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LORENTZIAN MANIFOLDS WITH SPECIAL HOLONOMY AND PARALLEL SPINORS

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ABSTRACT. We investigate the holonomy group of a simply connected, indecomposable and reducible Lorentzian spin manifold and the property that it should admit parallel spinors. After some algebraic consequences we show that such manifolds have to be Brinkmann waves and prove that they have abelian holonomy if and only if they are pp-manifolds. Further, we prove a theorem about the holonomy of special Brinkmann waves and construct examples of Lorentzian manifolds with parallel spinors starting from Riemannian Kähler and hyper-Kähler manifolds.

CONTENTS

The paper is in final form and no version of it will be submitted elsewhere.

1. Introduction

The most effective tool to decide whether a semi-Riemannian spin manifold admits parallel spinors is the holonomy group of the manifold because the space of parallel spinors of a simply connected spin manifold is isomorphic to the trivial subrepresentations of the spinor representation of the holonomy group. Therefore it is first necessary to classify the holonomy groups and then to test if they could have trivial spinor subrepresentations.

The first step is done for simply connected, irreducible manifolds by M. Berger [Ber55], [Ber57] and J. Simons [Sim62]. In the Riemannian case this classification becomes complete because of the splitting theorem of G. de Rham [dR52] which asserts that a simply connected, complete Riemannian manifold can be decomposed in a product of irreducible ones. Here M. Y. Wang [Wan89] did the second step and showed that the following holonomy groups from the Berger-list are those of simply connected, irreducible, non-locally symmetric m dimensional Riemannian spin manifolds admitting parallel spinors: $SU(k)$ for $m = 2k$, $Sp(k)$ for $m = 4k$, G_2 for $m = 7$ and Spin(7) for $m = 8$.

Although there is a generalization of Wangs theorem for semi-Riemannian simply connected, irreducible, not locally symmetric spin manifolds by H. Baum and I. Kath in [BK99] this does not solve the problem completely in the pseudo-Riemannian case. This is because the generalization of de Rhams splitting theorem by H. Wu in [Wu64] only asserts the decomposition of a simply connected, complete semi-Riemannian manifold into a product of indecomposable ones. The holonomy representation of these manifolds has no invariant subspace on which the metric is nondegenerate. In the Riemannian case irreducibility and indecomposability are the same.

So first the possible holonomy representations of indecomposable, reducible pseudo-Riemannian manifolds have to be found. This is a rather open question. The only investigations are made by A. Ikemakhen and L. Berard Bergery in the Lorentzian case ([Ike90], [BI93] and [Ike96]), in the case of index 2 in [Ike99], for index (r, r) in [BI97] and for general index in [Bou00].

In this paper we will deal with the Lorentzian case and — after some introductory sections — draw some conclusions from the results of [BI93] and [Ike96] for the existence of parallel spinors on simply connected, indecomposable Lorentzian spin manifolds.

In [BI93] the possible holonomy algebras of a simply connected, indecomposable but reducible Lorentzian manifold of dimension m are divided into four types of subalgebras of $(\mathbb{R} \oplus \mathfrak{so}(m-2)) \ltimes \mathbb{R}^{m-2}$. We will add the condition to the manifold to admit parallel spinors so that its holonomy algebra has to be a subalgebra of $\mathfrak{so}(m-2) \ltimes \mathbb{R}^{m-2}$ which leaves us with only two of those four types, the so called Brinkmann waves. We will then show, that the question whether these admit trivial subrepresentations of their spin representation can be reduced to the question if this is the case for their projections on the $\mathfrak{so}(m-2)$-component. Also, we will show that this projection can not be non-trivial abelian.

Further, we investigate the simply connected, indecomposable but reducible Lorentzian spin manifolds with abelian holonomy, i.e. with holonomy isomorphic to $\mathbb{R}^{m-2}$. These have parallel spinors. We will prove that this class of manifolds is equal to the so called pp-manifolds, which are an generalization of the plane-waves. As a result we obtain that pp-manifolds are the only Lorentzian manifolds which admit parallel spinors and have non- trivial abelian holonomy.

Finally, we prove a sufficient condition to a Brinkmann wave for having holonomy equal to Riemannian holonomy$\ltimes\mathbb{R}^n$. We construct examples with this holonomy and parallel spinors starting from Riemannian Kähler and hyper-Kähler manifolds.

2. Holonomy and parallel spinors

2.1. Principle fibre bundles, vector bundles and holonomy. First we will recall the notion of a holonomy group of a connection in a principle fibre bundle in general.

Let $\pi : P \to M$ a principle fibre bundle with structure group G, $\omega : TP \to \mathfrak{g} := LA(G)$ a connection in P and $\mathfrak{p}^\omega_\gamma$ the parallel displacement along a piecewise smooth curve γ in M from $P_{\gamma(0)}$ to $P_{\gamma(1)}$.

One defines the holonomy group to ω in a point $p \in P$ with $\pi(p) = x \in M$

$$Hol_p(\omega) := \left\{ g \in G \;\middle|\; \begin{array}{c}\text{there is a curve } \gamma \text{ in } M, \gamma(0) = \gamma(1) = x, \\ R_g(p) = \mathfrak{p}^\omega_\gamma(p)\end{array} \right\} \subset G.$$

The restricted holonomy group $Hol^0_p(\omega)$ is defined by restricting to homotopically trivial curves in M. This group is connected and lies in the connected component of the unity in G. Its Lie-algebra is denoted by $\mathfrak{hol}_p(\omega) := LA\left(Hol^0_p(\omega)\right)$.

The holonomy groups depend in the following way on the point $p \in P$: for two points p and q in the same fibre over $x \in M$ such that $q = R_g(p)$ for a $g \in G$ the holonomy groups are conjugated in G, i.e. $Hol_q(\omega) = g^{-1} Hol_p(\omega) g$; if there is a horizontal curve in P from p to q then the holonomy groups are the same. The holonomy group contains all the geometric information of P which is the assertion of the reduction theorem.

Theorem 2.1. *The subbundle of P, called the holonomy bundle,*

$$P^\omega(p) := \{q \in P \mid \textit{there exists a horizontal curve from } p \textit{ to } q\}$$

is a principle fibre bundle with structure group $Hol_p(\omega)$ which is a reduction of P and ω.

The second fundamental theorem on holonomy is the Ambrose-Singer holonomy theorem.

Theorem 2.2. [AS53] *Let M be connected. P a priciple fibre bundle over M with structure group G and ω a connection in P and Ω its curvature. Then*

$$\mathfrak{hol}_p(\omega) = \{\Omega_q(X,Y) \mid q \in P^\omega(p), X, Y \in Ker\ \omega_q \subset T_qP\} \subset \mathfrak{g}. \tag{1}$$

Let (V,ρ) be a representation of G. We now consider the vector bundle associated to P and ρ defined by $E := (P \times V)/G$. G acting on V is isomorphic to a subgroup of $Gl(E_x)$ acting on the fibre E_x for every fibre over x.

ω defines a covariant derivative ∇^ω in E. The reduction theorem then entails the following

$$E := (P \times V)/G \simeq (P^\omega(p) \times V)/Hol_p(\omega).$$

Now we can identify the parallel sections in E with invariant vectors under the holonomy representation:

$$
\begin{array}{lrcl}
(2) & V_{Hol_p(\omega)} := \{v \in V \,|\rho(Hol_p(\omega))(v) = v\} & \simeq & \{e \in \Gamma(E) | \nabla^{\omega} e = 0\} \\
& v & \mapsto & \left(y \mapsto \left[\mathfrak{p}^{\omega}_{\gamma(1)}(p), v\right]\right)
\end{array}
$$

where γ is a curve connecting $\pi(p)$ and y. For the right hand side we will use the notation $\left[\mathfrak{p}^{\omega}_{\gamma}(p), v\right]$ with γ running over curves starting at $\pi(p)$.

In case that M is simply connected we have on the Lie-algebra level

$$
(3) \qquad V_{\mathfrak{hol}_p(\omega)} := \left\{v \in V \,\middle|\rho_*(\mathfrak{hol}_p(\omega))(v) = 0\right\} \simeq \{e \in \Gamma(E) | \nabla^{\omega} e = 0\} \; .
$$

2.2. **Holonomy of semi-Riemannian manifolds.** Let (M, h) be a semi-Riemannian manifold of dimension $m = r + s$, index r and ∇ the Levi-Civita-connection.

Let $\mathcal{O}(M, h)$ be the bundle of orthonormal frames over M with structure group $O(r, s)$. Then one has as in the previous section $\mathcal{O}(M, h) \times_{O(r,s)} \mathbb{R}^m \simeq TM$.

The Levi-Civita connection ∇ defines a connection ω in $\mathcal{O}(M, h)$ with the local connection form

$$
\omega^s = \sum_{1 \le i < j \le n} h\left(\nabla s_i, s_j\right) E_{ij}
$$

where $s = (s_1, \dots s_m)$ is a section in $\mathcal{O}(M, h)$, that is a local orthormal frame-field and E_{ij} the standard basis in $\mathfrak{o}(r, s)$. Then it is $\nabla = \nabla^{\omega}$. One defines the holonomy groups

$$
\begin{array}{rcl}
(Hol_x(M, h), T_x M) & := & (Hol_p(\omega), \mathbb{R}^m) \\
(Hol^0_x(M, h), T_x M) & := & \left(Hol^0_p(\omega), \mathbb{R}^m\right) \; .
\end{array}
$$

The relation between parallel vector fields and invariant subspaces of the holonomy representation is given by (see for example [Bes87])

Theorem 2.3. *Let (M, h) be a semi-Riemannian manifold. Then the following holds*

1. *For $1 \le k < m$ these propositions are equivalent:*
   (i) *There is a k-dimensional distribution, which is invariant under parallel displacement.*
   (ii) *The holonomy representation leaves invariant a k-dimensional subspace.*
   This distribution has to be involutive.
2. *There is a parallel vectorfield on M if and only if the holonomy representation has a trivial subrepresentation, that means it leaves fixed a vector.*

The properties of the representation are used to characterize the manifold.

Definition 2.1. A semi-Riemannian manifold $(M^{r,s}, h)$ is called

1. (strictly) irreducible if the (reduced) holonomy representation has no invariant subspace,
2. (strictly) indecomposable if the (reduced) holonomy representation has no invariant subspace on which h is non-degenerate,
3. reducible/decomposable if it is not irreducible/indecomposable.

Irreducibility entails indecomposability and decomposability reducibility. For Riemannian manifolds both notions are the same. The tangent space of a semi-Riemannian manifold can be decomposed in a orthogonal sum of invariant subspaces which can be (in case of nontrivial signature of h) reducible with a degenerate, invariant subspace.

The holonomy representation of the product of two semi-Riemannian manifolds is the product of the holonomy representations (see again [Bes87])

$$\begin{aligned} Hol_{(x_1,x_2)}(M_1 \times M_2, h_1 \oplus h_2) &= Hol_{x_1}(M_1,h_1) \times Hol_{x_2}(M_2,h_2) \\ &= (Hol_{x_1}(M_1,h_1) \times 1) \oplus (1 \times Hol_{x_2}(M_2,h_2)). \end{aligned}$$

The de-Rham decomposition theorem asserts that locally the converse is true.

Theorem 2.4. [dR52] [Wu64] *Every simply-connected, complete semi-Riemannian manifold is isometric to a product of simply-connected, complete manifolds, of which one can be flat and all others are indecomposable.*

That means locally it is sufficient to know the indecomposable manifolds. For the subclass of irreducible ones there is the classification of Berger and Simons.

Theorem 2.5. [Ber55], [Sim62], [Ale68], [BG72] *and* [Bry87]. *Let $(M^{r,s},h)$ be a simply-connected, irreducible, non local-symmetric semi-Riemannian manifold of dimension $m = r+s$ and index r. Then the holonomy representation on $\mathbb{R}^m$ is one of the following (modulo conjugation in $O(r,s)$):*

$$\begin{array}{llll}
m = r+s \geq 2 & : & SO(r,s) & \\
m = 2p+2q \geq 4 & : & U(p,q) & \subset SO(2p,2q) \\
 & or & SU(p,q) & \subset SO(2p,2q) \\
m = 4p+4q \geq 8 & : & Sp(p,q) & \subset SO(4p,4q) \\
 & or & Sp(p,q)\cdot Sp(1) & \subset SO(4p,4q) \\
m = r+r \geq 4 & : & SO(r,\mathbb{C}) & \subset SO(r,r) \\
m = 2p+2p \geq 8 & & Sp(p,\mathbb{R})\cdot Sl(2,\mathbb{R}) & \subset SO(2p,2p) \\
m = 4p+4p \geq 16 & & Sp(p,\mathbb{C})\cdot Sl(2,\mathbb{C}) & \subset SO(4p,4p) \\
m = 7 = 0+7 & : & G_2 & \subset SO(7) \\
m = 7 = 4+3 & : & G^*_{2(2)} & \subset SO(4,3) \\
m = 14 = 7+7 & : & G_2^{\mathbb{C}} & \subset SO(7,7) \\
m = 8 = 0+8 & : & \mathrm{Spin}(7) & \subset SO(8) \\
m = 8 = 4+4 & : & \mathrm{Spin}_0(4,3) & \subset SO(4,4) \\
m = 16 = 8+8 & : & \mathrm{Spin}(7)^{\mathbb{C}} & \subset SO(8,8).
\end{array}$$

In case of symmetric spaces there is a classification of Berger [Ber57], which gives the following corollary for Lorentzian manifolds.

Corollary 2.1. *A simply-connected, complete and irreducible Lorentzian manifold has the trivial holonomy group or the full $SO_0(1,m-1)$.*

2.3. **Spinor representations and the spinor bundle.** We now want to define the spinor bundle of a m=(r+s)-dimensional semi-Riemannian manifold. (See [LM89], [Bau81] also [Bau94] for details.) Let $Cl_{r,s}$ be the Clifford-algebra of $(\mathbb{R}^m, \langle .,. \rangle_{r,s})$ where $\langle .,. \rangle_{r,s}$ is the standard scalarproduct of index r on $\mathbb{R}^m$. We define the following groups in $Cl_{r,s}$ ($\langle X, X\rangle_{r,s} =: \|X\|_{r,s}$):

$$\begin{aligned}
\mathrm{Spin}(r,s) &:= \{X_1 \cdot \ldots \cdot X_{2k} \mid \|X_i\|_{r,s} = \pm 1,\ k \geq 0\} \\
\mathrm{Spin}_0(r,s) &= \{X_1 \cdot \ldots \cdot X_{2k} \cdot Y_1 \cdot \ldots \cdot Y_{2l} \mid \|X_i\|_{r,s} = 1, \|Y_i\|_{r,s} = -1,\ k,l \geq 0\}
\end{aligned}$$

$$K = \left\{ X_1 \cdot \ldots \cdot X_{2k} \cdot Y_1 \cdot \ldots \cdot Y_{2l} \;\middle|\; \begin{array}{l} \|X_i\|_{r,s} = 1, \|Y_i\|_{r,s} = -1,\ \ k,l \geq 0 \\ X_i \in \mathrm{span}(e_1, \ldots, e_r),\ Y_j \in \mathrm{span}(e_{r+1}, \ldots, e_m) \end{array} \right\}$$

where $(e_1, \ldots, e_m)$ is a basis with $\langle e_i, e_j\rangle = \delta_{ij}\kappa_i$, $\kappa_1 = \ldots = \kappa_r = -1$ and $\kappa_{r+1} = \ldots = \kappa_m = 1$.

$\mathrm{Spin}_0(r,s)$ is the identity component and K its maximal compact subgroup.

Now let $\lambda : \mathrm{Spin}(r,s) \to SO(r,s)$ be the twofold covering of $SO(r,s)[SO_0(r,s)]$ by $\mathrm{Spin}(r,s)[\mathrm{Spin}_0(r,s)]$. λ_* is a Lie-algebra isomorphism between

$$\mathfrak{spin}(r,s) := LA(\mathrm{Spin}(r,s)) \subset Cl_{r,s} \quad \text{and} \quad \mathfrak{so}(r,s)\,.$$

Then

$$\mathfrak{spin}(r,s) = \mathrm{span}\{e_i \cdot e_j | 1 \leq i < j \leq m\}$$

and $\lambda_*(e_i \cdot e_j) = E_{ij}$ for matrices

$$(E_{ij})_{kl} = \begin{cases} -\kappa_j & \text{for} \quad (k,l) = (i,j) \\ \kappa_i & \text{for} \quad (k,l) = (j,i) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

the standard basis in $\mathfrak{so}(r,s)$.

We will now give an isomorphism between the complexification of the Clifford algebra $\mathbb{C}l_{(r,s)} := Cl_{r,s} \otimes \mathbb{C}$ and endomorphism algebras of complex vector spaces which yields complex representations of the spin group. First we consider the $\mathbb{C}^2$ with a basis $\left\{u(\varepsilon) := \frac{1}{\sqrt{2}}\binom{1}{-\varepsilon i} | \varepsilon = \pm 1\right\}$ and the isomorphisms of $\mathbb{C}^2$:

$$E := Id\ ,\ T := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ U := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ V := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then we have $T^2 = -V^2 = -U^2 = E$, $UT = -iV$, $VT = iU$, $UV = -iT$ and $Tu(\varepsilon) = -\varepsilon u(\varepsilon)$, $Uu(\varepsilon) = iu(-\varepsilon)$, $Vu(\varepsilon) = \varepsilon u(-\varepsilon)$. We define the isomorphisms as follows

1. In case m is even $\Phi_{(r,s)} : \mathbb{C}l_{(r,s)} \to \mathbb{C}\left(2^{\frac{m}{2}}\right)$ is defined by

$$\begin{aligned}
\Phi_{(r,s)}(e_{2k-1}) &:= \tau_{2k-1} E \otimes \ldots \otimes E \otimes U \otimes \underbrace{T \otimes \ldots \otimes T}_{(k-1)\text{-times}} \\
\Phi_{(r,s)}(e_{2k}) &:= \tau_{2k} E \otimes \ldots \otimes E \otimes V \otimes \underbrace{T \otimes \ldots \otimes T}_{(k-1)\text{-times}}
\end{aligned}$$

with $\tau_1 = \ldots = \tau_r = i$ and $\tau_{r+1} = \ldots = \tau_m = 1$ and $k = 1 \ldots \frac{n}{2}$.

2. In case m is odd $\Phi_{(r,s)} : \mathbb{C}l_{(r,s)} \to \mathbb{C}\left(2^{\frac{m-1}{2}}\right) \oplus \mathbb{C}\left(2^{\frac{m-1}{2}}\right)$ is defined by

$$\begin{aligned}
\Phi_{(r,s)}(e_k) &= \left(\Phi_{(m-2,1)}(e_k), \Phi_{(m-2,1)}(e_k)\right)\ ,\ k = 1 \ldots m-1 \\
\Phi_{(r,s)}(e_m) &= (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T)\,.
\end{aligned}$$

This yields representations of the spin group and algebra in case m even by restriction and in case m odd by restriction and projection onto the first component. The representation space $\Delta_{r,s} \doteq \mathbb{C}^{2^{[\frac{m}{2}]}}$ is called spinor module. We write $A \cdot v := \Phi_{(r,s)}(A)(v)$ for $A \in \mathbb{C}l_{r,s}$ and $v \in \Delta_{r,s}$. A useful basis in $\Delta_{r,s}$ is the following: $(u(\varepsilon_k, \dots, \varepsilon_1) := u(\varepsilon_k) \otimes \dots \otimes u(\varepsilon_1) | \varepsilon_i = \pm 1)$.

In case m is even the representation space $\Delta_{r,s}$ splits into two irreducible subspaces $\Delta^{\pm}_{r,s} := \mathrm{span}(u(\varepsilon_k, \dots, \varepsilon_1) | \varepsilon_k \cdot \dots \cdot \varepsilon_1 = \pm 1)$.

On $\mathbb{C}^{2^{[\frac{m}{2}]}}$ are given two scalar products, the standard scalar product $(.,.)$, which is positive definite and invariant under the action of the maximal compact subgroup K of $\mathrm{Spin}_0(r,s)$. The second is defined as follows

$$\langle u, v \rangle := i^{\frac{r(r-1)}{2}} (e_1 \cdot \dots \cdot e_r \cdot u, v)$$

and it is indefinite and invariant under $\mathrm{Spin}_0(r,s)$. The basis given by $u(\varepsilon_k, \dots, \varepsilon_1)$ with $k := [\frac{m}{2}]$ is an orthonormal basis for $(.,.)$.

We now consider a semi-Riemannian manifold which is assumed to be spin. Let $(\mathcal{Q}, f)$ be the λ-reduction for the orthonormal frame bundle $\mathcal{O}(M,h)$. Then we have $TM = \mathcal{O}(M,h) \times_{SO_0(r,s)} \mathbb{R}^m = \mathcal{Q} \times_{\mathrm{Spin}_0(r,s)} \mathbb{R}^m$. The vector bundle

$$S := \mathcal{Q} \times_{\mathrm{Spin}_0(r,s)} \Delta_{r,s}$$

is called spinor bundle. The Clifford multiplication is defined as follows

$$\begin{array}{rcl} TM \otimes S = \left(\mathcal{Q} \times_{\mathrm{Spin}_0(r,s)} \mathbb{R}^m\right) \otimes \left(\mathcal{Q} \times_{\mathrm{Spin}_0(r,s)} \Delta_{r,s}\right) & \to & S \\ X \otimes \varphi = [q,x] \otimes [q,v] & \mapsto & [q, X \cdot v] =: X \cdot \varphi . \end{array}$$

Since the indefinite scalar product $\langle .,. \rangle$ is invariant under $\mathrm{Spin}_0(r,s)$ it defines a scalar product on S. But one can also define the positive definite scalar product invariantly by fixing a maximal timelike (negativ definit) subbundle $\xi \subset TM$. Then there is a $K \subset \mathrm{Spin}_0(r,s)$-reduction of $\mathcal{Q}$ to a K-principle bundle $\mathcal{Q}_\xi$ for which

$$S = \mathcal{Q} \times_{\mathrm{Spin}_0(r,s)} \Delta_{r,s} = \mathcal{Q}_\xi \times_K \Delta_{r,s}$$

so that the K invariance of $(.,.)$ suffices to define a positive definite scalarproduct $(.,.)_\xi$ on S.

Let $\tilde{\omega}$ be the lift of the Levi-Civita connection ω into the spin structure $\mathcal{Q}$, $\mathfrak{p}^{\tilde{\omega}}_\gamma$ its parallel displacement. For its holonomy group we have the important relation to those of ∇

$$\lambda \left(Hol_q \left(\mathcal{Q}, \tilde{\omega} \right) \right) = Hol_{f(p)}(\mathcal{O}(M,h), \omega) \simeq Hol_x(M,h)$$

so that λ_* identifies the Lie algebras

$$\mathfrak{hol}_q \left(\mathcal{Q}, \tilde{\omega} \right) \overset{\lambda_*}{\simeq} \mathfrak{hol}_{f(p)}(\mathcal{O}(M,h), \omega) .$$

The covariant derivative corresponding to $\tilde{w}$ is given by

$$\nabla^S_X \varphi = X(\varphi) + \frac{1}{2} \sum_{i<j} \kappa_i \, \kappa_j \, h(\nabla^{LC}_{s_i} X, s_j) s_i \cdot s_j \cdot \varphi . \tag{5}$$

∇^S is metric with respect to $\langle .,. \rangle$ and distributive in the following way

$$\nabla^S (X \cdot \varphi) = (\nabla X) \cdot \varphi + X \cdot \nabla^S \varphi .$$

2.4. **Parallel spinors.** First let (M,h) be a semi-Riemannian spin manifold of dimension m. We define

Definition 2.2. A non-trivial spinor field $\varphi \in \Gamma(S)$ is called parallel spinor if $\nabla^S\varphi = 0$.

If we set $\tilde{H} := Hol_q(\mathcal{Q},\tilde{\omega}) = \lambda^{-1}(Hol_{f(q)}(\mathcal{O}(M,h),\omega))$ and $\tilde{\mathfrak{h}} := \mathfrak{hol}_q(\mathcal{Q},\tilde{\omega}) = \lambda_*^{-1}(\mathfrak{h}ol_{f(q)}(\mathcal{O}(M,h),\omega))$ then we have from the previous section

$$\begin{array}{rrcl} & V_{\tilde{H}} := \left\{ v \in \Delta_{r,s} \middle| (\Phi_{(r,s)})(\tilde{H})(v) = v \right\} & \simeq & \{\text{parallel spinors}\} \\ (6) & v & \mapsto & [\mathfrak{p}_\gamma^{\tilde{\omega}}(q), v] \end{array}$$

or in the simply connected case

$$\{\text{parallel spinors}\} \simeq V_{\tilde{\mathfrak{h}}} := \left\{ v \in \Delta_{r,s} \middle| (\Phi_{(r,s)})_*(\tilde{\mathfrak{h}})(v) = 0 \right\}.$$

We now want to find properties of manifolds admitting parallel spinors. One has the following

Lemma 2.1. ([Hit74] for the Riemannian case, [Bau81] for the general). *The existence of a parallel spinor entails a totally isotropic Ricci endomorphism, i.e.* $h\,(Ric(X), Ric(Y)) = 0$ *for all* $X, Y \in TM$.

If one studies the Berger-list of irreducible non-locally symmetric semi-Riemannian holonomy groups under the assumption that the manifold should admit parallel spinors one gets the following

Theorem 2.6. [Wan89, for the Riemannian case] [BK99, for the general] *Let (M,h) be a 1-connected, complete, irreducible semi-Riemannian spin manifold with parallel spinors. Then the holonomy representation on $\mathbb{R}^m$ is one of the following:*

$$\begin{array}{lcllcl} m = 2p+2q \geq 4 & : & SU(p,q) & \subset & SO(2p,2q) \\ m = 4p+4q \geq 8 & : & Sp(p,q) & \subset & SO(4p,4q) \\ m = 7 = 0+7 & : & G_2 & \subset & SO(7) \\ m = 7 = 4+3 & : & G^*_{2(2)} & \subset & SO(4,3) \\ m = 14 = 7+7 & : & G_2^{\mathbb{C}} & \subset & SO(7,7) \\ m = 8 = 0+8 & : & \mathrm{Spin}(7) & \subset & SO(8) \\ m = 8 = 4+4 & : & \mathrm{Spin}_0(4,3) & \subset & SO(4,4) \\ m = 16 = 8+8 & : & \mathrm{Spin}(7)^{\mathbb{C}} & \subset & SO(8,8). \end{array}$$

There are no irreducible Lorentzian manifolds with parallel spinors. We will illustrate this later.

For product manifolds holds the following (see for example [Lei00]).

Theorem 2.7. *On a semi-Riemannian manifold which is given as a product*

$$(M,h) \simeq (M_1,h_1) \times (M_2,h_2)$$

of two semi-Riemannian manifolds exist parallel spinors if and only if on both (M_i,h_i) exist parallel spinors.

2.5. **Decomposition of a Lorentzian manifold with parallel spinors.** From now on let (M, h) be a Lorentzian spin manifold of dimension m. We asign a vector field to every spinor field.

Definition 2.3. Let $\varphi \in \Gamma(S)$. The vector field $V_\varphi \in \Gamma(TM)$ defined by $h(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle$ for all $X \in TM$ is called φ-associated vector field or Dirac-current.

For the Dirac-current the following properties hold (see [Lei00]).

Lemma 2.2. *Let (M, h) a Lorentzian spin manifold and $\varphi \in \Gamma(S)$.*

1. *It is $h(V_\varphi, V_\varphi) \leq 0$ and $V_\varphi(x) = 0$ if and only if $\varphi(x) = 0$.*
2. *φ parallel entails V_φ parallel.*

From the second assertion of the lemma follows that a non-flat Lorentzian manifold with parallel spinors cannot be irreducible as we have seen in Corollary 2.1.

That means that we have two cases for a 1-connected Lorentzian manifold with parallel spinor φ:

1.) $h(V_\varphi, V_\varphi) < 0$, i.e. V_φ timelike. Since V_φ is parallel the holonomy group acts trivial on $\mathbb{R}\ V_\varphi(x) \subset T_xM$, and so the manifold decomposes due to Theorem 2.4 as follows

$$(M, h) \simeq (\mathbb{R}, -dt) \times (N, g)$$

with (N, g) a Riemannian manifold of dimension $m - 1$ which again can be decomposed in flat or irreducible Riemannian manifolds. Since irreducible locally symmetric Riemannian manifolds with parallel spinors have to be flat (as a conclusion of the Ricci-flatness, see [Bes87]) one obtains the following theorem from the result in the Riemannian case (see introduction).

Theorem 2.8. *(M, h) is a simply connected, complete Lorentzian manifold with parallel spinor whose associated vector field is timelike if and only if (M, h) is isometric to a product of $(\mathbb{R}, -dt^2)$ and 1-connected, irreducible Riemannian manifolds with one of the following holonomy groups $SU(k)$, $Sp(k)$, G_2, $\mathrm{Spin}(7)$ and possibly a flat factor.*

2.) Let V_φ lightlike. In this case (M, h) decomposes in a product of irreducible Riemannian manifolds with parallel spinors and an indecomposable Lorentzian manifold which is reducible (both are 1- connected), because of the parallel vector field coming from the parallel spinor. That means one has to investigate under which conditions such a manifold admits a parallel spinor field. This will be the aim of the following sections.

3. Parallel spinors on indecomposable, reducible Lorentzian manifolds

3.1. **Four types of indecomposable, reducible Lorentzian manifolds.** We will now cite some results of [BI93] and [Ike96] about holonomy groups of indecomposable but reducible Lorentzian manifolds.

Let (M, h) be a simply connected and complete Lorentzian manifold of dimension $m = n + 2 \geq 3$. We consider the tangent bundle as associated to the fibre bundle $\mathcal{L}(M, h)$ with fibres

$$\mathcal{L}_x(M, h) := \left\{ (t_0, X_1, \ldots X_n, t_{n+1}) : \begin{array}{c} \text{a basis in } T_xM \text{ with } h(t_0, t_{n+1}) = 1, \\ h(t_k, t_k) = h(t_k, X_i) = 0, h(X_i, X_j) = \delta_{ij} \end{array} \right\}$$

and structure group $SO(\eta)$ where $\eta := \begin{pmatrix} 0 & 0^t & 1 \\ 0 & E_n & 0 \\ 1 & 0^t & 0 \end{pmatrix}$.

If (M,h) now is indecomposable and reducible, not necessarily with a fixed lightlike vector under holonomy representation but only with an invariant degenerate subspace $E \subset T_xM$ then $E^\perp$ is invariant and one defines $T := E \cap E^\perp \neq \{0\}$ an invariant, isotropic, one-dimensional subspace. This subspace defines via theorem 2.3 an one dimensional isotropic parallel distribution $\mathcal{T}$ so that we have a reduction of $\mathcal{L}(M,h)$ to the bundle

$$\mathcal{N}(M,h) := \{(t_0, X_1, \ldots X_n, t_{n+1}) \in \mathcal{L} | t_0 \in \mathcal{T}\}$$

and structure group G, which has the Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0^t & -a \end{pmatrix} \middle| a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}. \tag{7}$$

In order to find those subgroups which do not leave any non degenerate subspace invariant one fixes some notations:

1. $\mathfrak{g}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}^n \oplus \mathfrak{so}(n)$ as vector space and with the commutator:

$$[(a,X,A),(b,Y,B)] = \Big(0, (A + a\, Id)\, Y - (B + b\, Id)\, X, [A,B]_{\mathfrak{so}(n)}\Big). \tag{8}$$

2. $\mathfrak{a} := \{(a,0,0)\, | a \in \mathbb{R}\} \subset \mathfrak{g}$ an abelian subalgebra of $\mathfrak{g}$.
3. $\mathfrak{n} := \{(0,X,0)\, | X \in \mathbb{R}^n\} \subset \mathfrak{g}$ an abelian ideal in $\mathfrak{g}$ isomorphic to $\mathbb{R}^n$.
4. $\mathfrak{k} := \{(0,0,A)\, | A \in \mathfrak{so}(n)\} \subset \mathfrak{g}$ a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{so}(n)$.
5. Further it is $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{n}$ and $[\mathfrak{g},\mathfrak{k}] \subset \mathfrak{k} \ltimes \mathfrak{n}$ as well as $[\mathfrak{k},\mathfrak{a}] = \{0\}$ and $[\mathfrak{a} \oplus \mathfrak{k}, \mathfrak{n}] \subset \mathfrak{n}$. The latter means that

$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{k}) \ltimes \mathfrak{n}.$$

Let now $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of $\mathfrak{g}$ and set

$$\begin{aligned} \mathfrak{b} &:= pr_{\mathfrak{k}}\mathfrak{h} \subset \mathfrak{k} \simeq \mathfrak{so}(n) \\ \mathfrak{t} &:= \mathfrak{z}(\mathfrak{b}) \text{ the center of } \mathfrak{b}, \text{ an abelian subalgebra of } \mathfrak{k} \\ \mathfrak{d} &= [\mathfrak{b},\mathfrak{b}] \text{ a subalgebra of } \mathfrak{k}. \end{aligned}$$

From a general theorem (see for example [KN63, Appendix 5]) which asserts that subalgebras of $\mathfrak{so}(n)$ are reductive the following decomposition follows

$$\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{d} \text{ and } \mathfrak{d} \text{ is semisimple.} \tag{9}$$

Under the assumption that $\mathfrak{h}$ is the holonomy algebra of a Lorentzian manifold with invariant degenerate subspace of the holonomy representation in [BI93] now an important fact about $\mathfrak{b}$ is proved which is an analogon to the complete reducibility in the Riemannian case.

Theorem 3.1. [BI93] *Let $\mathfrak{b} := pr_{\mathfrak{k}}(\mathfrak{hol}_x(M,h))$ the projection of the holonomy algebra of an indecomposable, reducible Lorentzian manifold onto the $\mathfrak{so}(n)$-component. Then $\mathfrak{b}$ is completely reducible. I.e. there exist decompositions of $\mathbb{R}^n$ in orthogonal subspaces and of $\mathfrak{b}$ in commuting ideals*

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_r \quad \textit{and} \quad \mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$$

where $\mathfrak{b}$ acts trivial on E_0, $\mathfrak{b}_i$ acts irreducible on E_i and $\mathfrak{b}_i(E_j) = \{0\}$ for $i \neq j$.

For the following we have to cite another general fact about subgroups of $SO(n)$.

Theorem 3.2. [KN63, Appendix 5] *Let $G \subset SO(n)$ connected.*

1. *If G is semisimple so it is closed in $SO(n)$ and therefore compact.*
2. *If G acts irreducible on $\mathbb{R}^n$ then the dimension of the center of G is smaller than two, i.e. $Z(G)$ is discrete or isomorphic to S^1.*

Together with (9) one concludes that an irreducible acting subgroup of $SO(n)$ is closed in $SO(n)$ and therefore compact.

For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with the property that it does not leave any non degenerate subspace invariant (e.g. the holonomy algebra of an indecomposable but reducible Lorentzian manifold) in [BI93] is proved a dinstinction in four exclusive types based upon the possible projections of $\mathfrak{h}$ on $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{k}$. (The corresponding Lie groups to the algebras are denoted by the corresponding latin capitals.)

1. $pr_{\mathfrak{n}}(\mathfrak{h}) = \mathfrak{n}$

   Type 1: $pr_{\mathfrak{a}}(\mathfrak{h}) = \mathfrak{a}$. Therefore $\mathfrak{h} = (\mathfrak{a} \oplus \mathfrak{b}) \ltimes \mathfrak{n}$ and so $H = (A \times B) \ltimes N$. $A \simeq \mathbb{R}^+$ and $N \simeq \mathbb{R}^n$ are closed and by theorem 3.2 also B is closed so that H is closed.

   Type 2: $pr_{\mathfrak{a}}(\mathfrak{h}) = 0$ i.e. $\mathfrak{h} = \mathfrak{b} \ltimes \mathfrak{n}$, i.e. $H = B \ltimes N$ closed.

   Type 3: Neither Type 1 nor Type 2.

   In that case exists a surjective homomorphism $\varphi : \mathfrak{t} \to \mathfrak{a}$, such that
   $$\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{d}) \ltimes \mathfrak{n}$$
   where $\mathfrak{l} := graph\ \varphi = \{(\varphi(T), T) | T \in \mathfrak{t}\} \subset \mathfrak{a} \oplus \mathfrak{t}$. That means $H = (L \times D) \ltimes N$. D and N are closed. Now one shows that L and therefore H are closed if and only if $Ker\ \varphi$ generates a compact subgroup of T.

2. **Type 4:** $pr_{\mathfrak{n}}(\mathfrak{h}) \neq \mathfrak{n}$ i.e. $\mathfrak{h}$ does not contain $\mathfrak{n}$. Then exists
   (a) a non-trivial decomposition $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{q} \simeq \mathbb{R}^p \oplus \mathbb{R}^q$, $0 < p, q < n$,
   (b) a surjective homomorphism $\varphi : \mathfrak{t} \to \mathfrak{q}$
   such that $\mathfrak{b} \subset \mathfrak{so}(p)$ and $\mathfrak{h} = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathfrak{p} \subset \mathfrak{g}$ where $\mathfrak{l} := \{(\varphi(A), A) | A \in \mathfrak{t}\} = graph\ \varphi \subset \mathfrak{q} \oplus \mathfrak{t}$. Or written as matrices:
   $$\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & 0 & X^t & \varphi(A)^t \\ 0 & 0 & 0 & 0 \\ 0 & -X & A+B & 0 \\ 0 & -\varphi(A) & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{t}, B \in \mathfrak{d}, X \in \mathbb{R}^p \right\}.$$
   So $H = (L \times D) \ltimes P$ with P isomorphic to $\mathbb{R}^p$ and again H and L are closed if and only if $Ker\ \varphi$ generates a compact subgroup of the torus T.

   Further one observes that the non-triviality of the decomposition of $\mathfrak{n}$ forces the center of $\mathfrak{b}$ to non-trivial.

Remark:

1. In [BI93] examples of metrics for all types and especially with non-closed holonomy groups of type 3 and 4 are given.
2. From the commutator relation (8) it is clear that the only abelian indecomposable, reducible holonomy algebra has to be of type 1 or 2, both with $pr_{\mathfrak{k}}\mathfrak{h} = 0$.

Now one separates the two types with a parallel vector field as follows.

Definition 3.1. A Lorentzian manifold with a lightlike parallel vector field is called Brinkmann- wave (see [Bri25], also [Eis38]).

Corollary 3.1. *An indecomposable, reducible Lorentzian manifold is of type 2 or 4 if and only if it is a Brinkmann-wave.*

The proof is clear since the condition for a Brinkmann-wave is equivalent to the existence of a one dimensional trivial subrepresentation of $\mathfrak{h}$ but this occurs only for the types 2 and 4.

3.2. **Conclusions for the existence of parallel spinors.** We will now prove some first, purely algebraic properties for the components of $\mathfrak{h} := \mathfrak{hol}_x(M,h)$ in relation to the existence of parallel spinors.

With respect to the four types from the previous section the existence of parallel spinors gives an obvious restriction.

Proposition 3.1. *Let (M,h) be an indecomposable Lorentzian manifold with lightlike parallel spinor. Then (M,h) is reducible of type 2 or 4.*

If further its holonomy algebra is abelian then it is of type 2, namely it equals to $\mathfrak{n}$.

The proof is clear because indecomposable manifolds with parallel spinors are Brinkmann-waves.

We consider the Lie-algebras of both types

$$\mathfrak{h}_1 = \mathfrak{b}_1 \ltimes \mathfrak{n} \text{ and } \mathfrak{h}_2 = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathfrak{p}$$

with $\mathfrak{b}_i := pr_{\mathfrak{k}}(\mathfrak{h}_i) \subset \mathfrak{k} \simeq \mathfrak{so}(n)$ for $n+2=m$ and $\mathfrak{d}$ a semisimple and $\mathfrak{t}$ an abelian subalgebra of $\mathfrak{so}(p)$ which commute with each other, $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{q} \simeq \mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ for $0 < p < n$ and $\mathfrak{l} := graph(\phi) \subset \mathfrak{t} \oplus \mathfrak{q}$ for $\phi : \mathfrak{t} \to \mathfrak{q}$ surjective. We ask under which condition

$$V_{\mathfrak{h}_i} := \left\{ v \in \Delta_{1,m-1} \,\middle|\, \lambda_*^{-1}(\mathfrak{h}_i) v = 0 \right\}$$

is non trivial.

First we can reduce the problem in the following way.

Proposition 3.2. *Let (M,h), $x \in M$, $t_0, X_1, \dots, X_n, t_{n+1}$ a basis in T_xM as in the previous section and $\left(u(\varepsilon_k, \dots, \varepsilon_0) | \varepsilon = \pm 1, k = [\frac{n}{2}]\right)$ a basis of $\Delta_{1,m-1} = \Delta_{1,n+1}$. Then the following holds*

1. $V_{\mathfrak{n}} = \mathrm{span}\left(\{u(\varepsilon_k, \dots, \varepsilon_1, 1) | \varepsilon_i = \pm 1\}\right)$.
2. *Let* $\mathfrak{b}_i = pr_{\mathfrak{k}}\mathfrak{h}_i$, $i = 1,2$. *Then* $\dim V_{\mathfrak{h}_i} = \frac{1}{2} \dim V_{\mathfrak{b}_i}$.

Proof. 1.) We consider the following orthonormal frame in $T_xM \simeq \mathbb{R}^{n+2}$:

$$\begin{aligned} e_- &:= \frac{1}{\sqrt{2}}(t_0 - t_{n+1}) \quad \text{timelike unit vector} \\ e_+ &:= \frac{1}{\sqrt{2}}(t_0 + t_{n+1}) \end{aligned}$$

and the $e_1, \dots, e_n$. In this basis the isotropy subalgebra $\mathfrak{g}$ of t_0 in $\mathfrak{so}(1, m-1) \supset \mathfrak{g} \supset \mathfrak{h}_i$ has the following shape

$$\begin{pmatrix} 0 & 0 & X^t \\ 0 & 0 & X^t \\ X & -X & A \end{pmatrix} \text{ with } X \in \mathbb{R}^n, A \in \mathfrak{so}(n) . \tag{10}$$

For the standard basis in $\mathfrak{so}(1, m-1)$ (see (4)) and $e_-, e_+, e_1, \dots e_n$ in $\mathbb{R}^{n+2}$ it is $\mathfrak{n} = \operatorname{span}(E_{-\,i} + E_{+\,i} | i = 1, \dots, n)$ and

$$\tilde{\mathfrak{n}} := \lambda_*^{-1}(\mathfrak{n}) = \operatorname{span}(e_- \cdot e_i + e_+ \cdot e_i = -e_i \cdot (e_- + e_+),\ i = 1, \dots, n) .$$

Since the e_i's are isomorphisms it is $V_{\tilde{\mathfrak{n}}} = V_{\mathbb{R} e_i \cdot (e_- + e_+)} = V_{\mathbb{R}(e_- + e_+)}$ and from

$$(e_- + e_+) \cdot u(\varepsilon_k, \dots, \varepsilon_0) = (\varepsilon_0 - 1)\, u(\varepsilon_k, \dots, \varepsilon_1, -\varepsilon_0)$$

follows the conclusion.

2.) Since $\tilde{\mathfrak{b}}_i \subset \mathfrak{spin}(n) = \operatorname{span}(e_i \cdot e_j | 1 \leq i < j \leq n)$ does not act on $u(\varepsilon_0)$ in $u(\varepsilon_k, \dots, \varepsilon_0)$ it is clear that $V_{\tilde{\mathfrak{b}}_i}$ has a basis of the form $(v_1^+, v_1^-, \dots v_l^+, v_l^-)$ with $v_i^\pm \in \operatorname{span}(\{u(\varepsilon_k, \dots, \varepsilon_1, \pm 1) | \varepsilon_i = \pm 1\})$.

Then the v_k^+ lie in $V_{\tilde{\mathfrak{h}}_i}$ for both i. We have to show that they are a basis of both.

For $i = 1$ this is clear because of $V_{\tilde{\mathfrak{h}}_1} = V_{\tilde{\mathfrak{n}}} \cap V_{\tilde{\mathfrak{b}}}$.

For $i = 2$ one takes a $v \in V_{\tilde{\mathfrak{h}}_2}$. Since it has to be in $V_{\tilde{\mathfrak{p}}}$ it must lie in $\operatorname{span}(\{u(\varepsilon_k, \dots, \varepsilon_1, 1) | \varepsilon_i = \pm 1\})$ and therefore in $V_{\tilde{\mathfrak{q}}}$. This entails $v \in V_{\tilde{\mathfrak{l}}}$. The assumption to v gives also $v \in V_{\tilde{\mathfrak{d}}}$ and therefore finally $v \in V_{\tilde{\mathfrak{b}}}$. Thats why it is $V_{\tilde{\mathfrak{h}}_2} = V_{\tilde{\mathfrak{b}}} \cap V_{\tilde{\mathfrak{n}}}$ which is spanned by the v_i^+. □

As a corollary we obtain a sufficient condition for the existence of parallel spinors

Corollary 3.2. *Let (M, h) be a simply connected, indecomposable, reducible Lorentzian manifold with $\mathfrak{hol}_x(M, h) = \mathfrak{n}$. Then (M, h) admits parallel spinors with light like Dirac current.*

Remark:

1. Since the indecomposable reducible Lorentzian symmetric spaces have holonomy $\mathfrak{n}$ (see [CW70], also [BI93] for the result) this corollary gives a first group of examples of indecomposable Lorentzian manifold with parallel spinors. In section 4.3 we give more examples and describe these spaces further.
2. The parallel spinors resulting from the basis of $V_{\tilde{\mathfrak{n}}}$ namely $\varphi_{(\varepsilon_k, \dots, \varepsilon_1)} := [q, u(\varepsilon_k, \dots, \varepsilon_1, 1)]$ are pure. That means the complex dimension of the space $\{Z \in TM^{\mathbb{C}} | Z \cdot \varphi_{(\varepsilon_k, \dots, \varepsilon_1)} = 0\}$ is maximal, that is equal to $[\frac{n+2}{2}]$ (see [Kat00]). A basis of this space is given by $\left(t_0, e_1 - i\varepsilon_1 e_2, \dots, e_{n-1} - i\varepsilon_{[\frac{n}{2}]} e_n\right)$.
3. The second point of the proposition reduces the problem to find subalgebras $\mathfrak{h}_i \subset \mathfrak{so}(1, m-1)$ with $V_{\tilde{\mathfrak{h}}_i} \neq 0$ to the problem to find subalgebras $\mathfrak{b} \subset \mathfrak{so}(n)$ with $V_{\tilde{\mathfrak{b}}} \neq 0$ where here λ_*^{-1} can be understood as the differential of the twofold covering $\lambda : \operatorname{Spin}(n) \to SO(n)$ since the representations are equivalent $\Delta_{1,m-1} \simeq \Delta_{1,1} \otimes \Delta_n$.

From Theorem 3.1 one obtains the following for $\mathfrak{b}$.

Proposition 3.3. *Let $\mathfrak{b} \subset \mathfrak{k} \simeq \mathfrak{so}(n)$.*

1. *Let $V_{\tilde{\mathfrak{b}}} \neq 0$, $\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_r$ and $\mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$ the decomposition in irreducible subrepresentations due to Theorem 3.1. Then for $i = 1 \ldots k$ it has to be* $\dim\ E_i = 4k_i$ *or* $\mathfrak{z}(\mathfrak{b}_i) = 0$.
2. *For $\mathfrak{b} = pr_{\mathfrak{k}}\mathfrak{h}$ and $\mathfrak{h}$ the holonomy algebra of an indecomposable Lorentzian manifold with a parallel spinor. If $\mathfrak{b}$ is abelian, then it is trivial.*

Proof. 1.) From Theorem 3.2 we have for irreducible acting $\mathfrak{b}_i$ that the center of $\mathfrak{b}_i$ is trivial or one-dimensional, i.e. $\dim\ E_i = 2k_i$ and $\mathfrak{t}_i = \mathfrak{z}(\mathfrak{b}_i) = \mathbb{R}J$ with $J^2 = -Id$. J is of the form

$$J = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & j \end{pmatrix} \text{ with } j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now it is $\tilde{J} := \lambda_*^{-1}(J) = \sum_{l=1}^{k_i} e_{2l-1} \cdot e_{2l}$ and therefore

$$\tilde{J}u\left(\varepsilon_{k_i}, \ldots, \varepsilon_1\right) = -\left(\varepsilon_{k_i} + \ldots + \varepsilon_1\right) u\left(\varepsilon_{k_i}, \ldots, \varepsilon_1\right).$$

In case that $V_{\tilde{\mathfrak{t}}_i} \neq 0$ the k_i's must be even.

2.) Let $\mathfrak{b}$ given as in the assumptions. From Theorem 3.1 we have $\mathfrak{b} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_r$ and $\mathbb{R}^n = E_1 \oplus \ldots \oplus E_r \oplus E_0$ such that $\mathfrak{b}$ acts trivial on E_0 and $(\mathfrak{b}_i, E_i), i = 1 \ldots r$ are irreducible representations. Since $\mathfrak{b}$ is abelian, so are the $\mathfrak{b}_i$'s, but this entails $\mathfrak{b}_i \simeq \mathfrak{so}(2)$ and $E_i = \mathbb{R}^2$, $i = 1, \ldots r$ or $\mathfrak{b}_i$ trivial. Now the second proposition follows from the first. □

4. Brinkmann-waves with special holonomy and parallel spinors

4.1. **The local form of the metric of an indecomposable, reducible Lorentzian manifold.** In [Ike96] the following theorem about the local form of the metric is proved. See also [Bri25] and [Eis38] for the second part.

Proposition 4.1. [Ike96] *Let (M, h) be an indecomposable, reducible Lorentzian manifold. Then for every point there exists a coordinate system $(U, \xi := (x_0, x_1, ..., x_n, x_{n+1}))$ such that the metric h has the following form on U*

$$h = 2\ dx_0 dx_{n+1} + \sum_{i=1}^{n} u_i dx_i dx_{n+1} + f dx_{n+1}^2 + \sum_{i,j=1}^{n} g_{ij} dx_i\ dx_j$$

where $f, u_i, g_{ij} \in C^\infty(U)$ with $\frac{\partial}{\partial x_0} g_{ij} = \frac{\partial}{\partial x_0} u_i = 0$.

If (M, h) is of type 2 or 4, that is a Brinkmann-wave, then one has in addition $\frac{\partial}{\partial x_0} f = 0$.

It is clear that in case of a Brinkmann-wave the vector field $\partial_0 := \frac{\partial}{\partial x_0}$ corresponds to the parallel lightlike vector field.

If one considers small n-dimensional submanifolds in U through $x = \xi^{-1}(x_0, x_1, \ldots, x_n, x_{n+1})$ defined by

$$W_{(x_0, x_{n+1})} := \{\xi^{-1}(x_0, y_1, \ldots y_n, x_{n+1}) | (y_1, \ldots y_n) \in \mathbb{R}^n \cap \xi(U)\}$$

then one can understand the g_{ij} as coefficients of a family of Riemannian metrics $g_{x_{n+1}}$ and the u_i as coefficients of a family of 1-forms $\phi_{x_{n+1}}$ on $W_{(x_0,x_{n+1})}$ which depends on a parameter x_{n+1}.

In this sense [Ike96] proves the following

Proposition 4.2. *Let $x = \xi^{-1}(x_0, x_1, \dots, x_n, x_{n+1}) \in W_{(x_0,x_{n+1})} \subset U$. Then it is $Hol_x(W_{(x_0,x_{n+1})}, g_{x_{n+1}}) \subset pr_K(Hol_x(M,h))$.*

Remark:

1. In general there is no equality. In [Ike96] is given an example of a 7-dimensional Lorentzian manifold for which $g_{x_{n+1}}$ is flat for all x_{n+1}, $\mathfrak{b}$ is 3-dimensional and acts irreducible on $\mathbb{R}^5$. But due to the Berger classification $\mathfrak{b}$ cannot be the holonomy algebra of a Riemannian manifold which has to be $\mathfrak{so}(5)$ in case of irreducible action. That means that there is no coordinate transformation such that the holonomy of the transformed g corresponds to $\mathfrak{b}$.
2. It is known (see [Eis38], [Sch74]) that every Brinkmann-wave can be transformed into the following shape

$$h = 2\ dx_0 dx_{n+1} + \sum_{i,j=1}^{n} g_{ij} dx_i\ dx_j \tag{11}$$

   where $g_{ij} \in C^\infty(U)$ with $\frac{\partial}{\partial x_0} g_{ij} = 0$. One can say that this makes the dependence of g_{ij} on x_{n+1} more complicated.

From proposition 4.2 one obtains the following corollary for the existence of parallel spinors.

Proposition 4.3. *Let (M,h) be a simply connected Brinkmann-wave with parallel spinor. Then all metrics $g_{x_{n+1}}$ from the family of metrics given by the local form of h admit parallel spinors.*

For the Ricci-tensor one can prove the following. The proof is due to [FO99].

Lemma 4.1. *Let (M,h) be a Brinkmann-wave given in the local form $h = h(f, u_i, g_{ij})$. Its Ricci-endomorphism is totally isotropic if and only if $Ric(\partial_i, .) = 0$, i.e. $Ric = r\ dx_{n+1} \circ dx_{n+1}$ for a function r.*

4.2. **Special Brinkmann waves.** In this section we will prove a theorem about a special class of Brinkmann waves (M,h). We set the following assumption on (M,h):

1. For every $x \in M$ exists a coordinate neighborhood, such that h has the following form on U

$$h = 2\ dx_0 dx_{n+1} + \phi_{x_{n+1}} dx_{n+1} + f dx_{n+1}^2 + c^2\ g$$

   where $\frac{\partial}{\partial x_0} f = 0$, $\phi_{x_{n+1}}$ a 1-form on $W_{(x_0,x_{n+1})}$ depending on x_{n+1}, g a metric in $W_{(x_0,x_{n+1})}$, not depending on x_{n+1} and c a function of x_{n+1}.

   Then it is clear that $Hol_x(W_{(x_0,x_{n+1})}, g) \overset{conjugated}{\simeq} Hol_y(W_{(y_0,y_{n+1})}, g)$ for both W's contained in the same coordinate neighborhood U.
2. We assume furthermore, that this must hold also for W's in different coordinate neighborhoods U and V.

3. Finally every $\phi_{x_{n+1}}$ should be closed: $d\phi_{x_{n+1}} = 0$. Since $\phi_{x_{n+1}}$ is understood as a family of forms on $W_{(x_o,x_{n+1})}$ the differential is taken with respect to the variables $x_1, \dots x_n$.

Then holds the following

Theorem 4.1. *For f sufficient general (e.g. $\partial_k^2 f \neq 0$) and $x = \xi^{-1}(x_0, x_1, \dots x_n, x_{n+1})$ it is*

$$Hol_x(M,h) \simeq Hol_x(W_{(x_0,x_{n+1})}, g) \ltimes \mathbb{R}^n.$$

Proof. First we will calculate the local curvature forms. We notice that for a Brinkmann wave with parallel vector field $T = \partial_0$ exists a reduction of the bundle of orthonormal frames and the Levi-Civita connection to the following bundle

$$\mathcal{P} := \{(X_-, X_+, X_1, \dots X_n) \in \mathcal{O}(M,h) | h(X_\mp, T) = \mp 1, h(X_i, T) = 0\} \tag{12}$$

and structure group $K \ltimes N \simeq SO(n) \ltimes \mathbb{R}^n$ realized as subgroup of $SO(1, n+1)$ as in formula (10). Now we formulate a

Lemma 4.2. *Let $s = (s_-, s_+, s_1, \dots, s_n) \in \Gamma(U, \mathcal{P}(M,h))$ be a local section around a point $x = \xi^{-1}(x_0, \hat{x}, x_{n+1})$ which lies in the coordinate neighborhood V_x. Set $W := U \cap W_{(x_0,x_{n+1})}$. Then $\hat{s} := (\hat{s}_1, \dots, \hat{s}_n)$ with $\hat{s}_i := pr_{TW}(s_i)$ is a local section in the bundle of orthonormal frames for (W,g). For the connection and curvature θ and Θ resp. ω and Ω of g resp. h holds*

1. $pr_{so(n)}(\omega^s(\partial_{n+1})) = pr_{so(n)}(\omega^s(\partial_0)) = 0$ *and*
2. $pr_{so(n)}(\omega^s(X)) = \theta^{\hat{s}}(X)$ *for $X \in T_xW$,*
3. $pr_{so(n)}(\Omega^s(\partial_{n+1}, .)) = pr_{so(n)}(\Omega^s(\partial_0, .)) = 0$ *and*
4. $pr_{so(n)}(\Omega^s(X,Y)) = \Theta^{\hat{s}}(X,Y)$ *for $X, Y \in T_xW$.*

Proof. First we get rid of the factor $c^2(x_{n+1})$ by a change of coordinates:

$$\bar{x}_0 := x_0 \quad \bar{x}_{n+1} := x_{n+1} \quad \bar{x}_i := c(x_{n+1})x_i\,.$$

In this coordinates $\bar{h}_{ij} = g_{ij}$ independent of x_{n+1} and $\bar{\phi} = \frac{1}{c}\sum_{i=1}^n u_i d\bar{x}_i$ if $\phi = \sum_{i=1}^n u_i dx_i$. Since $\frac{\partial}{\bar{x}_i} = \frac{1}{c}\frac{\partial}{x_i}$ one gets that $\bar{\phi}$ is closed, too. From now on we drop the bar.

Let $s = (s_-, s_+, s_1, \dots, s_n) \in \Gamma(U, \mathcal{P}(M,h))$. Because of $h(\partial_0, s_i) = 0$ it is clear that s_i has no ∂_{n+1}-part. We can write in the coordinate neighborhood

$$s_i = \underbrace{\sum_{k=1}^n \xi_{ik}\partial_k}_{=:\tilde{s}_i} + \zeta_i \partial_0\,.$$

Because of the local form of the metric the ξ_{ij} are independent of x_0 and x_{n+1}. So it is

$$[\partial_0, \tilde{s}_i] = [\partial_{n+1}, \tilde{s}_i] = 0\,. \tag{13}$$

On W one has $\hat{s}_i = \tilde{s}_i$.

We now consider families of n-dimensional submanifolds $W_{(y_0,y_{n+1})}$ near $W_{(x_0,x_{n+1})} = W$ and calculate the relevant terms for the components of ω^s in x:

1.) $h(\nabla_T s_i, s_j) = h(\nabla_T \tilde{s}_i, \tilde{s}_j) = h([T, \tilde{s}_i], \tilde{s}_j) = 0$.

2.) For $X \in TW_{(y_0,y_{n+1})}$: $h(\nabla_X s_i, s_j) = h(\nabla_X \tilde{s}_i, \tilde{s}_j) =$

$$\begin{aligned}
&= \frac{1}{2}\left(\tilde{s}_i\left(h(X,\tilde{s}_j)\right) - \tilde{s}_j\left(h(X,\tilde{s}_i)\right) - h\left([X,\tilde{s}_j],\tilde{s}_i\right) + h\left([X,\tilde{s}_i],\tilde{s}_j\right) + h\left([\tilde{s}_j,\tilde{s}_i],X\right)\right)\\
&= \frac{1}{2}\left(\tilde{s}_i\left(g(X,\tilde{s}_j)\right) - \tilde{s}_j\left(g(X,\tilde{s}_i)\right) - g\left([X,\tilde{s}_j],\tilde{s}_i\right) + g\left([X,\tilde{s}_i],\tilde{s}_j\right) + g\left([\tilde{s}_j,\tilde{s}_i],X\right)\right)\\
&= g(\nabla^g_X \tilde{s}_i, \tilde{s}_j)\,.
\end{aligned}$$

3.) Since $h(\nabla_{\partial_{n+1}}\tilde{s}_i, \partial_0) = 0$ it is $h(\nabla_{\partial_{n+1}} s_i, s_j) = h(\nabla_{\partial_{n+1}}\tilde{s}_i, \tilde{s}_j)$ and

$$\begin{aligned}
h(\nabla_{\partial_{n+1}}\tilde{s}_i, \tilde{s}_j) &= \frac{1}{2}\left(\tilde{s}_i\left(h(\partial_{n+1},\tilde{s}_j)\right) - \tilde{s}_j\left(h(\partial_{n+1},\tilde{s}_i)\right)\right)\\
&\quad -\frac{1}{2}\left(h\left([\partial_{n+1},\tilde{s}_j],\tilde{s}_i\right) - h\left([\partial_{n+1},\tilde{s}_i],\tilde{s}_j\right) - h\left([\tilde{s}_j,\tilde{s}_i],\partial_{n+1}\right)\right)\\
&= \frac{1}{2}\left(d\phi\left(\tilde{s}_i,\tilde{s}_j\right)\right) - h\left([\partial_{n+1},\tilde{s}_j],\tilde{s}_i\right) + h\left([\partial_{n+1},\tilde{s}_i],\tilde{s}_j\right)\big)\\
&= 0\,.
\end{aligned}$$

So we get on W:

$$\begin{aligned}
pr_{\mathfrak{so}(n)}\left(\omega^s(X)\right) &= \sum_{1\le i<j\le n} h\left(\nabla_X s_i, s_j\right) E_{ij}\\
&\overset{W}{=} \begin{cases} 0 & \text{for} \quad X = \partial_0\\ 0 & \text{for} \quad X = \partial_{n+1}\\ \theta^{\hat{s}}(X) & \text{for} \quad X \in TW\,. \end{cases}
\end{aligned}$$

We now prove the same for the local curvature form.

1. For $X, Y \in TW$ it holds (because of the result just proved and because of the commutator relations in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$):

$$\begin{aligned}
pr_{\mathfrak{so}(n)}\left(\Omega^s(X,Y)\right) &= pr_{\mathfrak{so}(n)}\left(d\omega^s(X,Y) + \frac{1}{2}[\omega^s(X),\omega^s(Y)]\right)\\
&= d\theta^{\hat{s}}(X,Y) + \frac{1}{2}[pr_{\mathfrak{so}(n)}\left(\omega^s(X)\right), pr_{\mathfrak{so}(n)}\left(\omega^s(Y)\right)]\\
&= d\theta^{\hat{s}}(X,Y) + \frac{1}{2}[\theta^{\hat{s}}(X),\theta^{\hat{s}}(Y)]\\
&= \Theta^{\hat{s}}(X,Y)\,.
\end{aligned}$$

2. For $X = \partial_0$ and $Y = \partial_k$ or $= \partial_{n+1}$:

$$\begin{aligned}
pr_{\mathfrak{so}(n)}\left(\Omega^s(\partial_0, Y)\right) &= \sum_{1\le i<j\le n}\left(\partial_0\left(h(\nabla_Y s_i, s_j)\right) - Y\left(h(\nabla_{\partial_0} s_i, s_j)\right)\right) E_{ij}\\
&= 0\,.
\end{aligned}$$

3. for $X = \partial_{n+1}$, $Y = \partial_k$:

$$\begin{aligned}
pr_{\mathfrak{so}(n)}\left(\Omega^s(\partial_{n+1}, \partial_k)\right) &= \sum_{1\le i<j\le n}\left(\partial_{n+1}\left(h(\nabla_{\partial_k} s_i, s_j)\right) - \partial_k\left(h(\nabla_{\partial_{n+1}} s_i, s_j)\right)\right) E_{ij}\\
&= 0\,.
\end{aligned}$$

This proves the assertion of the lemma. □

Now we prove the proposition. Let $x \in V_x$ a coordinate neighborhood and $x \in W_{(x_0,x_{n+1})}$. g^x denotes the Riemannian part of h in V_x.

The assumption to f ensures that the whole $\mathbb{R}^n$ is generated so that we have to show that

$$pr_{\mathfrak{so}(n)}\left(\mathfrak{hol}_x(M,h)\right) \simeq \mathfrak{hol}_x(W_{(x_0,x_{n+1})},g^x).$$

The inclusion $\supset$ is clear because of proposition 4.2. For the other direction we use the Ambrose-Singer holonomy theorem. Let $p \in \mathcal{P}_x(M,h)$ and $\mathcal{P}^\omega(p)$ the holonomy bundle of p with respect to ω the Levi-Civita connection. We have then by Ambrose-Singer

$$\mathfrak{hol}_x(M,h) \simeq \mathfrak{hol}_p(\omega) \simeq \{\Omega_q(X,Y) | q \in \mathcal{P}^\omega(p), X,Y \in Th_qP\}.$$

Let $q \in \mathcal{P}^\omega(p)$ fixed and $\pi(q) =: y \in M$. We now find a local section $s \in \Gamma(U_y, \mathcal{P}(M,h))$ such that

$$s(y) = q \quad \text{and} \quad (ds)_y(X) \in Th_q\mathcal{P}(M,h) \text{ for } X \in T_yM.$$

Now we choose a coordinate neighborhood V_y such that $V_y \subset U_y$. As in the previous lemma we have a section $\hat{s}$ in the bundle of frames over $W_{(y_0,y_{n+1})} =: W$ orthonormal with respect to g^y.

Because of the choosen properties of s one gets

$$\begin{aligned} pr_{\mathfrak{so}(n)}\left(\Omega_q(Y_1,Y_2)\right) &= pr_{\mathfrak{so}(n)}\left(\Omega^s_y(X_1,X_2)\right) \\ &\overset{Lemma 4.2}{=} \begin{cases} \Theta^{\hat{s}}_y(X_1,X_2) & \text{for } X_i = d\pi(Y_i) \in T_yW \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We consider now the holonomy bundle $\hat{P}^\theta(\hat{s}(y))$ of the bundle $\hat{P}$ of g-orthonormal frames over W with respect to θ. For $\hat{q} \in \left(\hat{P}^\theta(\hat{s}(y))\right)_y$ we have $\hat{q} = R_A(\hat{s}(y))$ with $A \in Hol_{\hat{s}(y)}(\theta)$. Now we can continue the equations from above

$$\begin{aligned} \Theta^{\hat{s}}_y(X_1,X_2) &= Ad(A^{-1})\left(\Theta_{\hat{q}}\left(dR_A \circ d\hat{s}(X_1), dR_A \circ d\hat{s}(X_2)\right)\right) \\ &= \underbrace{Ad(A^{-1})}_{\in Aut\left(\mathfrak{hol}_{\hat{s}(y)}(\theta)\right)} \underbrace{\Theta_{\hat{q}}\left(pr_{Th_{\hat{q}}\hat{P}}\left(dR_A \circ d\hat{s}(X_1)\right), pr_{Th_{\hat{q}}\hat{P}}\left(dR_A \circ d\hat{s}(X_2)\right)\right)}_{\in \mathfrak{hol}_{\hat{s}(y)}(\theta)} \\ &\in \mathfrak{hol}_{\hat{s}(y)}(\theta) \simeq \mathfrak{hol}_y(W,g^y) \overset{\text{2nd assumption}}{\simeq} \mathfrak{hol}_x(W_{(x_0,x_{n+1})},g^x). \end{aligned}$$

So we have for an arbitrary $q \in \mathcal{P}^\omega(p)$:

$$pr_{\mathfrak{so}(n)}\left(\Omega_q(Y_1,Y_2)\right) \hookrightarrow \mathfrak{hol}_x(W_{(x_0,x_{n+1})},g^x)$$

and therefore

$$pr_{\mathfrak{so}(n)}\left(\mathfrak{hol}_x(M,h)\right) \hookrightarrow \mathfrak{hol}_x(W_{(x_0,x_{n+1})},g^x)$$

which is the proposition. □

4.3. **Brinkmann waves with abelian holonomy.** We have seen that one class of examples of manifolds with abelian holonomy, i.e. $\simeq \mathbb{R}^n$ are locally symmetric Lorentzian spaces. Here we will give necessary and equivalent conditions for a Brinkmann wave to have abelian holonomy. Therefore we have to consider a very special class of Brinkmann waves which are generalization of the so called plane waves.

Definition 4.1. A Brinkmann-wave is called pp-manifold if for its curvature tensor holds the following trace condition

$$tr_{(3,5)(4,6)}(\mathcal{R}\otimes\mathcal{R}) = 0\,. \tag{14}$$

Schimming [Sch74] proved that this condition is equivalent to the existence of local coordinates such that the metric h has the following form

$$h = 2\ dx_0 dx_{n+1} + f dx_{n+1}^2 + \sum_{i=1}^{n} dx_i^2 \ \text{ with } \partial_0 f = 0\,.$$

We now can prove

Theorem 4.2. *A simply connected Brinkmann wave has abelian holonomy $\mathbb{R}^n$ if and only if it is a pp-manifold.*

Proof. Lets assume, that a Brinkmann wave (M,h) has holonomy $\mathbb{R}^n$. That means that the holonomy bundle $\mathcal{P}^\omega(p)$ has the structure group $\mathbb{R}^n$. For $p\in\mathcal{P}_x(M,h))$ the holonomy bundle has fibres

$$\left(\mathcal{P}^\omega(p)\right)_y = \{\mathfrak{p}^\omega_\gamma(p)|\gamma \text{ from } x \text{ to } y\}\,.$$

The action of $\mathbb{R}^n$ on $\left(\mathcal{P}^\omega(p)\right)_y$ is given by

$$(s_-,s_+,s_1,\ldots,s_n) \overset{(1,X)}{\mapsto} (\hat{s}_-,\hat{s}_+,\hat{s}_1,\ldots,\hat{s}_n)$$

with $\hat{s}_i = s_i + X_i(s_- + s_+) = s_i + X_i T$ and $\hat{s}_\mp = s_\mp \pm \left(\sum_{i=1}^n X_i s_i + \frac{1}{2}X^t X\ T\right)$. But this means for $(s_-,s_+,s_1,\ldots,s_n)\in\left(\mathcal{P}^\omega(p)\right)_y$ that

$$\nabla_{\dot\gamma(1)} s_i = f_i(\gamma(1))\ T \text{ which gives } \mathcal{R}(s_i,.,.,.) = 0$$

and

$$\nabla_{\dot\gamma(1)} s_\mp = \pm\sum_{i=1}^n g_i(\gamma(1)) s_i + h(\gamma(1))\ T \quad \text{which gives} \quad \mathcal{R}(X,Y)s_\mp \in \mathbb{R}\ T\,.$$

So we can check the trace condition with an element from $\left(\mathcal{P}^\omega(p)\right)_y$:

$$\begin{aligned} tr_{(3,5)(4,6)}(\mathcal{R}\otimes\mathcal{R})(X,Y,U,V) &= -h\left(\mathcal{R}(X,Y)s_-,\mathcal{R}(U,V)s_-\right) \\ &\quad +h\left(\mathcal{R}(X,Y)s_+,\mathcal{R}(U,V)s_+\right) \\ &= 0\,. \end{aligned}$$

This gives one direction.

The other direction follows from the local form of the metric of a pp-manifold and theorem 4.1. □

4.4. **More examples with parallel spinors.** In this last section we will construct some examples besides pp-manifolds, analogous to the results of theorem section 4.2. We will see that in case of holonomy groups which admit parallel spinors one can weaken the conditions of theorem 4.1.

Instead of local considerations in this section we start with a n-dimensional manifold N and consider $\mathbb{R}^2 \times N \ni (t,r,x)$ equipped with a family of Riemannian metrics g_r and of 1-forms ϕ_r on N and a function f of r and $x \in N$.

In a first step one constructs a Brinkmann-wave (M,h)

$$(M := \mathbb{R}^2 \times N, h = 2\ dr\ dt + f dr^2 + \phi_r dr + g_r) \tag{15}$$

with parallel lightlike vectorfield $T := \frac{\partial}{\partial t}$.

We denote by $p: M \to N$ the projection, by $dp: TM \to TN$ its differential.

Now we consider the following fibre bundle over $\mathbb{R} \times N$ with fibres

$$\begin{array}{ccl} \mathcal{O}(N) & := & \{(X_1,\dots,X_n,r) \mid \ r \in \mathbb{R} \text{ and } (X_1,\dots,X_n) \in \mathcal{O}(N,g_r)\} \\ \downarrow & & \quad\quad\downarrow \\ \mathbb{R} \times N & \ni & (r, \pi(X_1,\dots,X_n)) \end{array}$$

and structure group $SO(n)$. A section in this bundle can be understood as a family of sections s_r into $\mathcal{O}(N,g_r)$ with parameter r.

dp gives a map between $\mathcal{O}(M,h)$ and $\mathcal{O}(N)$ denoted also by dp defined fibrewise

$$(\mathcal{O}(M,h))_{(t,r,x)} \ni (X_-,X_+,X_1,\dots,X_n) \overset{dp}{\mapsto} (dp_{(t,r,x)}(X_1),\dots,dp_{(t,r,x)}(X_n),r) \in \mathcal{O}(N)$$

such that the diagramm commutes

$$\begin{array}{ccc} \mathcal{O}(M,h) & \overset{dp}{\to} & \mathcal{O}(N) \\ \downarrow & & \downarrow \\ M & \overset{(t,r,x)\mapsto(r,x)}{\to} & \mathbb{R} \times N . \end{array}$$

To a section $\sigma := ((\sigma_-,\sigma_+,\sigma_1,\dots,\sigma_n) \in \Gamma(\mathcal{O}(M,h))$ the map dp asigns a section denoted by $dp(\sigma) \in \Gamma(\mathcal{O}(N))$ and defined by

$$s_r := dp(\sigma) : (r,x) \mapsto (dp_{(t,r,x)}(\sigma_1(t,r,x)),\dots,dp_{(t,r,x)}(\sigma_n(t,r,x)),r) \ .$$

This definition is correct, i.e. independent of t because of the following considerations: If $\sigma_i = \sum_{j=1}^n \xi_{ij}\frac{\partial}{\partial x_j} + a_i\frac{\partial}{\partial r} + b_i\frac{\partial}{\partial t}$ then it is $dp_{(t,r,x)}(\sigma_i(t,r,x)) = \sum_{j=1}^n \xi_{ij}(t,r,x)\frac{\partial}{\partial x_j}(x)$. But from the local form of the metric follows that the ξ_{ij} are independent of t.

One can describe this situation in an equivalent way if one considers TN as a subbundle of TM by restricting to vectors with vanishing $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial t}$ components. A vectorfield $X \in \Gamma(TM)$ over M then defines a vectorfield $pr_{TN}(X) \in \Gamma(M,TN)$ also over M. So we can asign to a section $\sigma \in \Gamma(\mathcal{O}(M,h))$ a family of sections $(pr_{TN} \circ \sigma_1,\dots,pr_{TN} \circ \sigma_n)$ into $\mathcal{O}(N,g_r)$ since $pr_{TN} \circ \sigma_i$ is independent of t again.

Now we will give an equivalent condition to the existence of parallel spinors on (M,h). Therefore we denote as in the previous section the connection corresponding to g_r by θ_r and $\mathfrak{p}_\gamma^{\theta_r}$ the parallel displacement along a curve in $N \subset M$, i.e. with constant r and t component.

Furthermore we define a family of 2-forms on N. Let $s_r = (s_1,\dots,s_n)$ be a family of sections into $\mathcal{O}(N,g_r)$, s_i^* the dual vectorfields. Then we define a family of 2-forms on N depending on the parameter r

$$\delta_r := \frac{1}{2} \sum_{1 \le i < j \le n} \left(h([\frac{\partial}{\partial r}, s_i], s_j) - h([\frac{\partial}{\partial r}, s_j], s_i) \right) s_i^* \wedge s_j^* .$$

In the following the hat over the symbol of a section into orthonormal frame bundles denotes the lift into the corresponding spin bundle.

Proposition 4.4. *Let (M, h) be a Lorentzian spin manifold given in form (15) with arbitrary family g_r. Then (M, h) admits a parallel spinor φ if and only if φ is of the following form:*

$$\varphi = [\hat{\sigma}, v \otimes u(1)] \quad \textit{with the property that} \tag{16}$$

$\varphi_r := [\hat{s}_r, v]$ is a family of parallel spinors of (N, g_r) with $s_r := dp(\sigma)$ and $v \in C^\infty(\mathbb{R} \times N, \Delta_n)$ independent of t satisfying the equation

$$0 = \frac{\partial}{\partial r}(\varphi_r) + (d\phi_r + \delta_r) \cdot \varphi_r . \tag{17}$$

This is equivalent to the condition that the family of parallel spinors on N is given by

$$\varphi_r = [\widehat{(\mathfrak{p}_\gamma^{\theta_r}(p))}, v] \quad \textit{where} \tag{18}$$

1. *$p \in \mathcal{O}_{(r,x)}(N)$, γ curves in M with constant r and t component and*
2. *$v \in C^\infty(\mathbb{R}, V_{\mathfrak{hol}_x(N,g)})$ satisfying the equation*

$$0 = \frac{\partial}{\partial r}(v) + \sum_{1 \le i < j \le n} (d\phi_r(s_i, s_j) + \delta_r(s_i, s_j))\, e_i \cdot e_j \cdot v . \tag{19}$$

Proof. φ is a parallel spinor on (M, h) iff $\varphi = [\hat{\tau}, w \otimes u(1)]$ where $\tau = \mathfrak{p}_\gamma^\omega(p)$ with p an orthonormal basis in $T_{(t,r,x)}M$ and $w \in V_{\mathfrak{b}}$ (see proposition 3.2).

Since $V_{\tilde{\mathbb{R}}^n} = \operatorname{span}(u(\varepsilon_k, \dots, \varepsilon_1, 1))$ we have for $(A + X) \in \mathfrak{spin}(n) \ltimes \mathbb{R}^n$

$$(\Phi_{1,n+1})_* (A, X)\, (v \otimes u(1)) = (\Phi_n)_* (A)(v) \otimes u(1).$$

Since $\exp_{GL(\Delta_{1,n+1})} \circ (\Phi_{1,n+1})_* = \Phi_{1,n+1} \circ \exp_{\mathrm{Spin}(n) \ltimes \mathbb{R}^n}$ we have for $(A, X) \in \mathrm{Spin}(n) \ltimes \mathbb{R}^n$ the following

$$\Phi_{1,n+1}(A, X)(v \otimes u(1)) = \Phi_n(A)(v) \otimes u(1) .$$

For a section $\sigma = (\sigma_-, \sigma_+, \sigma_1, \dots, \sigma_n)$ we notice the relations $h(\sigma_\mp, T) = \mp 1$ i.e. $\frac{\partial}{\partial t} = T = \sigma_- + \sigma_+$ parallel and one gets

$$\begin{aligned} 0 &= h(\nabla_X(\sigma_- + \sigma_+), \sigma_i) \\ &= h(\nabla_X \sigma_-, \sigma_i) + h(\nabla_X \sigma_+, \sigma_i) \text{ and} && (20) \\ 0 &= h(\nabla_X(\sigma_- + \sigma_+), \sigma_+) \\ &= h(\nabla_X \sigma_-, \sigma_+) + h(\nabla_X \sigma_+, \sigma_+) \\ &= h(\nabla_X \sigma_-, \sigma_+) + \underbrace{X\,(h(\sigma_+, \sigma_+))}_{=0} . && (21) \end{aligned}$$

So we have for a parallel spinor with τ and w as above

$$\begin{aligned}\varphi &= [\hat{\tau}, w \otimes u(1)] \\ &= [\hat{\tau}\cdot(A,X), \Phi_n(A)(w)\otimes u(1)] \\ &=: [\hat{\sigma}, v\otimes u(1)]\,.\end{aligned}$$

The equation for the parallelity becomes (with $e_-, e_+, e_1, \dots e_n$ the standard basis of $\mathbb{R}^{n+2}$).

$$\begin{aligned}0 &= \nabla^S_X[\hat{\sigma}, v\otimes u(1)] \\ &= \Big[\hat{\sigma},\ X(v)\otimes u(1) - \frac{1}{2}\big(h(\nabla_X\sigma_-,\sigma_+)\, e_-\cdot e_+\cdot(v\otimes u(1))\big) \\ &\quad -\frac{1}{2}\left(\sum_{i=1}^n h(\nabla_X\sigma_-,\sigma_i)\, e_-\cdot e_i - \sum_{i=1}^n h(\nabla_X\sigma_+,\sigma_i)\, e_+\cdot e_i\right)\cdot(v\otimes u(1)) \\ &\quad + \frac{1}{2}\sum_{1\le i<j\le n} h(\nabla_X\sigma_i,\sigma_j)\, e_i\cdot e_j\cdot(v)\otimes u(1))\Big] \\ &\overset{(20),(21)}{=} \Big[\hat{\sigma}, X(v)\otimes u(1) + \frac{1}{2}\sum_{i=1}^n h(\nabla_X\sigma_+,\sigma_i)(e_- + e_+)\cdot e_i\cdot(v\otimes u(1)) \\ &\quad + \frac{1}{2}\sum_{1\le i<j\le n} h(\nabla_X\sigma_i,\sigma_j)\, e_i\cdot e_j\cdot(v)\otimes u(1)\Big] \\ &\overset{(e_-+e_+)\cdot u(1)=0}{=} \Big[\hat{\sigma}, X(v)\otimes u(1) + \frac{1}{2}\sum_{1\le i<j\le n} h(\nabla_X\sigma_i,\sigma_j)\, e_i\cdot e_j\cdot(v)\otimes u(1)\Big]\,.\end{aligned}$$

Since for $X\in TN$ holds $h(\nabla_X\sigma_i,\sigma_j) = h(\nabla_X s_i, s_j) = g(\nabla^{g_r}_X s_i, s_j)$ with $s_i = pr_{TN}\circ\sigma_i$ one gets

$$\begin{aligned}0 &= \nabla^S_X\varphi \\ &= \left[\hat{\sigma}, \left(X(v) + \frac{1}{2}\sum_{1\le i<j\le n} g(\nabla^{g_r}_X s_i, s_j)\, e_i\cdot e_j\cdot(v)\right)\otimes u(1)\right].\end{aligned}$$

But this is the equation for a family of spinors $\varphi_r := [\hat{s}, v]$, with $s_i = pr_{TN}\circ\sigma_i$ to be parallel with respect to g_r.

In case $X := T = \frac{\partial}{\partial t}$ it is $h(\nabla_T\sigma_i,\sigma_j) = 0$ because of $[T,\sigma_i] = 0$. So one gets $T(\varphi) = 0$.

In case $X := \frac{\partial}{\partial r}$ the parallelity of the spinor is equivalent to

$$\begin{aligned}0 &= \left[\hat{\sigma}, \left(\frac{\partial}{\partial r}(v) + \frac{1}{2}\sum_{1\le i<j\le n} h\left(\nabla_{\frac{\partial}{\partial r}}\sigma_i,\sigma_j\right) e_i\cdot e_j\cdot(v)\right)\otimes u(1)\right] \\ &\overset{\text{Koszul-formula}}{=} \left[\hat{\sigma}, \left(\frac{\partial}{\partial r}(v) + \sum_{1\le i<j\le n} (d\phi_r(s_i,s_j) + \delta_r(s_i,s_j))\, e_i\cdot e_j\cdot v\right)\otimes u(1)\right]\end{aligned}$$

where $s_i = pr_{TN}\circ\sigma_i$ again. But this is equivalent to the equation

$$\frac{\partial}{\partial r}(\varphi_r) + (d\phi_r + \delta_r)\cdot\varphi_r = 0$$

for a family of spinors $\varphi_r := [\hat{s}_r, v(r)]$ on N. This gives the proposition.

For the equivalent formulation one writes equation (17) with the representant $\varphi_r := [\widehat{(\mathfrak{p}^{\theta_r}_\gamma(p))}, v]$. Then v must satisfy $v \in C^\infty(\mathbb{R}, V_{\mathfrak{su}(n)})$ and equation (19). □

Now we will assume that g_r is a family of Kähler metrics with complex structures J_r on N^{2n}. Then the bundle $\mathcal{P}(M,h)$ defined in formula (12) reduces to the bundle $\mathcal{U}(M,h)$ with fibres over $y = (t,r,x)$

$$\mathcal{U}_y(M,h) = \{(X_-, X_+, X_1, \dots X_{2n}) \in \mathcal{P}_y(M,h) | dp_y(X_{2k}) = J_r\,(dp_y(X_{2k-1}))\}$$

and structure group $U(n) \ltimes \mathbb{R}^{2n}$. Here we use the same realisation of $U(n)$ in $SO(n)$ as in [BK99]. Analogously we define the bundles $\mathcal{U}(N)$ and $\mathcal{U}(N, g_r)$ which are the $U(n)$-bundles.

We now extend J_r on TM in a trivial way, i.e. $J_r\left(\frac{\partial}{\partial t}\right) = J_r\left(\frac{\partial}{\partial r}\right) = 0$. We consider the family of 2-forms χ_r on TN:

$$\chi_r(X,Y) := d\phi_r(J_rX, Y) + d\phi_r(X, J_rY) + \delta_r(J_rX, Y) + \delta_r(X, J_rY)\,. \tag{22}$$

with δ_r the above defined family of 2-forms on N. It is clear that

$$\chi_r\,(J_rX, J_rY) = -\chi_r\,(X,Y) \tag{23}$$
$$\chi_r\,(X, J_rY) = \chi_r\,(J_rX, Y) \tag{24}$$
$$\text{i.e. } \chi_r\,(X, J_rX) = 0\,. \tag{25}$$

That means that χ_r defines a family of 2-forms with complexification in $\Lambda^{(2,0)}N$.

We now show that the vanishing of the projection of $\nabla_X J$ on TN is equivalent to the vanshing of χ_r and to the fact that $Hol_x(M,h) \subset U(n) \ltimes \mathbb{R}^n$.

Lemma 4.3. *$\chi_r = 0$ if and only if $Hol_x(M,h) \subset U(n) \ltimes \mathbb{R}^n$.*

Proof. We consider an arbitrary local section $s = (s_-, s_+, s_1, \dots s_{2n}) \in \Gamma(\mathcal{U}(M,h))$. If we consider TN as a subbundle of TM over M we can define vector fields $\hat{s}_i := pr_{TN} \circ s_i : M \to TN$, $i = 1, \dots 2n$. We show that

$$\omega^s(X) \subset \mathfrak{u}(n) \ltimes \mathbb{R}^{2n} \text{ i.e. } pr_{\mathfrak{so}(n)}\,(\omega^s(X)) \subset \mathfrak{u}(n).$$

Again it is

$$pr_{\mathfrak{so}(n)}\,(\omega^s(X)) = \sum_{1 \le i < j \le n} h\,(\nabla_X s_i, s_j)\,E_{ij} = \sum_{1 \le i < j \le n} h\,(\nabla_X \hat{s}_i, \hat{s}_j)\,E_{ij}\,.$$

It is

$$\begin{array}{rrcl} & pr_{\mathfrak{so}(n)}\,(\omega^s(X)) & \subset & \mathfrak{u}(n) \\ \Longleftrightarrow & h\,(\nabla_X \hat{s}_{2k-1}, \hat{s}_{2l-1}) & = & h\,(\nabla_X \hat{s}_{2k}, \hat{s}_{2l}) \text{ and} \\ & h\,(\nabla_X \hat{s}_{2k-1}, \hat{s}_{2l}) & = & h\,(\nabla_X \hat{s}_{2l-1}, \hat{s}_{2k}) \\ & & \text{for} & \hat{s}_{2i} = J\hat{s}_{2i+1}, i = 1, \dots, n \\ \Longleftrightarrow & h\,((\nabla_X J)\,\hat{s}_i, \hat{s}_j) & = & 0\ (i = 1, \dots 2n), \text{ i.e. } pr_{TN}(\nabla_X J) = 0\,. \end{array}$$

Since $0 = [\frac{\partial}{\partial t}, \hat{s}_i] = \nabla_{\frac{\partial}{\partial t}} \hat{s}_i$ this is obviously true for $X = \frac{\partial}{\partial t}$.

Because J_r is integrable on N this holds also for $X \in TN$.

We show that for $X = \frac{\partial}{\partial r}$ this is equivalent to $\chi_r = 0$.

Since J_r is extended on TM in a trivial way, i.e. $J\left(\frac{\partial}{\partial t}\right) = J\left(\frac{\partial}{\partial r}\right) = 0$ we get

$$h\left(J\left(\nabla_X \hat{s}_i\right), \hat{s}_j\right) = -h\left(\nabla_X \hat{s}_i, J\left(\hat{s}_j\right)\right). \tag{26}$$

The rest of the proof is a direct calculation with the help of the Koszul-formula. □

If we now assume that (M, h) with a family of Kähler metric g_r admits parallel spinors then g_r must be a family of Ricci-flat Kähler metrics. In [BK99] also is shown that

$$V_{\mathfrak{su}(n)} = \left\{v \in \Delta_n \left| \lambda_*^{-1}(\mathfrak{su}(n))(v) = 0\right.\right\} = \operatorname{span}\{u(\varepsilon, \ldots, \varepsilon) \,|\, \varepsilon = \pm 1\}. \tag{27}$$

By the proposition parallel spinors are given by φ_v with $v \in C^\infty\left(\mathbb{R}, V_{\mathfrak{su}(n)}\right)$ and equation (17).

We now prove the following

Lemma 4.4. *Let $\varphi := [\hat{s}, v]$ a family of spinors on N with $s \in \Gamma(\mathcal{U}(N))$ a family of sections in $\mathcal{U}(N, g_r)$ and $v \in C^\infty(N \times \mathbb{R}, V_{\mathfrak{su}(n)})$. Then holds the following*

$$(d\phi_r + \delta_r) \cdot \varphi \;=\; i\varepsilon\left(\left(tr_{J_r} d\phi_r + tr_{J_r}\delta_r\right)\, \varphi - \frac{1}{2}\chi_r \cdot \varphi\right) \tag{28}$$

$$\begin{aligned} \text{where } tr_{J_r} d\phi_r &= \sum_{k=1}^n d\phi_r(s_{2k-1}, s_{2k}) \\ tr_{J_r}\delta_r &= \sum_{k=1}^n \delta_r(s_{2k-1}, s_{2k}). \end{aligned}$$

Proof. We show this equality for $\varphi_r = [\hat{s}_r, u(\varepsilon, \ldots, \varepsilon)]$ and $s_r := (s_1, J_r s_1, \ldots s_n, J_r s_n, r) \in \Gamma(\mathcal{U}(N))$.

First we notice that

$$e_{2k-1} \cdot e_{2l-1} u(\varepsilon, \ldots, \varepsilon) = -e_{2k} \cdot e_{2l} u(\varepsilon, \ldots, \varepsilon) =$$

$$= (-1)^{l-k-1}\varepsilon^{l-k} u(\underbrace{\varepsilon}_{n}, \ldots, \varepsilon, \underbrace{-\varepsilon}_{l}, \ldots \varepsilon, \ldots, \varepsilon, \underbrace{-\varepsilon}_{k}, \varepsilon, \ldots, \underbrace{\varepsilon}_{1})$$

$$e_{2k-1} \cdot e_{2l} u(\varepsilon, \ldots, \varepsilon) = e_{2k} \cdot e_{2l-1} u(\varepsilon, \ldots, \varepsilon) =$$

$$= -i\,(-1)^{l-k-1}\varepsilon^{l-k-1} u(\varepsilon, \ldots, \varepsilon, -\varepsilon, \varepsilon, \ldots, \varepsilon, -\varepsilon, \varepsilon, \ldots, \varepsilon).$$

Then we calculate

$$\begin{aligned} \chi_r \cdot \varphi &= \left[\hat{s}_r, \sum_{1 \le i < j \le 2n} \chi_r\left(s_i, s_j\right) e_i \cdot e_j \cdot u(\varepsilon, \ldots, \varepsilon)\right] \\ &= \left[\hat{s}_r, \sum_{1 \le k < l \le n} \chi_r\left(s_{2k-1}, s_{2l-1}\right)\left(e_{2k-1} \cdot e_{2l-1} - e_{2k} \cdot e_{2l}\right) \cdot u(\varepsilon, \ldots, \varepsilon)\right. \\ &\qquad \left. + \sum_{1 \le k < l \le n} \chi_r\left(s_{2k-1}, s_{2l}\right)\left(e_{2k-1} \cdot e_{2l} + e_{2k} \cdot e_{2l-1}\right) \cdot u(\varepsilon, \ldots, \varepsilon)\right] \end{aligned}$$

$$= 2 \sum_{1\le k<l\le n} (-1)^{l-k-1}\varepsilon^{l-k}$$
$$\Big(i\,\varepsilon\left(\delta\left(s_{2k-1},s_{2l-1}\right)-\delta\left(s_{2k},s_{2l}\right)+d\phi_r\left(s_{2k-1},s_{2l-1}\right)-d\phi_r\left(s_{2k},s_{2l}\right)\right)$$
$$+\,\delta\left(s_{2k-1},s_{2l}\right)+\delta\left(s_{2k},s_{2l-1}\right)+d\phi_r\left(s_{2k-1},s_{2l}\right)+d\phi_r\left(s_{2k},s_{2l-1}\right)\Big)$$
$$\Big[\hat{s}_r, u(\varepsilon,\dots\varepsilon,,-\varepsilon,\varepsilon,\dots,\varepsilon,-\varepsilon,\varepsilon,\dots,\varepsilon)\Big].$$

Now it is for $\varphi_r = [\hat{s}_r, u(\varepsilon,\dots,\varepsilon)]$

$$(d\phi_r+\delta_r)\cdot\varphi_r$$
$$= \Big[\hat{s}_r, \sum_{k=1}^{n}(d\phi_r(s_{2k-1},s_{2k})+\delta_r(s_{2k-1},s_{2k}))\underbrace{e_{2k-1}\cdot e_{2k}\cdot u(\varepsilon,\dots,\varepsilon)}_{=i\varepsilon u(\varepsilon,\dots,\varepsilon)\text{ for all }k}\Big]$$
$$+\sum_{1\le k<l\le n}(-1)^{l-k-1}\varepsilon^{l-k}$$
$$\Big(d\phi_r(s_{2k-1},s_{2l-1})-d\phi_r(s_{2k},s_{2l})+\delta_r(s_{2k-1},s_{2l-1})-\delta_r(s_{2k},s_{2l})$$
$$-\,i\,\varepsilon\left(d\phi_r(s_{2k-1},s_{2l})+d\phi_r(s_{2k},s_{2l-1})+\delta_r(s_{2k-1},s_{2l})+\delta_r(s_{2k},s_{2l-1})\right)\Big)$$
$$\left[\hat{s}_r, u(\varepsilon,\dots,\varepsilon,-\varepsilon,\varepsilon,\dots,\varepsilon,-\varepsilon,\varepsilon,\dots,\varepsilon)\right]$$
$$= i\,\varepsilon\,(tr_{J_r}d\phi_r+tr_{J_r}\delta_r)\,\varphi_r-\frac{i\,\varepsilon}{2}\chi_r\cdot\varphi_r\,.$$

This is the proposition. □

Now we can prove the following

Theorem 4.3. *Let $(M := \mathbb{R}^2\times N^{2n}, h := 2drdt + fdr^2 + \phi_r dr + g_r)$ be a Lorentzian manifold where N is a n-dimensional manifold with a family of Riemannian Kähler metrics g_r and a 1-form ϕ_r.*

1. *If (M,h) admits a parallel spinor then g_r is a family of Ricci-flat Kähler metrics and if $Hol_x(N,g_r) = SU(n)$ for all r then the parallel spinor is given via formula (16) by a family of parallel spinors φ_r with respect to g_r satisfying the equation*

$$0=\left(\frac{\partial}{\partial r}+i\,\varepsilon\,(tr_{J_r}d\phi_r+tr_{J_r}\delta_r)\right)\,\varphi_r-\frac{i\,\varepsilon}{2}\chi_r\cdot\varphi_r\,. \tag{29}$$

In particular this family is given by $\varphi_r := [\widehat{\mathfrak{p}^{\theta_r}_\gamma(p)}, v(r)]$ with $v\in C^\infty(\mathbb{R}, V_{\mathfrak{su}(n)})$ satisfying in case $n\neq 2,4$ the equations

$$0 = \left(\frac{\partial}{\partial r}+i\,\varepsilon\left(\sum_{k=1}^{n}d\phi_r(s_{2k-1},s_{2k})+\sum_{k=1}^{n}\delta_r(s_{2k-1},s_{2k})\right)\right)v \text{ and} \tag{30}$$

$$0 = \frac{i\,\varepsilon}{2}\sum_{1\le i<j\le 2n}\chi_r(s_i,s_j)e_i\cdot e_j\cdot v\,. \tag{31}$$

2. *If* $Hol_x(N, g_r) \subset SU(n)$ *then a spinor given by a family of parallel spinors with respect to* g_r *satisfying equation (29) is parallel.*

Proof. From the existence of a parallel spinor on (M, h) it is clear that the family g_r of Kähler metrics must be Ricci-flat.

The parallel spinor φ is given by a family of parallel spinors φ_r with respect to g_r due to equation (16) satisfying (17). Since $Hol_x(N, g_r) = SU(n)$ for all r it is $\varphi_r = [\hat{s}, v]$ with $v \in C^\infty(\mathbb{R}, V_{\mathfrak{su}(n)})$ so that we have by the previous lemma the formula of the proposition. By excluding the cases $n = 2$ and $n = 4$ the formula (29) is equivalent to both algebraic conditions (30) and (31) with φ_r written as in the proposition. The other direction is clear: a given φ_r family of parallel spinors over the family of Ricci-flat Kähler manifolds with the additional equation defines a parallel spinor on M by the previous lemma and proposition. □

In the following proposition we give sufficient conditions for the reduction of the Levi-Civita connection ω to the bundle $\mathcal{U}(M, h)$.

Corollary 4.1. *Let* (M, h) *be a Brinkmann-wave as just constructed with the additional conditions*

1. $g_r \equiv g$ *and* $J_r \equiv J$ *both independent of* r,
2. $(d\phi_r)_{\mathbb{C}} \in \Lambda^{(1,1)}N$ *i.e.* $d\phi_r(JX, JY) = d\phi_r(X, Y)$ *for* $X, Y \in TN$.

Then the Levi-Civita connection reduces to the $U(n) \ltimes \mathbb{R}^{2n}$ *bundle* $\mathcal{U}(M, h)$,
i.e. $Hol_x(M, h) \subset U(n) \ltimes \mathbb{R}^{2n}$.

Proof. Because of the first condition it is $[\frac{\partial}{\partial r}, s_i] = 0$ such that $\delta_r = 0$. The second gives $d\phi_r(JX, Y) + d\phi_r(X, JY) = 0$ such that $\chi_r = 0$. Then we apply lemma 4.3. □

Now we will give sufficient conditions in that setting for (M, h) to admit parallel spinors.

Corollary 4.2. *Let* (M, h) *be a Brinkmann wave as in the previous corollary with the additional condition that* g *is a Ricci-flat Kähler metric and*

$$tr_J\,(d\phi_r) := \sum_{k=1}^{n} d\phi_r(s_{2k-1}, Js_{2k-1}) = \vartheta$$

where ϑ *is a function of only* r. *Then* (M, h) *admits parallel spinors.*
If $Hol_x(N, g) = SU(n)$ *then* $Hol_x(M, h) = SU(n) \ltimes \mathbb{R}^n$.

Proof. The proof follows from the theorem. Equation (29) becomes

$$0 = \left(\frac{\partial}{\partial r} + i\,\varepsilon\,\vartheta\right)\varphi.$$

Writing $\varphi := \varphi_v = [\hat{s}, v]$ with $v \in V_{\mathfrak{su}(n)}$ this gives

$$0 = \frac{\partial v}{\partial r} + i\,\varepsilon\,\vartheta.$$

Written v in the basis $u(1, \ldots, 1)$ and $u(-1, \ldots, -1)$ of $V_{\mathfrak{su}(n)}$ this is a pair of ordinary differential equations in two functions of r which is solvable so that we get the proposition.

If the holonomy of g equals to $SU(n)$ so the holonomy of (M,h) contains $SU(n)\ltimes\mathbb{R}^{2n}$ and must properly contained in $U(n)\ltimes\mathbb{R}^{2n}$ which gives $Hol_x(M,h) = SU(n)\ltimes\mathbb{R}^{2n}$ since dim $U(n) = $ dim $SU(n)+1$. □

By formula (11) in an above remark it is clear that there exists a coordinate transformation such that

$$h = dt\ d\hat{r} + \hat{f}d\hat{r}^2 + g_{\hat{r}}$$

with $g_{\hat{r}}$ a family of Ricci-flat Kähler metrics.

Easily we get another class of manifolds with parallel spinors.

Corollary 4.3. *Let (M,h) be a Brinkmann-wave constructed as above with additional condition that (N,g) is a hyper-Kähler manifold with complex structures $I,J,K=IJ$ and*

$$d\phi_r(X,Y) = d\phi_r(IX,IY) = d\phi_r(JX,JY).$$

Then $Hol_x(M,h)\subset Sp(n)\ltimes\mathbb{R}^{4n}$.

Necessary conditions for the existence of parallel spinors on a Brinkmann-wave determining the holonomy group until now are only known in low dimensions. Bryant ([Bry99a] resp. [Bry00], see also [FO99]) has recently proved that for dimensions $m\leq 11$ the maximal groups admiting parallel spinors are all of the form

$$\text{Riemannian holonomy with parallel spinors} \ltimes \mathbb{R}^n,$$

i.e.: $\mathbb{R},\mathbb{R}^2,\mathbb{R}^3, Sp(1)\ltimes\mathbb{R}^4, (Sp(1)\times 1)\ltimes\mathbb{R}^5, SU(3)\ltimes\mathbb{R}^6, G_2\ltimes\mathbb{R}^7, \mathrm{Spin}(7)\ltimes\mathbb{R}^8$, $(\mathrm{Spin}(7)\times 1)\ltimes\mathbb{R}^9$, but it is not known whether all subgroups can be obtained in this way. The example of Ikemakhen in [Ike96] which is not of this form does not admit parallel spinors.

Bryant also shows in [Bry99b], resp. [Bry00] the following. For an 11-dimensional Brinkmann-wave $(M, h = dt\ dr + f dr^2 + g_r)$ with a family of Riemannian metrics, the condition to h to have holonomy $(\mathrm{Spin}(7)\times\mathbb{R})\ltimes\mathbb{R}^9$ is equivalent to the fact that g_r is a conformal anti-selfdual family. This means that the family of the Spin(7)-defining 4-form θ_r satisfies $\frac{\partial}{\partial r}\theta_r = \alpha\ \theta_r + \Upsilon$ with a function α and an anti-selfdual 4-form Υ. This condition is locally trivial, i.e. for every given family of Spin(7) metrics one can find diffeomorphisms such that the family of the transformed metrics with Spin(7) holonomy is conformal anti-selfdual.

Another class of metrics was obtained by Figueroa O'Farril who proved in [FO99] the following. Let (M,h) be a Brinkmann-wave and $(\hat{M},\hat{g})$ a Riemannian manifold and $\sigma\in C^\infty(M)$ but only dependent on x_{n+1}. Then for the holonomy group of the product manifold and the warped product metric $\hat{h} = h+\sigma\hat{g}$ holds the following

$$Hol_{(x,y)}(M\times N,\hat{h}) = \Big(Hol_x(M,h)\times Hol_y(\hat{M},\hat{g})\Big)\ltimes\mathbb{R}^n.$$

So by choosing both factors with parallel spinors — in [FO99] this is done in dimension 11 — one gets a new class of metrics with parallel spinors, but no new examples of holonomy groups admitting trivial subrepresentations which cannot be constructed by the method of proposition 4.1.

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