Ercüment Ortaçgil

A remark on the topology of higher order frames

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 23rd Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2004. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 72. pp. [171]--176.

Persistent URL: http://dml.cz/dmlcz/701733

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 72 (2004), pp. 171-176

A REMARK ON THE TOPOLOGY OF HIGHER ORDER FRAMES

ERCÜMENT ORTAÇGIL

ABSTRACT. We study the problem whether 1^{st} order flatness of a 2^{nd} order G-structure implies its 2^{nd} order flatness. We give a sufficient condition for the existence of 2^{nd} order flat lifts. We indicate that certain characteristic classes can be defined which seem to be topological obstructions for the existence of 2^{nd} order flat lifts. The problem whether these classes can be nontrivial requires further study.

Let M be a differentiable manifold and F^k be the set of k-jets of *all local* diffeomorphisms of M with source at some fixed point $o \in M$ and target at any point of M. Projecting elements of F^k to their targets, we obtain a right principal bundle $F^k \to M$, whose group GL_k is the set of all k-jets with source and target at o and GL_k acts on F^k by jet composition on the source. $F^k \to M$ can be identified with the k^{th} order frame bundle of M. A k^{th} order G-structure on M may be viewed as a subbundle of $F^k \to M$. Now let $P^2 \to M$ be a 2^{nd} order G^2-structure on M, $G^2 \subset GL_2$. $P^2 \to M$ defines a 1^{st} order G^1-structure $P^1 \to M$, $G^1 \subset GL_1$. We have the projection homomorphism $G^2 \to G^1$ and also the bundle projection $P^2 \to P^1$. We will be interested here in two questions:

Q1: Is the bundle $P^2 \to P^1$ always trivial?

A connection on $P^2 \to M$ projects to a connection on $P^1 \to M$. Converse is also true: A connection on $P^1 \to M$ lifts to some connection on $P^2 \to M$. To see this, we start with a connection ω on $P^1 \to M$ and piece together the local lifts of ω by a partition of unity which recovers ω on $P^1 \to M$ and gives a lift of ω to $P^2 \to M$. We refer to [8] for technical details.

Q2: Let ω_1 be a flat connection on $P^1 \to M$. Does ω_1 always lift to some flat ω_2 on $P^2 \to M$?

To give more substance to the above questions, we will first consider a rich source of examples supplied by homogeneous spaces $M = G/H$. Some $a \in G$ defines a diffeomorphism $\overline{a}$ of M which maps xH to axH and the map $G \to Diff(M)$ sending a to $\overline{a}$ is a continuous homomorphism with kernel K. If K is trivial, the action of G

2000 *Mathematics Subject Classification*: 53C10, 57R20.
Key words and phrases: G-structures, gauge sequence, characteristic classes.
The paper is in final form and no version of it will be submitted elsewhere.

on M is called effective and (G, H) is called an effective pair. K is the largest normal subgroup of G contained in H. Let $j_k(\overline{a})(xH, axH)$ denote the k-jet of $\overline{a}$ with source at xH and target axH. In this way we obtain a k^{th} order transitive Lie groupoid Θ^k on M. Note that the elements of Θ^k are induced by global diffeomorphisms which may not be the case for an arbitrary Lie groupoid. Fixing the source of jets of Θ^k at $o = H$, we obtain a k^{th} order G^k-structure $P^k \to M$. For a more explicit description of $P^k \to M$, first assume $k = 1$. Now $\overline{h}$ fixes o and we obtain the 1^{st} order isotropy representation $h \to j_1(\overline{h})(o, o)$ with kernel $K_1 \subset H$. Clearly $K \subset K_1$.

Proposition 1. *The bundle $P^1 \to M$ coincides with the homogeneous bundle $G/K_1 \to G/H$.*

Note that $G/L \to G/H$ is always a principal bundle with group H/L if $L \lhd H$. The new ingredient in Proposition 1 is that this bundle can be realized as a 1^{st} order G-structure if $L = K_1$.

Proposition 1 is contained, for instance, in [6] but we will recall its proof here to see that it generalizes to the case $k \geq 1$ in a straightforward way. We first define a map $i : G/K_1 \to F^1$ by $i(aK_1) = j_1(\overline{a})(o, aH)$. Using definitions, it is easily checked that i is well defined, injective and its image is clearly P^1 defined above. Further, i injects H/K_1 isomorphically into GL_1 with image G^1. H/K_1 acts on G/K_1 on the right by $(aK_1)(hK_1) = ahK_1$. It is easily checked that this action is well defined and i commutes with the actions of H/K_1 on G/K_1 and GL_1 on F^1 respectively. Finally, we check that i commutes with projections $G/K_1 \to G/H$ and $F^1 \to M$ respectively and i is continuous.

If K_1 is trivial, then all diffeomorphisms $\overline{a}$ are uniquely determined by their 1^{st} order jets and the 1^{st} order isotropy representation is faithful. This is the case, for instance, if i) H is discrete ii) H is compact iii) G/H is reductive (see [3], pg. 190–199). However, for many important homogeneous spaces K_1 is not trivial as we will henceforth assume. Consider now the 2^{nd} order isotropy representation $h \to j_2(\overline{h})(o, o)$ with kernel $K_2 \subset K_1$. Clearly $K \subset K_2$. As in the proof of Proposition 1, we define a map $j : G/K_2 \to F^2$ by $j(aK_2) = j_2(\overline{a})(o, aH)$. Repeating the steps in the proof of Proposition 1 with i replaced by j, we see that $G/K_2 \to G/H$ is a 2^{nd} order H/K_2-structure and we also have the projection of principal bundles $G/K_2 \to G/K_1$ with group K_1/K_2. Iterating this construction, we obtain a decreasing sequence of normal subgroups $H \supset K_1 \supset K_2 \supset \cdots \supset K$ and this sequence stabilizes at some $K_{m+1} = K$ (see [10], pg. 161–164). From now on we will assume that (G, H) is an effective pair so that $K = \{e\}$. In this way we obtain the following sequence of principal bundles

$$
\begin{array}{ccccccccccc}
G & \to & G/K_m & \to & G/K_{m-1} & \to & \dots & \to & G/K_1 & \to & G/H \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
F^{m+1} & \to & F^m & \to & F^{m-1} & \to & \dots & \to & F^1 & \to & M
\end{array} \tag{1}
$$

where the vertical arrows are injections of principal bundles. In particular, $G \to G/H$ can be realized as a $(m+1)^{\text{th}}$ order H-structure and a connection on $G \to G/H$ is a $(m+1)^{\text{th}}$ order connection. We therefore see that even the most classical examples of principal bundles sometimes implicitly involve higher order jets. The natural question arises whether these higher order jets have any topological implication.

As for the groups of these principal bundles, we have the commutative diagram

$$
\begin{array}{ccccccccccc}
H & \rightarrow & H/K_m & \rightarrow & \dots & \rightarrow & H/K_2 & \rightarrow & H/K_1 & \rightarrow & e \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
GL_{m+1} & & GL_m & \rightarrow & \dots & \rightarrow & GL_2 & \rightarrow & GL_1 & \rightarrow & e
\end{array} \tag{2}
$$

where the vertical arrows are injective homomorphisms.

Let KL_i be the kernel of $GL_i \rightarrow GL_{i-1}$, $i \geq 2$. Then KL_i is a vector space and K_i/K_{i+1} is a closed subgroup of KL_i. Therefore K_i/K_{i+1} is either a lattice, or a subspace or a direct sum of two such subgroups.

For simplicity, we now assume $m = 1$, i.e., $K_2 = 0$. We will consider two cases:

a) K_1 is a lattice,

b) K_1 is a subspace.

An effective pair satisfying a) (it is not difficult to give explicit examples) gives a remarkable subgroup $H \subset GL_2$: In this case (2) reduces to

$$
\begin{array}{ccccccccc}
0 & \rightarrow & K_1 & \rightarrow & H & \rightarrow & H/K_1 & \rightarrow & e \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & KL_2 & \rightarrow & GL_2 & \rightarrow & GL_1 & \rightarrow & e
\end{array} \tag{3}
$$

and the principal bundle $H \rightarrow H/K_1$ is a covering map which is not trivial if H is connected. Thus we arrive at an example where 2^nd^ order jets carry nontrivial topology. Observe that the upper sequence in (3) can not split in this case for otherwise H/K_1 would be a subgroup of H and translating an element in the fiber by the subgroup H/K_1 would trivialize the covering map and disconnect H. This problem of splitting will be more important below when we consider b).

As for the corresponding G-structures, (1) reduces to

$$
\begin{array}{ccccc}
G & \rightarrow & G/K_1 & \rightarrow & G/H \\
\downarrow & & \downarrow & & \downarrow \mathrm{id} \\
F^2 & \rightarrow & F^1 & \rightarrow & M
\end{array} \tag{4}
$$

and again $G \rightarrow G/K_1$ is a nontrivial covering if G is connected. It follows therefore that the answer to **Q1** is negative. However, in this case, it is easy to see that the answer to **Q2** is affirmative: In fact, all lifts of a flat ω_1 are flat in this case.

We now assume b). Note that the fibers of $G \rightarrow G/K_1$ are contractible and the bundle admits a cross section in this case. We will first state

Proposition 2. *Any flat connection on $F^1 \rightarrow M$ lifts to a flat connection on $F^2 \rightarrow M$.*

M is not necessarily a homogeneous space in Proposition 2. To prove it, we observe that the lower sequence in (3) splits and GL_1 is a subgroup of GL_2. This splitting allows us to construct sections of $F^2 \rightarrow F^1$ which are special: They are invariant under the action of GL_1. Such sections are called ε-connections in the literature (ε standing for Ehresmann) and are in 1-1 correspondence with torsion free connections on $F^1 \rightarrow M$ (see [1], [4], [7], [11], [5] pg. 105-115). Consequently, a choice of an

ε-connection imbeds F^1 in F^2 as a principal subbundle and Proposition 2 follows by extending the flat connection on F^1 to a flat connection on F^2.

Note that an arbitrary section of $F^2 \to F^1$ does not help to find a flat lift. To see this, suppose we fix such a section s. Let $f : U \to F^1$ be a local section of F^1 such that $f(U)$ is an integral manifold of the flat connection ω_1 on F^1. If $g : U \to F^1$ is another such section, then $f(x) = g(x)a$ for some $a \in GL_1$ which does not depend on $x \in U$. It is now natural to consider the compositions of such sections with s and consider sections of $F^2 \to F^1$ of the form $s \circ f$. Now $s \circ f(x) = (s \circ g(x))a(x)$ for some $a(x) \in GL_2$, however there is no a priori reason why $a(x)$ should be constant even though its projection in GL_1 is. Thus we see that the problem of lifting a flat ω_1 involves more than the triviality of $F^2 \to F^1$.

The above argument gives also a proof of the following

Proposition 3. *If the upper sequence in (3) splits, then the answer to* ***Q2*** *is affirmative.*

The proof of Proposition 3 works for general G-structures and not necessarily for homogeneous spaces. The construction of an ε-connection with the assumption of the existence of a splitting of $0 \to K^2 \to G^2 \to G^1 \to e$ is straightforward ([8]). Note, however, that an ε-connection does not necessarily define a torsion free connection on $P^1 \to M$ unless P^2 is contained in the 1st jet extension J^1P^1. In fact, $P^1 \to M$ may not admit a torsion free connection. In this direction, we have

Proposition 4. *If $P^2 \subset J^1P^1$, then the answer to* ***Q2*** *is affirmative.*

The proof of Proposition 4 consists in showing that the assumption $P^2 \subset J^1P^1$ forces the sequence $0 \to K^2 \to G^2 \to G^1 \to e$ to split which is not difficult. In fact, if $P^1 \to M$ is uniformly 1-flat (see [1], [8] for the definition of uniform 1-flatness), then we can define its prolongation $\mathrm{prP}^1 \subset J^1P^1$ and any (flat) connection on P^1 uniquely prolongs to some (flat) connection on prP^1 ([8]). If $P^2 \subset J^1P^1$, then P^1 is uniformly 1-flat and $P^2 \subset \mathrm{prP}^1$ by the definition of prP^1. Now the unique flat lift reduces to P^2.

However, note that the flat lifts given by ε-connections are very special: If ω_2 lifts ω_1, then the holonomy group $\mathrm{Hol}_1 \subset G^1$ of ω_1 is isomorphic to the holonomy group $\mathrm{Hol}_2 \subset G^2$ because the fiber of $\mathrm{Hol}_2 \to \mathrm{Hol}_1$ consists of a single point, which need not be the case in general.

The above arguments suggest, we believe, that the answer to **Q2** is not affirmative either. We will now fix a flat ω_1 on P^1, lift it it to some ω_2 and define certain characteristic classes from the curvature R_2 of ω_2 which do not depend on the lift. By construction, these classes vanish if there exists a flat lift. These classes seem to be richer than the ones given by the usual Chern-Weil construction by means of an arbitrary ω_2 on P^2 as they will be defined only when ω_1 is flat.

For this purpose, consider $gkg^{-1}$where $k \in K^2$ and $g \in G^2$. Then $gkg^{-1} \in K^2$ since $K^2 \lhd G^2$. Since K^2 is also abelian, gkg^{-1} depends only on the image $\pi(g)$ of g where $\pi : G^2 \to G^1$ is the projection. Thus we have an action of G^1 on K^2 and therefore an action $*$ of G^1 on the abelian Lie algebra k^2 of K^2. Now let ω_1 be some fixed flat connection on $P^1 \to M$ and ω_2 be some arbitrary lift to $P^2 \to M$. The curvature R_2 of ω_2 takes values in the Lie algebra g^2 of G^2 and projects to the curvature R_1 of ω_1. Since $R_1 = 0$, R_2 takes values actually in k^2. By the definition of $*$, R_2 transforms by $\pi(g) * R_2$ under the action of $g \in G^2$. Let $I^m(G^1, k^2)$ be the space of symmetric m-linear maps $k^2 \times ... \times k^2 \to \mathbf{R}$ which are invariant under $*$. Now any $Q \in I^m(G^1, k^2)$ defines a $2m$-form $Q(\omega_2)$ in a standard way. It is not difficult to check that $Q(\omega_2)$ is horizontal and pulls back to closed form on M which is independent of the lift ω_2.

Let $I^m(G^2, g^2)$ be the space of symmetric m-linear maps $g^2 \times ... \times g^2 \to \mathbf{R}$ which are invariant under the action of $Ad(G^2)$. Note that there is a restriction map $\theta : I^m(G^2, g^2) \to I^m(G^1, k^2)$ and the surjectivity of θ implies that the above classes are nothing but the ones obtained by the Chern-Weil construction in terms of an arbitrary connection on P^2. In this case, it seems to us that the above classes vanish.

The following question therefore seems to be of significance:

Q3: Can the above classes be nontrivial?

More explicit constructions are possible in some special cases, for instance, when G^1 is contained in the orthogonal group $O(n)$ (see [9]).

The above construction of characteristic classes becomes particularly transparent and elementary once formulated in terms of the gauge sequence worked out in [7], [8]. We will shortly recall this sequence here to formulate a more general form of **Q2** which has been our starting point.

Any transitive Lie groupoid of order k on M defines a k^{th} order G-structure $P^k \to M$ by fixing the source (or target) of jets. All G-structures arise in this way. For any such principal bundle we have the exact sequence

$$P^k \to CA^k \to \wedge^2(T^*) \otimes g^k \tag{5}$$

called gauge sequence. The sequence (5) can be read as

$$\text{bundle} \to \text{connection} \to \text{curvature} \tag{6}$$

In fact, (5) can be taken as the definition of connection and curvature for such principal bundles. For the details for this sequence we refer to [7], [8] and the references therein. The curvature operator in (5) has components $(R_1, \dots, R_k)$ in local coordinates and R_1 is the classical curvature tensor. For simplicity, let $k = 2$. If 2^{nd} order jets contain any topology which is not contained in 1^{st} order jets which can be deduced from a 2^{nd} order connection, it is natural to expect that the new information is contained in R_2. However, R_2 has no invariant meaning but only (R_1, R_2) does. However, R_2 becomes an object by itself if $R_1 = 0$ which explains **Q2**. We therefore have

Q4: Can a 2^{nd} order connection contain any topology which is not contained in a 1^{st} order connection?

As we hinted above, it is not possible to formulate **Q4** within the framework of general principal bundles and connections because in this framework a connection is unable to detect the order of jets but treats them all together as a single object.

Finally, we will shortly indicate the relation of **Q2** to secondary characteristic classes (see, for instance, [2]). Let $P \to M$ be an arbitrary principal bundle and $Q \to M$ be a subbundle. A flat connection ω on $P \to M$ defines certain (P, Q)-characteristic classes which vanish if ω reduces to Q. Now, the flat connection ω_1 which we assumed on $P^1 \to M$ extends to a flat $\widetilde{\omega}_1$ on $F^1 \to M$ which lifts to some $\widetilde{\omega}_2$ on $F^2 \to M$ by Proposition 2. **Q2** is therefore whether any such $\widetilde{\omega}_2$ reduces to $P^2 \to M$.

References

[1] Guillemin, V., *The integrability problem for G-structures*, Trans. A.M.S. **116** (1965), 544–560.

[2] Kamber, F., Tondeur, P., *Foliated Bundles and Characteristic Classes*, Lecture Notes in Mathematics, **493** Springer Verlag, 1975.

[3] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*. Vol. II, J. Wiley-Interscience, 1969.

[4] Kolar, I., *Torsionfree connections on higher order frame bundles*, New Developments in Differential Geometry, Debrecen 1994, 233–241.

[5] Morita, S., *Geometry of Characteristic Classes*, Translations of Mathematical Monographs, A.M.S., Vol. **199** (1999).

[6] Ochiai, T., *Geometry associated with semisimple flat homogeneous spaces*, Transactions of A.M.S., Vol: **152**(1970), 159–193.

[7] Ortaçgil, E., *On a digram in the theory of connections*, Proc. Conf. of DGA 1998, Brno, Czech Republic 1999, 273–283.

[8] Ortaçgil, E., *Second order connections via gauge sequence*, preprint.

[9] Ortaçgil, E., *On a cohomology class defined by a flat hermitian structure*, preprint.

[10] Sharpe, R. W., *Differential Geometry, Cartan's Generalization of Klein's Erlangen Program*, Springer Verlag, Graduate Texts in Mathematics, **166**, 1997.

[11] Yuen, P. C., *Higher order frames and linear connections*, Cahiers Topol. Geom. Diff. **12** (1971), 333–371.

Boğaziçi University, Mathematics Department
Bebek, 34342, Istanbul, Türkiye
E-mail: ortacgil@boun.edu.tr