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ON THREE-DIMENSIONAL HYPERSURFACES WITH TYPE NUMBER TWO IN $\mathbb{H}^4$ AND $\mathbb{S}^4$ TREATED IN INTRINSIC WAY

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ABSTRACT. The analogous problem for $\mathbb{R}^4$ was treated in [2, Chapter 10], under the key-words "local rigidity problems". In the present paper we consider three-dimensional Riemannian manifolds of c-conullity two as in [12] with their natural classification according to the number of asymptotic foliations. We limit ourselves to the case $c < 0$, and we study the isometric immersions of the corresponding classes of manifolds in $\mathbb{H}^4$. We show how the maximal number of nontrivial isometric deformations of the immersed manifold corresponds to its number of asymptotic foliations. The case $c > 0$ (corresponding to $\mathbb{S}^4$) is very similar and not treated explicitly. The original study of isometric deformations of the hypersurfaces in title using the purely extrinsic methods can be traced back to V. Sbrana and E. Cartan.

INTRODUCTION

The Riemannian manifolds of *constant conullity two* are of special interest. These are manifolds (M, g) such that the tangent space T_xM at any point $x \in M$ admits an orthogonal decomposition

$$T_xM = T_x^1M + T_x^0M\,,$$

where $\dim T_x^1M = 2$ and T_x^0M is the nullity space of the curvature tensor R_x.

First, it is known that all Riemannian manifolds of conullity two are *semi-symmetric spaces, i.e.,* $R(X,Y)\cdot R = 0$ holds for every two vector fields X and Y, where the dot denotes the derivation on the algebra of all tensor fields on M. Because the nullity distribution $\{T_x^0M\}_{x\in M}$ determines a totally geodesic and locally Euclidean foliation, we call these spaces *foliated semi-symmetric spaces.* (Other types of semi-symmetric spaces are locally symmetric spaces, two-dimensional surfaces and the "Szabó cones"— see the fundamental papers by Z. Szabó [18], [19], [20].) The foliated semi-symmetric spaces have been studied deeper and in a much more explicit form by the second

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author [9], later by E. Boeckx [1] and finally, the basic results have been summarized in [2].

The second interesting feature is the following one: a classical result (known already by W. Killing) says that a hypersurface $M \subset \mathbb{R}^{n+1}$ $(n \geq 3)$ with type number $t(p) \geq 3$ everywhere is locally rigid (*cf.* [8, Chapter VII]). The same result holds if one replaces $\mathbb{R}^{n+1}$ by any space of constant curvature $\tilde{c}$. Now, the complete connected hypersurfaces of $\mathbb{R}^{n+1}$ with type number zero are hyperplanes and those with type number one are generalized cylinders. They are always isometrically deformable (in a continuous way). On the other hand, the local metric properties of hypersurfaces with type number two were studied thoroughly by E. Cartan [3]. He proved (by purely extrinsic methods) that just the following three possibilities can occur:

a) M is locally isometrically deformable (in a continuous way),
b) M is locally rigid,
c) M is locally deformable in a unique way: to every sufficiently small domain $\Sigma \subset M$ there is, up to rigid motions and reflections, exactly one hypersurface $\bar{\Sigma} \subset \mathbb{R}^{n+1}$ which is isometric to Σ and not congruent to it.

In the doctoral Thesis by the first author [6] (see also [2, Chapter 10]) a different approach was used to construct new classes of examples satisfying a), b) or c). The method is based on the explicit description of three-dimensional Riemannian manifolds with conullity two as studied in [9]. Later in the same year, Dajczer, Florit and Tojeiro [4] have given a revised version of the Cartan's paper (with the generalization to arbitrary space forms) and they also constructed examples of hypersurfaces of space forms with the property c) using purely extrinsic methods.

Now, the natural extension of [6] is the intrinsic study of hypersurfaces with type number two in general space forms. The induced metrics of such hypersurfaces make them Riemannian manifolds of $\tilde{c}$-conullity two (see Definition A in the next section). The subject of extensive research of the present authors was the study of *three-dimensional Riemannian manifolds of $\tilde{c}$-conullity two with respect to some nonzero constant* $\tilde{c} \in \mathbb{R}$. This class is, of course, much broader than that coming from immersed hypersurfaces with type number two in four-dimensional space forms of constant curvature $\tilde{c}$. It can be also characterized by the property that two principal Ricci curvatures ρ_1 and ρ_2 are equal, and may depend on the point, and the last one is equal to the constant $2\tilde{c}$. (All these spaces belong to a broader class of so-called pseudo-symmetric spaces.) The second and the third authors have calculated explicit formulas for these metrics in the cases of so-called nonelliptic types (see [12]–[16]).

The aim of the present paper is to use these formulas and the Cauchy-Kowalewski Theorem for deriving existence theorems concerning the number of hypersurfaces of type number two in four-dimensional space forms. Here the different types a), b) and c) from the Cartan's classification occur explicitly.

As we will repeatedly apply the Cauchy-Kowalewski Theorem for partial differential equations in the present paper, we will always consider the real analytic case.

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1. Classical results

We start by recalling some fundamental results on hypersurfaces in spaces of constant curvature.

Let $\tilde{M}^{n+1}(\tilde{c}) = (\tilde{M}^{n+1}, \tilde{g})$ be a $(n+1)$-dimensional Riemannian space of constant curvature $\tilde{c}$, and $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{M}^{n+1}(\tilde{c})$. Then the Riemannian curvature tensor $\tilde{R}$ of $\tilde{M}^{n+1}(\tilde{c})$ satisfies

$$\tilde{R}(X,Y)Z = \tilde{c}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\}$$

for all vector fields X, Y and Z on $\tilde{M}^{n+1}$.

Now let (M,g) be a connected hypersurface of $\tilde{M}^{n+1}(\tilde{c})$ with the Riemannian metric g induced from $\tilde{g}$, and U a local unit normal vector field of (M,g) in $\tilde{M}^{n+1}(\tilde{c})$. Then the *shape operator* S of M derived from U is a linear operator of the tangent bundle TM defined by

$$SX = -\tilde{\nabla}_X U$$

for all $X \in TM$. It is a local tensor field of type (1,1) on M (determined uniquely up to a sign), which is symmetric with respect to the metric g. We denote by ∇ and R the Levi-Civita connection and the Riemannian curvature tensor of (M,g), respectively. The shape operator S satisfies the *Gauss equation*

$$\begin{aligned} R(X,Y)Z = \tilde{c}\{g(Y,Z)X - g(X,Z)Y\} \\ + g(SY,Z)SX - g(SX,Z)SY \end{aligned} \tag{1.1}$$

and the *Codazzi equation*

$$(\nabla_X S)Y = (\nabla_Y S)X \tag{1.2}$$

for all $X, Y, Z \in TM$. Conversely, prescribing a local tensor field S satisfying (1.1) and (1.2) on an n-dimensional Riemannian manifold (M,g) defines locally an isometric embedding of (M,g) into an $(n+1)$-dimensional Riemannian space $\tilde{M}^{n+1}(\tilde{c})$ of constant curvature $\tilde{c}$. Such an embedding is unique up to an isometry of $\tilde{M}^{n+1}(\tilde{c})$. See [8, Chapter VII] for more details.

Given a Riemannian manifold (M,g), it is natural to ask in how many different (*i.e.*, non-congruent) ways it can be locally embedded as a hypersurface in $\tilde{M}^{n+1}(\tilde{c})$.

Definition A. Let (M,g) and (M',g') be two hypersurfaces embedded in $\tilde{M}^{n+1}(\tilde{c})$. An isometry φ of (M,g) to (M',g') is an *isometric deformation* of (M,g) if there is no isometry Φ of $\tilde{M}^{n+1}(\tilde{c})$ onto itself such that $\Phi|_M = \varphi$. The hypersurface (M,g) is a (*locally*) *rigid hypersurface* of $\tilde{M}^{n+1}(\tilde{c})$ if there are no (local) deformations of (M,g).

Definition B. For any Riemannian manifold (M,g) and any constant $\tilde{c}$ define a tensor $R_{\tilde{c}}$ by the formula

$$R_{\tilde{c}}(X,Y)Z = \tilde{c}\{g(Y,Z)X - g(X,Z)Y\},$$

where $X, Y, Z \in TM$. Then, for any Riemannian manifold (M,g), we shall define the *$\tilde{c}$-nullity space at $p \in M$ (with respect to $\tilde{c} \in \mathbb{R}$)* as the linear subspace

$$T_{p,\tilde{c}}M = \{X \in T_pM \mid (R - R_{\tilde{c}})(X,Y)Z = 0 \quad \text{for all} \quad Y, Z \in T_pM\}.$$

(Recall that $R - R_{\tilde{c}} = 0$ if and only if (M, g) is a space of constant curvature $\tilde{c}$). Then the dimension $\nu_{\tilde{c}}(p)$ of $T_{p,\tilde{c}}M$ will be called the *$\tilde{c}$-nullity of (M, g) at p*, and the number $\dim M - \nu_{\tilde{c}}(p)$ the *$\tilde{c}$-conullity of (M, g) at p*.

The *type number* $t(p)$ at a point p of an embedded hypersurface $(M, g) \subset \tilde{M}^{n+1}(\tilde{c})$ is the rank of the shape operator S_p. Clearly, the type number does not depend on the choice of the local unit normal vector field and thus it is uniquely determined at each point of M.

It is easy to prove the following

Theorem C. Let (M, g) be a hypersurface with the metric g induced from the metric of a space $\tilde{M}^{n+1}(\tilde{c}) = (\tilde{M}^{n+1}, \tilde{g})$ of constant curvature $\tilde{c}$. Then for any point $p \in M$,

(1) $t(p) = 0$ or 1 if and only if (M, g) has constant curvature $\tilde{c}$ at p.
(2) If $t(p) \geq 2$, then $t(p) = n - \nu_{\tilde{c}}(p) =$ be the $\tilde{c}$-conullity at p.

In particular, this implies that at each point in M where R is different from $R_{\tilde{c}}$, the type number is the same for every local isometric embedding of (M, g) into $\tilde{M}^{n+1}(\tilde{c})$.

2. Fundamental equations

Let (M, g) be a three-dimensional Riemannian manifold of $\tilde{c}$-conullity two, $\tilde{c} \neq 0$, which is called *a pseudo-symmetric space of constant type with constant $\tilde{c}$*. Then its Ricci tensor $\hat{R}$ has eigenvalues $\rho_1 = \rho_2 \neq \rho_3$, $\rho_3 = 2\tilde{c}$. Now let $\{E_1, E_2, E_3\}$ be a local orthonormal moving frame such that the Ricci tensor $\hat{R}$ is expressed in the form $\hat{R}_{ij} = \rho_i \delta_{ij}$, $i, j = 1, 2, 3$, where δ_{ij} is the Kronecker's delta and let $\{\omega^1, \omega^2, \omega^3\}$ be the coframe dual to $\{E_1, E_2, E_3\}$. Then, in a normal neighborhood $\mathcal{U}$ of any point $p \in M$, there exists a local coordinate system (w, x, y) such that

$$
(2.1) \qquad \begin{cases} \omega^1 = f\,\mathrm{d}w\,, \\ \omega^2 = A\,\mathrm{d}x + C\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + H\,\mathrm{d}w\,, \end{cases}
$$

where f, A and C are smooth functions of the variables w, x and y, $fA \neq 0$, and H is a smooth function of the variables w and x ([14] and [13]). The frame $\{E_1, E_2, E_3\}$ is given in terms of the local coordinates (w, x, y) by

$$
(2.2) \qquad \begin{cases} E_1 = \dfrac{1}{f}\dfrac{\partial}{\partial w} - \dfrac{C}{fA}\dfrac{\partial}{\partial x} - \dfrac{H}{f}\dfrac{\partial}{\partial y}\,, \\ E_2 = \dfrac{1}{A}\dfrac{\partial}{\partial x}\,, \\ E_3 = \dfrac{\partial}{\partial y}\,, \end{cases}
$$

and the covariant differentiation is given by

$$
(2.3)\qquad \begin{cases}
\nabla_{E_1}E_1 = -\dfrac{f'_x}{fA}E_2 - aE_3\,, & \nabla_{E_1}E_2 = \dfrac{f'_x}{fA}E_1 - cE_3\,,\\
\nabla_{E_2}E_1 = \alpha E_2 - bE_3\,, & \nabla_{E_2}E_2 = -\alpha E_1 - eE_3\,,\\
\nabla_{E_1}E_3 = aE_1 + cE_2\,, & \nabla_{E_2}E_3 = bE_1 + eE_2\,,\\
\nabla_{E_3}E_1 = -bE_2\,, & \nabla_{E_3}E_2 = bE_1\,,\\
\nabla_{E_3}E_3 = 0\,,
\end{cases}
$$

where

$$
(2.4)\qquad \alpha = \frac{1}{fA}(A'_w - C'_x - HA'_y)\,,
$$

$$
(2.5)\qquad \beta = \frac{1}{2fA}(H'_x + AC'_y - CA'_y)\,,
$$

$$
(2.6)\qquad a = \frac{f'_y}{f}\,,\quad b = \beta\,,\quad c = \beta - \frac{h}{fA}\,,\quad e = \frac{A'_y}{A}
$$

with $h = H'_x$.

For the eigenvalue ρ_1 we have

$$
(2.7)\qquad \rho_1 = -\frac{1}{fA}\Big((A\alpha)'_w + \Big(\frac{f'_x}{A} - C\alpha + H\beta\Big)'_x + (Af'_y + AC\beta)'_y\Big)\,.
$$

The last formula in (2.3) implies that the trajectories of the unit vector field E_3 are geodesics of (M,g). We call them *principal geodesics* of (M,g). A smooth surface $N \subset M$ is called an *asymptotic leaf* if it is generated by the principal geodesics and its tangent planes are parallel along these principal geodesics with respect to the Levi-Civita connection ∇ of (M,g). In a neighbourhood of each asymptotic leaf one has an "asymptotic foliation". The corresponding tangent "asymptotic distribution" is determined by the following quadratic Pfaffian equation (see [11]–[16] for more details):

$$
(2.8)\qquad c(\omega^1)^2 + (e-a)\omega^1\omega^2 - b(\omega^2)^2 = 0\,.
$$

Consider the discriminant $\Delta = (e-a)^2 + 4bc$ of (2.8). We see that there are just four possibilities: $\Delta < 0$, $\Delta > 0$, $\Delta = 0$ and (2.8) is nonvanishing, and finally, (2.8) vanishes identically, *i.e.*, $e - a = b = c = 0$. Hence there are four possibilities for a fixed point in M:

(E) There is no asymptotic leaf through this point.
(H) There are exactly two asymptotic leaves through this point.
(P) There is exactly one asymptotic leaf through this point.
(Pℓ) There are infinitely many asymptotic leaves through this point.

We call these points *elliptic, hyperbolic, parabolic* and *planar* ones, respectively. In the following we are occupied only with the Riemannian manifolds of the "pure" type, *i.e.*, with the same kind of points. We call them manifolds *of type* (E), (H), (P) and (Pℓ) accordingly. The non-elliptic manifolds are also called "asymptotically foliated".

On each manifold of types (H) and (P), there exists a local coordinate system in the form (2.1) annihilating the function β ([15, Theorem 3.6] or [16, Theorem 4.8]). As concerns the type (Pℓ), we have $\beta = 0$ by definition. Thus for every foliated pseudo-symmetric space of constant type we can assume $\beta = 0$. On every such space, at least one of the asymptotic foliations is totally geodesic if $\alpha = 0$ ([15, Proposition 3.8] or [16, Proposition 4.10]). We call a metric *generic* if $\alpha \neq 0$ and *singular* if $\alpha = 0$. Finally, if $h = 0$, then our space is of type (H), (P) or (Pℓ). On a space of type (H), $h = 0$ means that the asymptotic foliations are mutually orthogonal ([15, Proposition 3.7] or [16, Proposition 4.7]).

Since the conformal curvature tensor of type (1,3) vanishes identically on any three-dimensional Riemannian manifold, the Riemannian curvature tensor R of (M,g) satisfies

$$\begin{aligned} R(X,Y)Z = {} & \hat{R}(Y,Z)X - \hat{R}(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \frac{\mathrm{Sc}(g)}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.9}$$

for all $X, Y, Z \in TM$, where $\hat{R}$, Q and $\mathrm{Sc}(g)$ denote the Ricci form, the Ricci operator and the scalar curvature, respectively. Since

$$QE_i = \rho_i E_i\,, \quad i = 1,2,3, \tag{2.10}$$

hold, the scalar curvature $\mathrm{Sc}(g)$ of (M,g) is given by $\mathrm{Sc}(g) = 2\rho_1 + \rho_3 = 2\rho_1 + 2\tilde{c}$.

Now, suppose that (M,g) is embedded as a hypersurface in a four-dimensional space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ and let S be the shape operator of the embedding (determined by a local unit normal vector field). Then, by (2.9), the Gauss equation (1.1) takes on the form

$$\begin{aligned} & \hat{R}(Y,Z)X - \hat{R}(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \Big(\frac{\mathrm{Sc}(g)}{2} + \tilde{c}\Big)\{g(Y,Z)X - g(X,Z)Y\} \\ & - g(SY,Z)SX + g(SX,Z)SY = 0 \end{aligned} \tag{2.11}$$

for all $X, Y, Z \in TM$. As the $\tilde{c}$-nullity of (M,g) equals to one, the type number of (M,g) in $\tilde{M}^4(\tilde{c})$ is two and hence the kernel of S is one-dimensional.

Let X be a unit vector field of eigenvectors of the shape operator S corresponding to the eigenvalue 0. Then from (2.11) we get

$$\begin{aligned} & \hat{R}(Y,Z)X - \hat{R}(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \Big(\frac{\mathrm{Sc}(g)}{2} + \tilde{c}\Big)\{g(Y,Z)X - g(X,Z)Y\} = 0\,. \end{aligned} \tag{2.12}$$

Putting $Y = E_2$ and $Z = E_1$ in (2.12), we obtain $(\rho_1 - \rho_3)g(X,E_1) = 0$, which implies that X is perpendicular to E_1 because $\rho_1 \neq \rho_3$. Similarly we show that X is also perpendicular to E_2. Hence $X = E_3$ up to a sign and E_3 is a vector field of eigenvectors of S corresponding to the eigenvalue 0. Thus, taking into account that S

is a symmetric endomorphism of rank two on each tangent space, we can write

$$\begin{cases} SE_1 = \mathrm{L}E_1 + \mathrm{M}E_2\,, \\ SE_2 = \mathrm{M}E_1 + \mathrm{N}E_2\,, \\ SE_3 = 0\,, \end{cases} \tag{2.13}$$

where $\mathrm{L} = \mathrm{L}(w,x,y)$, $\mathrm{M} = \mathrm{M}(w,x,y)$ and $\mathrm{N} = \mathrm{N}(w,x,y)$ are functions of the variables w, x and y satisfying $\mathrm{LN} - \mathrm{M}^2 \neq 0$.

Putting $X = E_1$, $Y = E_2$ and $Z = E_1$ in (2.11), and using (2.10) and (2.13), we see that

$$\mathrm{Sc}(g) - 6\tilde{c} = 2(\mathrm{LN} - \mathrm{M}^2)\,. \tag{2.14}$$

Conversely, one can see easily that (2.14) is equivalent to the fulfilling of the Gauss equation *in general.*

Next, we express the Codazzi equation (1.2) on the basis of (2.13), using the local orthonormal frame $\{E_1, E_2, E_3\}$. We get the system of six partial differential equations for the components L, M and N of the shape operator S:

$$(f\mathrm{L})'_y + \Big(3\beta - \frac{h}{fA}\Big)f\mathrm{M} = 0\,, \tag{2.15}$$

$$(A\mathrm{M})'_y + \beta A\mathrm{N} = 0\,, \tag{2.16}$$

$$(A\mathrm{N})'_y - \beta A\mathrm{M} = 0\,, \tag{2.17}$$

$$(f\mathrm{L})'_x - A\mathrm{M}'_w + C\mathrm{M}'_x + AH\mathrm{M}'_y - 2\alpha fA\mathrm{M} - f'_x\mathrm{N} = 0\,, \tag{2.18}$$

$$(f\mathrm{M})'_x - A\mathrm{N}'_w + C\mathrm{N}'_x + AH\mathrm{N}'_y + \alpha fA\mathrm{L} + f'_x\mathrm{M} - \alpha fA\mathrm{N} = 0\,, \tag{2.19}$$

$$\beta fA\mathrm{L} + (fA'_y - f'_yA)\mathrm{M} - fA\Big(\beta - \frac{h}{fA}\Big)\mathrm{N} = 0\,. \tag{2.20}$$

In reality, we receive one more equation:

$$\mathrm{M}'_y - \beta\mathrm{L} + \frac{f'_y}{f}\mathrm{M} + \Big(2\beta - \frac{h}{fA}\Big)\mathrm{N} = 0\,.$$

Yet, this is a consequence of (2.16) and (2.20).

Conversely, if L, M and N are functions of the variables w, x and y satisfying the partial differential equations (2.15)–(2.20) together with the algebraic equation (2.14), then the tensor field S of type (1,1) defined by (2.13) is symmetric and satisfies the Gauss and Codazzi equations. Hence, it defines a unique (local) embedding of (M,g) into $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c}$ (up to an isometry of the ambient space).

3. Results and their proofs

The formulas and arguments in this section are often the same as those by V. Hájková [6] and therefore some more delicate technical details of the proofs will be omitted. The reader is advised to see [2, Chapter 10].

We shall present the proofs of the main theorems in the hyperbolic case $\tilde{c} < 0$ only. For the elliptic case $\tilde{c} > 0$ we need only slight modifications of these proofs. In the following formulas the symbol λ will always denote the positive number such that $\lambda^2 = -\tilde{c}$ in the hyperbolic case and $\lambda^2 = \tilde{c}$ in the elliptic case.

3.1. **Orthogonally foliated spaces of type (H).** The metrics of three-dimensional orthogonally foliated generic spaces of type (H) are locally determined, in the hyperbolic case, by an orthonormal coframe

$$\tag{3.1} \begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)]\,\mathrm{d}x \\ \omega^3 = \mathrm{d}y\,, \end{cases}$$

and, in the elliptic case, by an orthonormal coframe

$$\tag{3.2} \begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\,\mathrm{d}x \\ \omega^3 = \mathrm{d}y\,, \end{cases}$$

where $t = t(w,x)$, $u = u(w,x)$, $p = p(w,x)$ and $q = q(w,x)$ are functions of the variables w and x such that

$$\tag{3.3} \begin{cases} up'_w - tq'_w = 0\,, \\ pu'_x - qt'_x = 0\,, \\ pu - qt \neq 0\,. \end{cases}$$

(The system (3.3) can be solved in the form where the functions t and u are explicitly expressed through p and q, see [15] or [16]).

Theorem 3.1. *The three-dimensional orthogonally foliated generic spaces of type* (H) *which can be locally realized as hypersurfaces in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ depend on one arbitrary function of two variables, six arbitrary functions of one variable and five real parameters. For the proper choice of the involved functions* (*which still remain "arbitrary"*), *these spaces can be realized as hypersurfaces in $\tilde{M}^4(\tilde{c})$ in exactly two qualitative different ways* (*up to an isometry of $\tilde{M}^4(\tilde{c})$*).

Proof. We have

$$\tag{3.4} \begin{aligned} f &= t\cosh(\lambda y) + u\sinh(\lambda y)\,, \\ A &= p\cosh(\lambda y) + q\sinh(\lambda y) \end{aligned}$$

and $\alpha = A'_w/(fA)$, $\beta = H = C = 0$.

Condition (2.20) is reduced to $(fA'_y - Af'_y)\mathrm{M} = 0$. But we know that $fA'_y - Af'_y = \lambda(qt - pu) \neq 0$, so it follows

$$\tag{3.5} \mathrm{M} = 0\,.$$

Then (2.16) is satisfied identically. Conditions (2.15) and (2.17) imply

$$\tag{3.6} \mathrm{L} = \frac{\bar{L}}{f}\,, \quad \mathrm{N} = \frac{\bar{N}}{A}\,,$$

where $\bar{L} = \bar{L}(w,x)$ and $\bar{N} = \bar{N}(w,x)$ are functions of the variables w and x.

Next, we substitute from (3.5) and (3.6) in (2.18) to obtain $A(\bar{L})'_x = f'_x\bar{N}$. Substituting for f and A the expression from (3.4), and comparing the corresponding coefficients, we find

$$p\bar{L}'_x = t'_x\bar{N}\,, \tag{3.7}$$

$$q\bar{L}'_x = u'_x\bar{N}\,. \tag{3.8}$$

These two equations are linearly dependent because of $(3.3)_2$. Substitution from (3.5) and (3.6) in (2.19) gives $f\bar{N}'_w = A'_w\bar{L}$, or equivalently,

$$t\bar{N}'_w = p'_w\bar{L}\,, \tag{3.9}$$

$$u\bar{N}'_w = q'_w\bar{L}\,. \tag{3.10}$$

Again, these equations are dependent because of $(3.3)_1$.

Finally, using expressions (2.7), we write the Gauss equation (2.14) in the form

$$\bar{L}\bar{N} = \frac{f'_wA'_w}{f^2} + \frac{f'_xA'_x}{A^2} - \frac{A''_{ww}}{f} - \frac{f''_{xx}}{A} + \lambda^2(pt - qu)\,. \tag{3.11}$$

Because the requirement $\alpha \neq 0$ is equivalent to $pq'_w - qp'_w \neq 0$, we have $p'_w \neq 0$ or $q'_w \neq 0$. Now we suppose $p'_w \neq 0$. (We can treat the case $q'_w \neq 0$ similarly.) By $(3.3)_1$, we obtain

$$u = \frac{tq'_w}{p'_w}\,. \tag{3.12}$$

Since the function t satisfies the equality $t = (pu - qt)p'_w/(pq'_w - qp'_w)$ and $pu - qt \neq 0$, we get $t \neq 0$. Substituting (3.12) into $(3.3)_2$, we obtain

$$q''_{wx} + \left(\frac{t'_x}{t} - \frac{p''_{wx}}{p'_w}\right)q'_w - \frac{t'_xp'_w}{pt}q = 0\,. \tag{3.13}$$

So, if we choose arbitrary non-zero (real analytic) functions p and t of the variables w and x such that $p'_w \neq 0$, then q is a solution of (3.13) and u is determined by (3.12).

Now, we turn again to condition (3.11). We substitute from (3.4), using also $q'_w = up'_w/t$ and $u'_x = qt'_x/p$ (which are coming from (3.3)) to obtain

$$pt\bar{L}\bar{N} = \frac{p}{t}p'_wt'_w + \frac{t}{p}t'_xp'_x - pp''_{ww} - tt''_{xx} + \lambda^2(pt - qu)pt\,, \tag{3.14}$$

$$\begin{aligned}(qt + pu)\bar{L}\bar{N} = {} & \frac{p'_w}{t}(qt'_w + pu'_w) + \frac{t'_x}{p}(up'_x + tq'_x) - qp''_{ww} - pq''_{ww} \\ & - ut''_{xx} - tu''_{xx} + \lambda^2(pt - qu)(qt + pu)\,,\end{aligned} \tag{3.15}$$

$$qu\bar{L}\bar{N} = \frac{q}{t}p'_wu'_w + \frac{u}{p}q'_xt'_x - qq''_{ww} - uu''_{xx} + \lambda^2(pt - qu)qu\,. \tag{3.16}$$

Using the identities

$$u'_wp'_w + up''_{ww} - t'_wq'_w - tq''_{ww} = 0\,,$$
$$p'_xu'_x + pu''_{xx} - q'_xt'_x - qt''_{xx} = 0\,,$$

which are obtained by differentiation of (3.3), we see easily that equations (3.14)–(3.16) are dependent. Thus, we are left with the equations (3.7), (3.9) and (3.14) *i.e.*, the basic system of partial differential equations is given by

$$pL'_x = t'_x \bar{N}, \tag{3.17}$$

$$t\bar{N}'_w = p'_w \bar{L}, \tag{3.18}$$

$$\bar{L}\bar{N} = F, \tag{3.19}$$

where we have put, for simplicity of notation,

$$F = \frac{p'_w t'_w}{t^2} + \frac{t'_x p'_x}{p^2} - \frac{p''_{ww}}{t} - \frac{t''_{xx}}{p} + \lambda^2(pt - qu).$$

Note that only the functions p and t occur explicitly in the system because the functions q and u can be calculated from p and t by (3.13) and (3.12).

From (3.19), we express $\bar{N}$ as $\bar{N} = F/\bar{L}$ and substitute in (3.17) and (3.18). In this way, we obtain

$$(\bar{L}^2)'_x = 2\frac{t'_x F}{p}, \tag{3.20}$$

$$(\bar{L}^2)'_w = -2\frac{p'_w}{Ft}\bar{L}^4 + 2\frac{F'_w}{F}\bar{L}^2. \tag{3.21}$$

The integrability condition for this system of partial differential equations is given by

$$\Big(-\frac{p'_w}{Ft}\bar{L}^4 + \frac{F'_w}{F}\bar{L}^2\Big)'_x = \Big(\frac{t'_x F}{p}\Big)'_w,$$

which is written (using (3.20)) as

$$G_1\bar{L}^4 + G_2\bar{L}^2 + G_3 = 0 \tag{3.22}$$

with functions G_1, G_2 and G_3 defined by

$$G_1 = \Big(\frac{p'_w}{Ft}\Big)'_x,$$

$$G_2 = 4\frac{p'_w t'_x}{pt} - \Big(\frac{F'_w}{F}\Big)'_x,$$

$$G_3 = -2\frac{F'_w t'_x}{p} + \Big(\frac{Ft'_x}{p}\Big)'_w.$$

The roots of the quadratic equation (3.22) in $\bar{L}^2$ are of the form

$$\bar{L}^2 = \frac{-G_2 \pm \sqrt{{G_2}^2 - 4G_1G_3}}{2G_1}. \tag{3.23}$$

Now we can see that, in a neighborhood of a fixed point (w_0, x_0, y_0), the functions p and t can be chosen in such a way that

$$G_1G_3 > 0, \quad G_1G_2 < 0, \quad {G_2}^2 - 4G_1G_3 > 0, \quad Fpt \neq 0.$$

Moreover, the function p still remains "general" and the function t can be calculated as a general solution of a fifth order partial differential equation. We refer to pp. 201–205 of [2] for more formal (and a bit complicated) details. Hence, we can calculate two different branches for the function $\bar{L}^2$ from (3.23), and hence two different solutions for $\bar{L}$ which are not just opposite. The function $\bar{N}$ is then given by $\bar{N} = F/\bar{L}$. So, from (2.13) we get just two essentially distinct shape operators S and the corresponding metrics are locally realizable as hypersurfaces of $\tilde{M}^4(-\lambda^2)$ in exactly two non-equivalent ways. □

The metrics of three-dimensional orthogonally foliated singular spaces of type (H) belong to the following two classes ([15, Theorem 4.9] or [16, Theorem 5.9]):

Class I. The orthonormal coframe is given, in the hyperbolic case, by

$$\begin{cases} \omega^1 = [t\cosh(\lambda y) + u\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [\cosh(\lambda y) + q\sinh(\lambda y)]\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.24}$$

and, in the elliptic case, by

$$\begin{cases} \omega^1 = [t\cos(\lambda y) + u\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [\cos(\lambda y) + q\sin(\lambda y)]\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.25}$$

where $t = t(w,x)$, $u = u(w,x)$ and $q = q(x)$ are arbitrary functions such that

$$u'_x - qt'_x = 0\,, \tag{3.26}$$

Class II. The orthonormal coframe is given, in the hyperbolic case, by

$$\begin{cases} \omega^1 = [\cosh(\lambda y) + u\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = \sinh(\lambda y)\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.27}$$

and, in the elliptic case, by

$$\begin{cases} \omega^1 = [\cos(\lambda y) + u\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = \sin(\lambda y)\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.28}$$

where $u = u(w,x)$ is an arbitrary function of the variables w and x.

Theorem 3.2. (1) *The three-dimensional orthogonally foliated singular spaces in the class* I *of type* (H) *which can be locally realized as hypersurfaces in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ depend on six arbitrary functions of one variable and five real parameters. For every such realization, the corresponding hypersurface is locally rigid.*
(2) *The three-dimensional orthogonally foliated singular spaces in the class* II *of type* (H) *which can be locally realized as hypersurfaces in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ depend on four arbitrary functions of one variable and five real parameters. For every such realization, the corresponding hypersurface is locally rigid.*

Proof. (1) In the class I, we have

$$
\begin{aligned}
f &= t\cosh(\lambda y) + u\sinh(\lambda y)\,, \\
A &= \cosh(\lambda y) + q\sinh(\lambda y)
\end{aligned}
\tag{3.29}
$$

and $\alpha = \beta = h = H = C = 0$.

Conditions (3.5)–(3.10) and (3.14) are reduced to

$$
\mathrm{M} = 0\,, \quad \mathrm{L} = \frac{\bar{L}}{f}\,, \quad \mathrm{N} = \frac{\bar{N}}{A}\,, \tag{3.30}
$$

$$
\bar{L}'_x = t'_x \bar{N}\,, \quad \bar{N}'_w = 0\,, \tag{3.31}
$$

$$
\bar{L}\bar{N} = -t''_{xx} + \lambda^2(t - qu)\,. \tag{3.32}
$$

We denote $F = -t''_{xx} + \lambda^2(t - qu)$ and we substitute $\bar{N} = F/\bar{L}$ into (3.31). This yields

$$
\begin{aligned}
(\bar{L}^2)'_x &= 2Ft'_x\,, \\
(\bar{L}^2)'_w &= \frac{2F'_w}{F}\bar{L}^2\,.
\end{aligned}
$$

The integrability condition for this system of partial differential equations is written as

$$
\bar{L}^2 = \frac{Ft''_{xw} - F'_w t'_x}{(F'_w/F)'_x}\,. \tag{3.33}
$$

Condition (3.33) implies that the function $\bar{L}$ is determined (up to the sign) by the metric g. The function $\bar{N}$ is given by $\bar{N} = F/\bar{L}$. So, the shape operator (2.13) is determined up to a sign and a realizable metric can be locally embedded in $\tilde{M}^4(-\lambda^2)$ by a unique way (up to an isometry of $\tilde{M}^4(-\lambda^2)$). The rest of the proof is same as that in p. 206 of [2].

(2) In the class II, we have

$$
\begin{aligned}
f &= \cosh(\lambda y) + u\sinh(\lambda y), \\
A &= \sinh(\lambda y)
\end{aligned}
\tag{3.34}
$$

and $\alpha = \beta = h = H = C = 0$.

Conditions (3.5)–(3.10) and (3.14) are reduced to

$$
\begin{aligned}
\mathrm{M} = 0\,, \quad \mathrm{L} &= \frac{\bar{L}}{f}\,, \quad \mathrm{N} = \frac{\bar{N}}{A}\,, \\
\bar{L}'_x &= u'_x \bar{N}\,, \quad \bar{N}'_w = 0\,, \\
\bar{L}\bar{N} &= -u''_{xx} - \lambda^2 u\,.
\end{aligned}
$$

The rest of the proof is the same as above. □

3.2. **Spaces of type (P).** The metrics of three-dimensional foliated generic spaces of type (P) are given, in the hyperbolic case, by an orthonormal coframe

$$(3.35)\qquad \begin{cases} \omega^1 = \xi[p\cosh(\lambda y) + q\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [p\cosh(\lambda y) + q\sinh(\lambda y)]\,\mathrm{d}x \\ \qquad\qquad + [r\cosh(\lambda y) + s\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w\,, \end{cases}$$

and, in the elliptic case, by an orthonormal coframe

$$(3.36)\qquad \begin{cases} \omega^1 = \xi[p\cos(\lambda y) + q\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^2 = [p\cos(\lambda y) + q\sin(\lambda y)]\,\mathrm{d}x \\ \qquad\qquad + [r\cos(\lambda y) + s\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w\,, \end{cases}$$

where $p = p(w,x)$ and $q = q(w,x)$ are arbitrary functions of the variables w and x such that $qp'_x - pq'_x \neq 0$, and

$$(3.37)\qquad \begin{cases} r = \dfrac{p'_x - pE}{D}\,, \quad s = \dfrac{q'_x - qE}{D}\,, \quad \xi = \left[\dfrac{p'_w - r'_x - \lambda qx}{pD}\right]^{1/2}, \\ D = \lambda(p'_x q - pq'_x)\,, \\ E = \lambda[pq'_w - p'_w q + \lambda(\epsilon p^2 + q^2)x]\,, \quad \epsilon = \mathrm{sgn}(\tilde{c})\,. \end{cases}$$

(See [15, Theorem 4.10] or [16, Theorem 5.10].)

Theorem 3.3. *The three-dimensional foliated generic spaces of type* (P) *which can be locally realized as hypersurfaces in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ depend on one arbitrary function of two variables and six arbitrary functions of one variable. For each such realization, the corresponding hypersurface is locally rigid, i.e., it cannot be* (*locally*) *isometrically deformed.*

Proof. For the metrics of foliated generic spaces of type (P), we have

$$(3.38)\qquad \begin{aligned} f &= \xi[p\cosh(\lambda y) + q\sinh(\lambda y)]\,, \\ A &= p\cosh(\lambda y) + q\sinh(\lambda y)\,, \\ C &= r\cosh(\lambda y) + s\sinh(\lambda y)\,, \end{aligned}$$

where ξ depends only on w and x.

Since we have $f'_y A - A'_y f = 0$ and $h = H'_x = 1$, condition (2.20) reduces to

$$\mathrm{N} = 0\,.$$

Then (2.17) is satisfied identically and from (2.16) we see that M can be written in the form

$$(3.39)\qquad \mathrm{M} = \frac{\bar{M}}{A}\,,$$

where $\bar{M} = \bar{M}(w,x)$ is a function of the variables w and x. Because $f = \xi A$, where ξ does not depend on y, the equation (2.15) can be written as

$$(A\mathrm{L})'_y = \frac{\bar{M}}{\xi A^2}\,.$$

From here the function L can be calculated by an elementary procedure (after replacing A by its expression from (3.38)) in the form

$$\mathrm{L} = \frac{\bar{L}}{A} - \frac{1}{\lambda}\frac{\cosh(\lambda y)\bar{M}}{q\xi A^2}\,, \tag{3.40}$$

where $\bar{L} = \bar{L}(w,x)$ is a function of the variables w and x.

Next we look at remaining equations (2.18) and (2.19). Using the formulas

$$f = \xi A\,, \quad h = 1\,, \quad \alpha = \frac{\mathcal{D}\cosh(\lambda y) + \mathcal{E}\sinh(\lambda y)}{\xi A^2}\,,$$

where

$$\mathcal{D} = p'_w - r'_x - \lambda q x\,, \quad \mathcal{E} = \frac{q}{p}\mathcal{D}\,,$$

(see [15, Proposition 4.2] or [16, Proposition 5.2]), and substituting for L and M from (3.39) and (3.40) in equation (2.18), we obtain

$$\begin{aligned} A^2\bar{M}'_w &- AA'_w\bar{M} - CA\bar{M}'_x + CA'_x\bar{M} - AA'_y x\bar{M} \\ &= A^2(\xi\bar{L})'_x - \frac{1}{\lambda}\frac{A\cosh(\lambda y)}{q}\bar{M}'_x + \frac{1}{\lambda}\frac{Aq'_x\cosh(\lambda y)}{q^2}\bar{M} \\ &+ \frac{1}{\lambda}\frac{A'_x\cosh(\lambda y)}{q}\bar{M} - \frac{2\mathcal{D}A^2}{p}\bar{M}\,. \end{aligned}$$

Substituting here explicit expressions for A and C from (3.38) and comparing coefficients of $\sinh^2(\lambda y)$, $\cosh(\lambda y)\sinh(\lambda y)$ and $\cosh^2(\lambda y)$, we obtain three equalities:

$$\begin{aligned} pq^2(\xi\bar{L})'_x &= pq^2\bar{M}'_w - pqs\bar{M}'_x + 2q^2p'_w\bar{M} - pqq'_w\bar{M} \\ &\quad + psq'_x\bar{M} - 2q^2r'_x\bar{M} - \lambda q(2q^2 - p^2)x\bar{M}. \end{aligned} \tag{3.41}$$

$$\begin{aligned} 2pq^2(\xi\bar{L})'_x &= 2pq^2\bar{M}'_w - q\Big(ps + qr - \frac{1}{\lambda}\Big)\bar{M}'_x + 3q^2p'_w\bar{M} \\ &\quad + qsp'_x\bar{M} - pqq'_w\bar{M} + \Big(qr - \frac{2}{\lambda}\Big)q'_x\bar{M} \\ &\quad - 4q^2r'_x\bar{M} - \lambda q(3q^2 - p^2)x\bar{M}, \end{aligned} \tag{3.42}$$

$$\begin{aligned} p^2q^2(\xi\bar{L})'_x &= p^2q^2\bar{M}'_w - pq\Big(qr - \frac{1}{\lambda}\Big)\bar{M}'_x - pq^2(p-2)p'_w\bar{M} \\ &\quad + q\Big(qr - \frac{1}{\lambda}\Big)p'_x\bar{M} - \frac{1}{\lambda}pq'_x\bar{M} - 2pq^2r'_x\bar{M} - \lambda pq^3x\bar{M}\,. \end{aligned} \tag{3.43}$$

Now, if we substitute the expression for $(\xi\bar{L})'_x$ from (3.41) into (3.42) and (3.43) and if we use the formulas $\lambda(qr - ps) = 1$ and $E = \lambda(rq'_x - sp'_x)$ (which are easy consequences of (3.37)), we see that these last two equations are satisfied identically. Thus we are left with the single equation (3.41). In a similar way, we treat the equation (2.19). This can be expressed in the form

$$\lambda pqA(\xi^2\bar{M})'_x = \cosh(\lambda y)\mathcal{D}\bar{M} - \lambda\xi qA\mathcal{D}\bar{L} - \lambda\xi^2 pqA'_x\bar{M}\,.$$

Substituting here the explicit expression for A from (3.38), and comparing coefficients of $\sinh(\lambda y)$ and $\cosh(\lambda y)$, we now obtain the following two equalities:

$$\lambda pq(\xi^2\bar{M})'_x = -\lambda\xi q\mathcal{D}\bar{L} - \lambda\xi^2 pq'_x\bar{M}\,, \tag{3.44}$$

$$\lambda p^2q(\xi^2\bar{M})'_x = \mathcal{D}\bar{M} - \lambda\xi pq\mathcal{D}\bar{L} - \lambda\xi^2 pqp'_x\bar{M}\,. \tag{3.45}$$

Multiplying (3.44) by p and then subtracting both equations, we get

$$\big(\lambda\xi^2 p(p'_xq - pq'_x) - \mathcal{D}\big)\bar{M} = 0\,.$$

This is satisfied identically due to $(3.37)_4$ and due to the formula $\mathcal{D} = \xi^2 pD$, which is only rewritten formula $(3.37)_3$. Thus, we are left with only one equation (3.44).

Finally, we turn to the Gauss equation (2.14). This can be rewritten in the form

$$\bar{M}^2 = -\frac{1}{2}(A^2\,\mathrm{Sc}(g) + 6\lambda^2A^2)\,,$$

which implies that $\bar{M}$ is uniquely determined, up to a sign, by the metric g. Further, (3.44) implies that $\bar{L}$ is completely determined by $\bar{M}$. So, the shape operator S is uniquely determined up to a sign by the given metric.

For the rest of the proof we refer pp. 192–194 of [2]. Every realizable metric can be locally embedded in $\tilde{M}^4(-\lambda^2)$ by a unique way up to an isometry of $\tilde{M}^4(-\lambda^2)$. □

The metrics of three-dimensional foliated singular spaces of type (P) are given, in the hyperbolic case, by an orthonormal coframe

$$\begin{cases} \omega^1 = \xi p\cosh(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = p\cosh(\lambda y)\,\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w \end{cases} \tag{3.46}$$

or

$$\begin{cases} \omega^1 = \xi q\sinh(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = q\sinh(\lambda y)\,\mathrm{d}x + [r\cosh(\lambda y) + s\sinh(\lambda y)]\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w\,, \end{cases} \tag{3.47}$$

and, in the elliptic case, by an orthonormal coframe

$$\begin{cases} \omega^1 = \xi p\cos(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = p\cos(\lambda y)\,\mathrm{d}x + [r\cos(\lambda y) + s\sin(\lambda y)]\,\mathrm{d}w\,, \\ \omega^3 = \mathrm{d}y + x\,\mathrm{d}w\,, \end{cases} \tag{3.48}$$

where $\xi = \xi(w,x)$, $p = p(w,x)$, $q = q(w,x)$, $r = r(w,x)$ and $s = s(w,x)$ are non-zero functions of the variables w and x, such that, in cases (3.46) and (3.48),

$$\tag{3.49} \begin{cases} sp'_x = -\epsilon\lambda p^2 x, \quad \epsilon = \operatorname{sgn}(\tilde{c}), \\ r'_x = p'_w, \\ \lambda ps = -1, \end{cases}$$

and in case (3.47),

$$\tag{3.50} \begin{cases} rq'_x = \lambda q^2 x, \\ s'_x = q'_w, \\ \lambda qr = 1. \end{cases}$$

(See [15, Theorem 4.11] or [16, Theorem 5.11].)

Theorem 3.4. *A three-dimensional foliated singular space of type* (P) *and of the form* (3.46), (3.47) *or* (3.48) *can be realized as a hypersurface in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ if and only if the non-zero function $\xi = \xi(w,x)$ of the variables w and x is given by the formula*

$$\tag{3.51} \xi^2 = \frac{1}{p^2}\Big(\mu_1 x^2 - \mu_2 \frac{1}{p^2} + 2\mu_3 \frac{x}{p}\Big),$$

where $\mu_i = \mu_i(w)$, $i = 1,2,3$, are arbitrary functions of the variable w satisfying the inequalities $\mu_1 > 0$ and $\mu_1\mu_2 + {\mu_3}^2 < 0$. These metrics depend on five arbitrary functions of one variable. The corresponding hypersurface can be locally isometrically deformed in a continuous way.

Proof. The metric (3.46) in the hyperbolic case satisfies $\alpha = \beta = \mathcal{D} = \mathcal{E} = D = 0$. As in the case of the generic spaces, we deduce from (2.20) and (2.15)–(2.17) that

$$\tag{3.52} \mathrm{N} = 0, \quad \mathrm{M} = \frac{\bar{M}}{A}, \quad \mathrm{L} = \frac{\bar{L}}{A} + \frac{1}{\lambda}\frac{\sinh(\lambda y)\bar{M}}{p\xi A^2},$$

where $\bar{L} = \bar{L}(w,x)$ and $\bar{M} = \bar{M}(w,x)$ are functions of the variables w and x.

After we substitute in (2.18) the expression (3.52) for N, M and L, we perform analogous computations as before and we find the single equation

$$\tag{3.53} (\xi\bar{L})'_x = -\frac{r}{p}\bar{M}'_x + \bar{M}'_w - \Big(\frac{r}{p}\Big)'_x \bar{M}.$$

We treat similarly the condition (2.19) and we obtain

$$\tag{3.54} (p\xi^2\bar{M})'_x = 0.$$

Finally, the Gauss equation (2.14) is equivalent to

$$\tag{3.55} \bar{M}^2 = \Big(\frac{p'_x}{p}\Big)'_x + \frac{(p\xi'_x)'_x}{p\xi} - \lambda^2 p^2,$$

which implies that $\bar{M}$ is determined, up to a sign, by the metric g. The function $\bar{L}$ can be calculated from the *differential* equation (3.53). So the shape operator S for a realizable metric depends on one arbitrary function of one variable.

The rest of the proof is a minor modification of the procedure in pp. 196–197 of [2]. The associated hypersurface can be locally isometrically deformed in a continuous way. □

Remark. Three of the five arbitrary functions in Theorem 3.4 are μ_1, μ_2 and μ_3.

3.3. **Spaces of type (Pℓ).** The metrics of three-dimensional foliated spaces of type (Pℓ) are locally determined, in the hyperbolic case, by an orthonormal coframe

$$\begin{cases} \omega^1 = \xi \sinh(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = \sinh(\lambda y)\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y \end{cases} \tag{3.56}$$

or

$$\begin{cases} \omega^1 = \xi \cosh(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = \cosh(\lambda y)\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.57}$$

and, in the elliptic case, by an orthonormal coframe

$$\begin{cases} \omega^1 = \xi \sin(\lambda y)\,\mathrm{d}w\,, \\ \omega^2 = \sin(\lambda y)\,\mathrm{d}x\,, \\ \omega^3 = \mathrm{d}y\,, \end{cases} \tag{3.58}$$

where $\xi = \xi(w,x)$ is a non-zero function of the variables w and x. (See [15, Theorem 4.12] or [16, Theorem 5.12].)

Remark. There are no three-dimensional foliated singular spaces of type (Pℓ).

We have the following existence theorem:

Theorem 3.5. *Every three-dimensional foliated space of type* (Pℓ) *can be* (*locally*) *isometrically embedded as a hypersurface in a space* $\tilde{M}^4(\tilde{c})$ *of constant curvature* $\tilde{c} \neq 0$. *The corresponding hypersurface is* (*locally*) *isometrically deformable in a continuous way.*

Proof. We consider the metric (3.56) in the hyperbolic case (we can treat the other cases similarly). We have

$$\begin{aligned} f &= \xi \sinh(\lambda y)\,, \\ A &= \sinh(\lambda y) \end{aligned} \tag{3.59}$$

and $\alpha = \beta = h = H = C = 0$.

According (2.15)-(2.17) we see that

$$\mathrm{L} = \frac{\bar{L}}{\sinh(\lambda y)}\,, \quad \mathrm{M} = \frac{\bar{M}}{\sinh(\lambda y)}\,, \quad \mathrm{N} = \frac{\bar{N}}{\sinh(\lambda y)}\,, \tag{3.60}$$

where $\bar{L} = \bar{L}(w,x)$, $\bar{M} = \bar{M}(w,x)$ and $\bar{N} = \bar{N}(w,x)$ are functions of the variables w and x. Then equation (2.20) is identically satisfied because of $(f/A)'_y = 0$.

Next, the equations (2.18) and (2.19) are reduced to

$$\begin{aligned} \bar{M}'_w &= (\xi\bar{L})'_x - \xi'_x\bar{N}\,, \\ \bar{N}'_w &= (\xi\bar{M})'_x + \xi'_x\bar{M}\,. \end{aligned} \tag{3.61}$$

Finally, the Gauss equation (2.14) is written as

$$-\frac{\xi''_{xx}}{\xi} - \lambda^2 = \bar{L}\bar{N} - \bar{M}^2. \tag{3.62}$$

In the following we distinguish two cases: $\bar{N} \neq 0$ and $\bar{N} = 0$.

Case I. Here we assume $\bar{N}$ different from 0, and hence

$$\bar{L} = \frac{1}{\bar{N}}\Big(\bar{M}^2 - \lambda^2 - \frac{\xi''_{xx}}{\xi}\Big)\,. \tag{3.63}$$

Suppose that the function $\xi = \xi(w, x)$ is a fixed non-zero real analytic function. Then we can apply the Cauchy-Kowalewski theorem for the system (3.61), after substitutions from (3.63) for $\bar{L}$. For the given function ξ, the solutions of (3.61) for $\bar{M}$ and $\bar{N}$ depend on the choice of two arbitrary functions of one variable. Thus the shape operator S depends on two arbitrary functions of one variable. This implies that the hypersurface in $\tilde{M}^4(-\lambda^2)$ can be locally isometrically deformed in a continuous way. □

Case II. Let the embedding admit a shape operator S of the form (2.13) such that N $= 0$ holds identically. This happens if and only if the second fundamental form vanishes on the distribution span$\{E_2, E_3\}$, *i.e.*, on the asymptotic foliation defined by $w =$ constant.

Proposition 3.6. *If a three-dimensional foliated space of type* (Pℓ) *of the form* (3.56), (3.57) *or* (3.58) *admits locally an isometric embedding in a space $\tilde{M}^4(\tilde{c})$ of constant curvature $\tilde{c} \neq 0$ for which the second fundamental form vanishes along the asymptotic foliation defined by $w =$* constant, *then the function $\xi = \xi(w, x)$ can be given in the form*

$$\xi = \sqrt{\cos(2\lambda x + \theta) + \gamma}\,, \quad \gamma = \gamma(w) > 1\,, \tag{3.64}$$

or $\xi = 1$. Conversely, a metric of the form (3.56), (3.57) *or* (3.58) *with a function ξ of form* (3.64) *can be realized on a hypersurface in $\tilde{M}^4(\tilde{c})$ such that the second fundamental form vanishes along the asymptotic foliation defined by $w =$* constant.

Proof. We see from (3.62) that $\bar{N} = 0$ implies

$$\frac{\xi''_{xx}}{\xi} + \lambda^2 = \bar{M}^2 \tag{3.65}$$

and (3.63) implies

$$|\bar{M}| = \frac{\varphi}{\xi^2}\,, \tag{3.66}$$

where $\varphi = \varphi(w)$ is a positive function of the variable w. Substituting (3.66) in (3.65), we obtain

$$\xi''_{xx} + \lambda^2\xi = \frac{\varphi}{\xi^3}\,. \tag{3.67}$$

Now we suppose that $\xi'_x \neq 0$ and multiply (3.67) by $2\xi'_x$. Then, by the usual integration procedure, we obtain the *general solution* in the form

$$\xi = \zeta\sqrt{\cos(2\lambda x + \theta) + \gamma},$$

where $\zeta = \zeta(w)$, $\theta = \theta(w)$ and $\gamma = \gamma(w)$ are arbitrary functions of the variable w. An easy calculation shows that $\varphi^2 = \lambda^2\zeta^4(\gamma^2 - 1)$ and hence $\gamma > 1$ because $\gamma < -1$ is impossible. By a change of the coordinate w to $\bar{w}$ satisfying $\mathrm{d}\bar{w}/\mathrm{d}w = \zeta$, we can make $\zeta = 1$. Thus ξ is given by the final form (3.64).

Next if $\xi'_x = 0$ in the given domain, we obtain the *singular solution* of equation (3.67) in the form $\xi = \xi(w)$. Again, after making a change of the coordinate w if necessary, we may suppose that $\xi = 1$.

The converse statement follows easily. □

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