coframe $\{e_i^\star\}_{i=1}^n = \{e^i\}_{i=1}^n$. The symbol of the operator $\mathcal{R}_S$ is the composition of maps

$$Symb(\mathcal{R}_S) = \sum_i \Pi^{\mp}_{\frac{3}{2}} \circ (1 \otimes e_i) \circ i_{\frac{3}{2}} \tag{8}$$

where $(1 \otimes e_j)$ is Spin-group action given by Clifford multiplication on the spinor module in the tensor product (and trivial action on the vector module) followed by projection on Rarita-Schwinger module of opposite chirality:

$$\begin{aligned} \mathbb{V} &\to \mathrm{End}(\mathbb{V} \otimes S_{\frac{1}{2}}) \\ e_i &\to \rho(e_i) : \mathbb{V} \otimes S^{\pm}_{\frac{1}{2}} \xrightarrow{1\otimes e_i} \mathbb{V} \otimes S^{\mp}_{\frac{1}{2}}, \end{aligned} \tag{9}$$

Definition 3.1. Rarita-Schwinger operator is the map

$$\begin{aligned} \mathcal{R}_S &: S^{\pm}_{\frac{3}{2}} \to S^{\mp}_{\frac{3}{2}} \\ \mathcal{R}_S &:= \sum_i \Pi^{\mp}_{\frac{3}{2}} \circ (1 \otimes e_i) \circ i_{\frac{3}{2}} \circ \nabla_i . \end{aligned} \tag{10}$$

4. Algebraic operators on $S_{\frac{3}{2}}$-valued differential forms

Let us fix canonical global ON-frame in $\mathbb{R}^n$ and restrict it to a domain $D \subset \mathbb{R}^n$.

First we define algebraic operators X, Y acting on $\Lambda^\star(D) \otimes S^{\pm}_{\frac{3}{2}}$ by

$$\begin{aligned} X &= \sum_i \epsilon(e^i) \otimes \rho(e_i), \\ Y &= \sum_i \iota(e^i) \otimes \rho(e_i), \end{aligned} \tag{11}$$

where $\epsilon(e^i)$ denotes wedge product with covector e^i and $\iota(e^i)$ denotes contraction by dual of covector e^i.

Lemma 4.1. *The algebraic operators $X, Y : \Lambda^\star(D) \otimes S^{\pm}_{\frac{3}{2}} \to \Lambda^{\star+1}(D) \otimes S^{\mp}_{\frac{3}{2}}$ are* Spin*-invariant, i.e. they intertwine canonical action of* Spin*-group on tensor product $\Lambda^\star(D) \otimes S_{\frac{3}{2}}$.*

Proof. It is sufficient to consider decomposable $S_{\frac{3}{2}}$-valued differential form $\omega\otimes s$. The action of $g\in$ Spin

$$(\lambda^\star(g)\otimes\rho(g))(\omega\otimes s)=\lambda^\star(g)\omega\otimes\rho(g)s \tag{12}$$

yields ($\lambda^\star$ denotes action on wedge product of fundamental vector representation and ρ denotes the representation on Rarita-Schwinger module)

$$\begin{aligned}
(\lambda^\star(g)\otimes\rho(g))X(\omega\otimes s) &= (\lambda^\star(g)\otimes\rho(g))\sum_i(\epsilon(e^i)\otimes\rho(e_i))(\omega\otimes s)\\
&=\sum_i\lambda^\star(g)(\epsilon(e^i))\lambda^\star(g)\omega\otimes\rho(g)\rho(e_i)\rho(g^{-1})\rho(g)s\\
&=\sum_i\lambda^\star(g)(\epsilon(e^i))\lambda^\star(g)\omega\otimes\rho(ge_ig^{-1})\rho(g)s\\
&=\sum_{i,j}\lambda^\star(g)(\epsilon(e^i))\lambda^\star(g)\omega\otimes\lambda_{ij}(g)\rho(e_j)\rho(g)s\\
&=\sum_{i,j,k}\lambda_{ki}(g^{-1})\lambda_{ij}(g)\epsilon(e^k)\lambda^\star(g)\omega\otimes\rho(e_j)\rho(g)s\\
&=(\sum_i\lambda_{ki}(g^{-1})\lambda_{ij}(g)=\delta_{kj})\\
&=\sum_j\epsilon(e^j)\lambda^\star(g)\omega\otimes\rho(e_j)\rho(g)s=X(\lambda^\star(g)\omega\otimes\rho(g)s)
\end{aligned}\tag{13}$$

which is the desired property. The proof for Y goes along the same lines. □

These algebraic operators allow (via their kernels and images), similarly to the case of spinor valued differential forms, projection on Spin-invariant subspace of $\Lambda^\star(D)\otimes S_{\frac{3}{2}}$. The difference with spinor case is that the corresponding spaces are (generally) reducible.

Let us focus on the case of $S_{\frac{3}{2}}$-valued 0-forms and 1-forms.

Lemma 4.2. *The algebraic operators acting on $S_{\frac{3}{2}}$-valued 0-forms and 1-forms*

$$\begin{aligned}
X|_{\Lambda^0(D)\otimes S^\pm_{\frac{3}{2}}}\ &:\ \Lambda^0(D)\otimes S^\pm_{\frac{3}{2}}\to\Lambda^1(D)\otimes S^\mp_{\frac{3}{2}},\\
Y|_{\Lambda^1(D)\otimes S^\pm_{\frac{3}{2}}}\ &:\ \Lambda^1(D)\otimes S^\pm_{\frac{3}{2}}\to\Lambda^0(D)\otimes S^\mp_{\frac{3}{2}}
\end{aligned}\tag{14}$$

fulfill

$$\begin{aligned}
\mathrm{Ker}\,(Y)|_{\Lambda^1(D)\otimes S^\pm_{\frac{3}{2}}} &\simeq \left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2}\right)\oplus\left(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2}\right)\\
&\qquad\oplus\left(\frac{5}{2},\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2}\right),\\
\mathrm{Im}\,(X)|_{\Lambda^1(D)\otimes S^\pm_{\frac{3}{2}}} &\simeq \left(\frac{3}{2},\frac{1}{2},\dots,\frac{1}{2},\mp\frac{1}{2}\right).
\end{aligned}\tag{15}$$

Proof. This result on isomorphism of Spin-modules is the consequence of Spin-invariance of X,Y and of the decomposition of $\Lambda^0(D)\otimes S_{\frac{3}{2}},\Lambda^1(D)\otimes S_{\frac{3}{2}}$ on irreducible Spin-modules. □

We would like to emphasize, that due to the Spin-invariance descend X, Y to algebraic operators on Spin-bundles on Spin-manifold M. In other words, the decompositions (15) hold true globally on manifolds.

Now we are prepared to define the notion of (higher) monogenic section of Rarita-Schwinger bundle.

Definition 4.3. A section $s \in \Gamma(M, S^{\pm}_{\frac{3}{2}})$ is called $S^{\pm}_{\frac{3}{2}}$ (left) monogenic iff fulfills

$$Y(\mathcal{R}_S f) = 0\,. \tag{16}$$

Similar definition holds true for right invariant monogenic functions.

5. Stokes theorem for $S_{\frac{3}{2}}$-valued monogenic functions

We shall generalize Stokes theorem, [1], Ch.II, §.0, from the case of spinor valued (left, right) monogenic functions to the case of $S_{\frac{3}{2}}$-valued (left, right) monogenic functions. We use the convention $(\,,) := (\,,)_{S_{\frac{3}{2}}}$ for Spin-invariant scalar product on $S_{\frac{3}{2}}$.

Let us first start with a domain $D \subset \mathbb{R}^{m+1}$, i.e. the Spin-module $S_{\frac{3}{2}}$ of values of ω_1, ω_2 is a fixed vector space. For two smooth (decomposable) elements $\omega_1 = \overline{\omega_1} \otimes v_1 \in \mathcal{C}_1(D, \Lambda^\star \otimes S_{\frac{3}{2}}), \omega_2 = \overline{\omega_2} \otimes v_2 \in \mathcal{C}_1(D, \Lambda^\star \otimes S_{\frac{3}{2}})$, we define

$$(\omega_1 \wedge \omega_2) := \overline{\omega_1} \wedge \overline{\omega_2} \otimes (v_1, v_2) \in \mathcal{C}_1(D, \Lambda^\star)\,. \tag{17}$$

The definition of action of exterior differential d on forms valued in Rarita-Schwinger module $S_{\frac{3}{2}}$

$$d(\overline{\omega} \otimes v) := (d\overline{\omega} \otimes v)\,, \quad \omega \in \Lambda^\star(D),\ v \in S_{\frac{3}{2}} \tag{18}$$

easily yields

Lemma 5.1. *If* $\omega_1 \in \mathcal{C}_1(D, \Lambda^\star \otimes S_{\frac{3}{2}})$, $\omega_2 \in \mathcal{C}_1(D, \Lambda^\star \otimes S_{\frac{3}{2}})$, *then*

$$d(\omega_1 \wedge \omega_2) = (d\omega_1 \wedge \omega_2) + (-)^{|\omega_1|}(\omega_1 \wedge d\omega_2)\,. \tag{19}$$

For $\Sigma \subset D$ a $(m+1)$-dimensional compact oriented submanifold with smooth boundary $\partial\Sigma$ we define $\mathrm{End}(S_{\frac{3}{2}})$-valued volume element on $\partial\Sigma$

$$d\sigma = \sum_{i=0}^{m} (-)^i \rho(e_i)\, d\hat{x}_i \in \Lambda^m(D) \otimes \mathrm{End}(S_{\frac{3}{2}})\,, \tag{20}$$

where for $\rho(e_k) \in \mathrm{End}(S_{\frac{3}{2}}), e_i \otimes s_j \in S_{\frac{3}{2}}$:

$$\rho(e_k) \,:\, e_i \otimes s_j \xrightarrow{1 \otimes e_k} e_i \otimes e_k s_j$$

such that in the second part of tensor product the Clifford algebra structure on spinor module is used. For each $i = 0, ..., m$,

$$d\hat{x}_i := dx_0 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_m\,. \tag{21}$$

Note that on the domain D we could use any possible product on $S_{\frac{3}{2}}$-values (instead of the scalar product). This is however not the case on Spin-manifolds, where the scalar product on $S_{\frac{3}{2}}$ and (Spin-) covariant derivative ∇^S of Levi-Civita connection on associate vector bundle instead of differential must be used (in such a way that $d\sigma$ is Spin-invariant endomorphism of Rarita-Schwinger bundle $\mathbf{S}_{\frac{3}{2}}$).

The Stokes theorem for $S_{\frac{3}{2}}$-monogenic functions will be formulated directly on Spin-manifolds instead (as has been done in [1]) on the domain D.

Theorem 5.2 (Stokes). *Let $\Sigma, \dim\Sigma = m+1$, be a Riemannian Spin-manifold with boundary $\partial\Sigma$, and let $(\,,)_{S_{\frac{3}{2}}}$ a Spin-invariant scalar product on associated Spin-bundle $\mathbf{S}_{\frac{3}{2}}$. Let $f, g \in \mathcal{C}_1(\Sigma, S_{\frac{3}{2}})$ be smooth sections of $\mathbf{S}_{\frac{3}{2}} = B_{\mathrm{Spin}} \times_{\mathrm{Spin}} S_{\frac{3}{2}}$. Then*

$$\int_{\partial\Sigma}(g, d\sigma f) = \int_{\Sigma}(\mathcal{R}_S g, f)d\Sigma + \int_{\Sigma}(g, \mathcal{R}_S f)d\Sigma\,. \tag{22}$$

where $d\Sigma$ is volume form of Σ and $d\sigma \in \Gamma(\Sigma, \Lambda^m T^\star\Sigma \otimes \mathrm{End}(\mathbf{S}_{\frac{3}{2}}))$.

Proof. For $f, g \in \mathcal{C}_1(D, S_{\frac{3}{2}})$, we have

$$d(g, d\sigma f) = (\nabla^S g, d\sigma f) + (-)^m (g, \nabla^S(d\sigma f))\,, \tag{23}$$

where the analogy of (19) for Riemannian manifolds (given by the lift of Levi-Civita connection to spinor bundle)

$$d(\omega_1 \wedge \omega_2) = (\nabla\omega_1 \wedge \omega_2) + (-)^{|\omega_1|}(\omega_1 \wedge \nabla\omega_2)\,. \tag{24}$$

has been used. We understand the action of ∇^S on $S_{\frac{3}{2}}$-valued form $d\sigma f$ in the sense of unique extension of spinor covariant derivative ∇^S from $S_{\frac{3}{2}}$-valued functions (sections) to $S_{\frac{3}{2}}$-valued forms $\Lambda^\star(T^\star\Sigma) \otimes S_{\frac{3}{2}}$, i.e.

$$\nabla^S(d\sigma f) \in \Gamma(\Lambda^{m+1}(T^\star\Sigma) \otimes S_{\frac{3}{2}}) \simeq \Gamma(S_{\frac{3}{2}}) \tag{25}$$

due to the orientability of Σ. Using the notation, in local chart $\{x_0, \ldots, x_m\}$, $dx = dx_0 \wedge \cdots \wedge dx_m$ we have for each $i = 0, \ldots, m$

$$\begin{aligned} dx_i \wedge d\sigma &= \rho(e_i)\, dx\,, \\ d\sigma \wedge dx_i &= (-)^m \rho(e_i)\, dx\,, \end{aligned} \tag{26}$$

which implies immediately

$$\begin{aligned} (dg \wedge d\sigma, f) &= (\mathcal{R}_S g, f)\, d\Sigma\,, \\ (-)^m (g d\sigma \wedge df) &= (g, \mathcal{R}_S f)\, d\Sigma\,. \end{aligned} \tag{27}$$

Now Spin-invariance of the scalar product, $(g, \rho f) = (\rho g, f)$, immediately implies Eq.(22). □

The Stokes Theorem implies analogy of Cauchy's Theorem for $S_{\frac{3}{2}}$-valued monogenic functions.

Corollary 5.3 (Cauchy). *Let us assume the same as in the Stokes Theorem and in addition let $\mathcal{R}_S g = 0 = \mathcal{R}_S f$. Then*

$$\int_{\partial\Sigma}(g, d\sigma f) = 0\,. \tag{28}$$

Remark 5.4. In the case of the domain $D \subset \mathbb{R}^n, \dim(D) = n$ (i.e. instead of the manifold M), where the valuation Rarita-Schwinger module $S_{\frac{3}{2}}$ is just a fixed vector space, we need not use the scalar product of f, g, but can use any Spin-equivariant pairing. The integration in Stokes Theorem is then component-wise in $S_{\frac{3}{2}}$. The corollaries of Stokes Theorem are then formulated in terms of $S_{\frac{3}{2}}$-monogenic functions.

6. Residue theory for $S_{\frac{3}{2}}$-valued monogenic function

This is a generalization of Leray-Norguet residue theory for real submanifolds of any codimension in the sense of spinor monogenic functions ([1], Chapter IV, p.382) towards $S_{\frac{3}{2}}$-valued monogenic functions.

6.1. **Leray-Norguet residues.** In this subsection we describe Leray-Norguet residues for $\mathbb{C}$-valued differential forms. In the next subsection this approach will be generalized to $S_{\frac{3}{2}}$-valued differential forms. All cohomology groups appearing in this subsection are de Rham cohomology groups (as a dual to $\mathbb{C}$-valued singular homology groups).

Let $M, \dim(M) = m$, be an oriented manifold and $N \subset M, \dim N = n$, be its oriented submanifold. An oriented tubular neighborhood of N in M is a pair (f, B), where (B, π, N) is an oriented vector bundle over N and $f : B \to M$ is an embedding with properties:

- f preserves the orientation;
- $f|_N = Id_S$ (S is identified with zero section of E);
- $f(B)$ is an open neighborhood of $N \subset M$.

An oriented closed tubular neighborhood of radius $\epsilon > 0$ of $N \subset M$ is the image $f(B_\epsilon)$ determined by the tubular neighbourhood (f, B_ϵ), where $B_\epsilon := \{b \in B \,:\, ||b|| \leq \epsilon,\, \epsilon > 0\}$ is ϵ-disk bundle of the oriented vector bundle B.

There exists an oriented tubular neighborhood and any two oriented closed tubular neighborhoods are isotopic, see for example [1], Appendix A, p. 435.

Let us denote by $S_\epsilon := \{b \in B \,:\, ||b|| = \epsilon,\, \epsilon > 0\}$ sphere bundle of B (it should be understood as $S_\epsilon = \partial B_\epsilon$). Let $p \geq m - n - 1$ and $U_\epsilon := f(B_\epsilon)$. Then the map $\pi \circ f^{-1} : U_\epsilon \to N$ induces push-forward map on smooth sections

$$(\pi f^{-1})_\star \,:\, \Gamma(\partial U_\epsilon, \Lambda^p) \longrightarrow \Gamma(N, \Lambda^{p-(m-n-1)}) \tag{29}$$

and due to the fact that $d(\pi f^{-1})_\star = (\pi f^{-1})_\star d$ also induces a map on de Rham cohomologies

$$(\pi f^{-1})_\star \,:\, H^p(\partial U_\epsilon) \longrightarrow H^{p-(m-n-1)}(N)\,. \tag{30}$$

Let us define canonical inclusion

$$i \,:\, \partial U_\epsilon \longrightarrow M \setminus N\,. \tag{31}$$

Then it is easy to see that the composition $\mathrm{Res} := \pi_\star i^\star$ is independent of all possible choices mentioned above.

Definition 6.1. Let $p \geq m - n - 1$. The map

$$\mathrm{Res} := \pi_\star i^\star \;:\; H^p(M \setminus N) \longrightarrow H^{p-(m-n-1)}(N) \tag{32}$$

is called the Leray-Norguet residue. The dual (boundary) map to Res (for $q \leq n$)

$$\delta \,:\, H_q(N) \longrightarrow H_{q+(m-n-1)}(M \setminus N) \tag{33}$$

is called Lerray-Norguet cobord.

The Lerray-Norguet residue theorem utilizes all previous results and definitions.

Theorem 6.2 (Lerray-Norguet residue theorem). *Let* $0 \leq p \leq n$. *Then for each closed* $(p+m-n-1)$*-form* ω *on* $M \setminus N$ *and each p-dimensional cycle* $\Sigma \subset N$

$$\int_{\delta\Sigma} \omega = \int_{\Sigma} \mathrm{Res}(\omega) . \tag{34}$$

holds true.

6.2. **Leray-Norguet residues for $S_{\frac{3}{2}}$-valued monogenic function.** We shall restrict to the formulation of residue theory for $S_{\frac{3}{2}}$-valued monogenic functions. However, note that no special property concerning Spin-module $S_{\frac{3}{2}}$ will be needed. In other words, presented machinery can be equally applied to any other Spin-module.

Let f be a (left) $S_{\frac{3}{2}}$-valued monogenic function. Then $\omega = d\sigma f$ is closed $S_{\frac{3}{2}}$-valued $(m-1)$-form.

Definition 6.3. Let $D \subset \mathbb{R}^m$ be a domain and $\Sigma \subset D$ a compact smooth oriented n-dimensional submanifold (n=0, ... ,m-2). If $\omega = d\sigma f$ for $S_{\frac{3}{2}}$-valued monogenic function $f \in M(D \setminus \Sigma)$ and $\mathrm{Res}\,(\omega) \in H^n(\Sigma)$ its Leray-Norguet residue, then the $\mathbb{C}$-number

$$\mathrm{res}_{\Sigma}(\omega) := \int_{\Sigma} \mathrm{Res}\,(\omega) \tag{35}$$

is called the residue of $S_{\frac{3}{2}}$-valued form ω on the submanifold Σ.

Note that the residues $\mathrm{Res}\,(\omega)$ and $\mathrm{res}\,(\omega)$ carry the same information, because $H^n(\Sigma) \simeq \mathbb{C}$.

Now we are able to state residue theorem, which computes global characteristic of ω in terms of local information stored in residues of ω.

Theorem 6.4 (Residue theorem). *Let* $D \subset \mathbb{R}^m$ *be a domain and* $D' \subset D$ *a relatively compact subdomain with a smooth closure* $\partial D'$. *Let* Σ_i, $i \in I$, $|I| < \infty$ *be a finite family of pairwise disjoint compact submanifolds of* D', *whose dimensions belong to the set* $\{0, \ldots, m-2\}$. *Let* $\omega = d\sigma f$, $f \in M(D \setminus (\cup_i \Sigma_i))$ *be* $S_{\frac{3}{2}}$*-valued monogenic function. Then*

$$\int_{\partial\Sigma'} \omega = \sum_{i \in I} \mathrm{res}_{\Sigma_i}(\omega) \tag{36}$$

Proof. The proof is standard and follows the case of spinor valued monogenic functions, see [1], p.384. Using suitable tubular neighborhoods of Σ_i $(i \in I)$, one applies Stokes theorem and the rest is trivial. □

Remark 6.5. Note that we have formulated Residue theorem only on a domain D and not on manifolds as in the case of Stokes theorem. The reason is that the residue is not invariant w.r. to the action of group Spin (in the sense of transition functions etc.).

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