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A REMARK ON NATURAL QUANTUM LAGRANGIANS AND NATURAL GENERALIZED SCHRÖDINGER OPERATORS IN GALILEI QUANTUM MECHANICS

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ABSTRACT. The natural quantum Lagrangians which appear in Galilei general relativistic quantum mechanics are classified by using methods of natural operators. It is proved that all 1st order natural quantum Lagrangians are linear combinations (with real coefficients) of the canonical quantum Lagrangian and the product of the scalar curvature of the spacetime vertical connection and the Hermitian product. The classification of all natural generalized Schrödinger operators is given and it is proved that all natural generalized Schrödinger operators can be induced from natural quantum Lagrangians.

INTRODUCTION

In [9] we have classified natural quantum Lagrangians which appear in the covariant quantum mechanics proposed by Jadczyk and Modugno, [5, 6], for curved Galilei spacetime. In this classification the metric structure of spacetime was used only with zero order. This fact implies that in the generalized Schrödinger equations related with quantum Lagrangians there is no term containing the scalar curvature as we can find in classical geometric quantization over a curved manifold, [1, 2, 15, 17].

In this paper we classify all 1st and 2nd order natural quantum Lagrangians and all natural generalized Schrödinger operators in the context of revisited covariant quantum mechanics proposed in [4]. Moreover, we use physical dimensions of all objects which in fact simplifies the results. We prove that all natural generalized Schrödinger operators are linear and can be induced from natural quantum Lagrangians.

In the paper we deal with geometrical properties of operators only. We do not discuss their physical interpretation which can be found in [3, 13, 14] and references quoted there.

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We assume the following fundamental unit spaces, which are positive 1–dimensional "semi–vector spaces" over $\mathbb{R}^+$:

(1) the space $\mathbb{T}$ of *time intervals*,

(2) the space $\mathbb{L}$ of *lengths*,

(3) the space $\mathbb{M}$ of *masses*.

A *time unit of measurement* is defined to be an element $u_0 \in \mathbb{T}$, or its dual $u^0 \in \mathbb{T}^*$.

Moreover, we assume the *Planck constant* to be an element $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$. We refer to a particle with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$.

In the paper $\mathcal{S}(Q)$ denotes the sheaf of local sections of a fibred manifold $\boldsymbol{Q} \to \boldsymbol{E}$, $\mathcal{F}(\boldsymbol{E}, \mathbb{U})$ denotes the sheaf of local $\mathbb{U}$-valued functions on $\boldsymbol{E}$.

1. Gravitational and quantum structure of spacetime

We assume the classical *(Galilei) spacetime* to be a 4-dimensional orientable manifold $\boldsymbol{E}$, the *absolute time* to be a 1-dimensional oriented affine space $\boldsymbol{T}$ associated with the vector space $\mathbb{T} \otimes \mathbb{R}$ and the *time map* to be a surjective map $t : \boldsymbol{E} \to \boldsymbol{T}$ of rank 1. Moreover, we assume the fibres of spacetime to be equipped with a "scaled" Riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E})$ or inverse metric $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{*2} \otimes (V\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E})$.

Thus, we have the *time-form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$. Given a mass $m \in \mathbb{M}$, it is convenient to introduce the "normalized" metric $G := \frac{m}{\hbar} g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes_{\boldsymbol{E}} V^*\boldsymbol{E})$ or its inverse $\bar{G} := \frac{\hbar}{m} \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes_{\boldsymbol{E}} V\boldsymbol{E})$. We stress that the normalized metric and all objects which will be derived from it incorporate the chosen mass and the Planck constant.

We choose an orientation of spacetime. We shall refer to *spacetime charts* (x^λ), which are adapted to the fibring t and to the chosen orientation of $\boldsymbol{E}$, and such that x^0 is a Cartesian chart of $\boldsymbol{T}$ associated with a time unit of measurement u_0. The index 0 will refer to the base space, Latin indices $i, j, \dots = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both to the base space and the fibres. For short, we denote the local bases of vector fields and forms of $\boldsymbol{E}$ induced by a spacetime chart by (∂_λ) and (d^λ). The chart on the tangent space $T\boldsymbol{E}$ induced by a spacetime chart (x^λ) will be denoted by $(x^\lambda, \dot{x}^\lambda)$. In general, the check symbol "ˇ" will indicate vertical restriction.

We have the coordinate expressions $dt = u_0 \otimes d^0$ and $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$.

The metric g and the spacetime orientation yield the *space-like vertical volume form* and the dual *space-like vertical volume vector*

$$\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}, \qquad \eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 ,$$

$$\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{*3} \otimes \Lambda^3 V\boldsymbol{E}, \qquad \bar{\eta} = \frac{1}{\sqrt{|g|}} \check{\partial}_1 \wedge \check{\partial}_2 \wedge \check{\partial}_3 ,$$

where $|g| := \det(g_{ij})$. Then, we obtain well defined *spacetime volume form* and the dual *spacetime volume vector*

$$\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E}, \qquad \upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 ,$$

$$\bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{*3}) \otimes \Lambda^4 T\boldsymbol{E}, \qquad \bar{\upsilon} = \frac{1}{\sqrt{|g|}} u^0 \otimes \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 .$$

The classical *phase space* is defined to be the first jet space $t_0^1 : JE \equiv J_1E \to E$ of sections. A spacetime chart (x^λ) induces on the phase space the chart (x^0, x^i, x_0^i). We can naturally identify the phase space with the affine subbundle $JE \subset \mathbb{T}^* \otimes TE$ characterized by $x_0^i = u_0 \otimes \dot{x}^i$. Thus, we obtain the well defined *jet contact maps* $д : JE \to \mathbb{T}^* \otimes TE$ and $\theta : JE \to T^*E \otimes_E VE$, with coordinate expressions $д \equiv u^0 \otimes д_0 = u^0 \otimes (\partial_0 + x_0^i \partial_i)$ and $\theta \equiv \theta^i \otimes \partial_i = (d^i - x_0^i d^0) \otimes \partial_i$.

The *gravitational field* is defined to be a torsion free linear connection $K^\natural : TE \to T^*E \otimes_{TE} TTE$ of the bundle $TE \to E$ such that $\nabla dt = 0$. Its coordinate expression is of the type $K^\natural = d^\mu \otimes (\partial_\mu + K^\natural{}_\mu{}^i{}_\nu \, \dot{x}^\nu \, \dot{\partial}_i)$, with $K^\natural{}_\mu{}^i{}_\nu = K^\natural{}_\nu{}^i{}_\mu \in \mathcal{F}(E, \mathbb{R})$.

The *electromagnetic field* is a closed scaled 2-form $f : E \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*E$. Given a particle of charge q it is convenient to consider the re-scaled electromagnetic field $F := \frac{q}{\hbar} f : E \to \Lambda^2 T^*E$ which can be "added", in a covariant way (for details see [4], to the gravitational connection $K^\natural$ yielding a *(total) spacetime connection* K, with coordinate expression $K_i{}^h{}_j = K^\natural{}_i{}^h{}_j$, $K_0{}^h{}_j = K^\natural{}_0{}^h{}_j + \frac{1}{2} G_0^{hp} F_{pj}$, $K_0{}^h{}_0 = K^\natural{}_0{}^h{}_0 + \frac{1}{2} G_0^{ph} F_{p0}$. It turns out to be a time preserving torsion free linear connection of the tangent space of spacetime.

A spacetime connection K is said to be *metric* if $\nabla g = 0$. A spacetime connection K is metric if and only if the vertical restriction of K is the Levi-Civita connection $\varkappa$ given by g and $g_{pi} K_0{}^p{}_j + g_{pi} K_0{}^p{}_i = -\frac{1}{2} \partial_0 g_{ij}$.

A *phase connection* is defined to be a torsion free affine connection $\Gamma : JE \to T^*E \otimes_{JE} TJE$ of the bundle $t_0^1 : JE \to E$. Its coordinate expression is of the type $\Gamma = d^\mu \otimes (\partial_\mu + \Gamma_{\mu}{}^{i}_{0}\, \partial_i^0)$, where $\Gamma_{\mu}{}^{i}_{0} = \Gamma_{\mu}{}^{i}_{0}{}^{0}_{0} + \Gamma_{\mu}{}^{i}_{0}{}^{0}_{j}\, x_0^j$ with $\Gamma_{\mu}{}^{i}_{0}{}^{0}_{\nu} \in \mathcal{F}(E, \mathbb{R})$.

We can prove, [10], that there is a natural bijection $K \mapsto \Gamma$ between spacetime connections and phase connections characterized in coordinates by $\Gamma_{\mu}{}^{i}_{0}{}^{0}_{\nu} = K_\mu{}^i{}_\nu$.

A phase connection Γ on JE and the vertical metric G yield the *phase 2-form* $\Omega := \nu[\Gamma] \bar{\wedge} \theta : JE \to \Lambda^2 T^* JE$, whose coordinate expression is

$$\Omega = G_{ij}^0 \, (d_0^i - \Gamma_{\mu}{}^{i}_{0} d^\mu) \wedge \theta^j \,, \tag{1.1}$$

where $\nu[\Gamma] : JE \to \mathbb{T}^* \otimes (T^* JE \otimes_{JE} VE)$ is the vertical valued form associated with Γ and the contracted wedge product $\bar{\wedge}$ is taken with respect to G. In [8] it has been proved that the above form is the only non trivial natural 2-form which can be obtained from Γ and G.

The form Ω is non-degenerate, in fact $dt \wedge \Omega \wedge \Omega \wedge \Omega : JE \to \mathbb{T} \otimes \Lambda^7 T^* JE$ is a scaled volume form on JE. Moreover, the phase 2-form Ω is assumed to be closed, i.e. cosymplectic. This condition is expressed by field equations which relates the metric g and the spacetime connection K. One of the field equations is expressed by the fact that K is the metric connection, [4].

The vertical metric g and the metric spacetime connection define the *gravitational structure* of spacetime.

We assume, according to [4], the *quantum bundle* to be a Hermitian line bundle over spacetime

$$\pi : Q \to E,$$

i.e., $\pi : Q \to E$ is a Hermitian complex vector bundle with one-dimensional fibres. Let us denote by $h : Q \times_E Q \to \mathbb{C} \otimes \Lambda^3 V^* E$ the Hermitian product with values in vertical volume forms. Let $\mathbf{b} : E \to Q$ be a (local) base of Q such that $h(\mathbf{b}, \mathbf{b}) = \eta$. Such a local base is said to be *normal* and the fibred coordinate chart (x^0, x^i, z), $z \in \mathcal{F}(Q, \mathbb{C}\otimes\mathbb{L}^{*3/2})$, induced by a normal base of Q is said to be a *normal coordinate chart* on Q. In any fibred normal coordinate chart $h(\Psi, \Psi) = \bar{\psi}\psi\eta$ for every section $\Psi = \psi \mathrm{b} \in \mathcal{S}(Q)$.

A linear connection on Q is said to be *Hermitian* if it preserves the Hermitian fibred product h. In a normal fibred coordinate chart Hermitian connections are expressed in the form, [4],

$$\text{ч} = d^\lambda \otimes (\partial_\lambda + i\,\text{ч}_\lambda \mathbf{1}), \qquad \text{ч}_\lambda \in \mathcal{F}(E, \mathbb{R}), \tag{1.2}$$

where $\mathbf{1} = z \otimes \mathrm{b}$ is the Liouville vector field on Q. Let us consider a new fibred normal coordinate chart $(\underline{x}^0, \underline{y}^i, \underline{z})$ on Q, then the transformation relations are

$$\underline{x}^0 = ax^0 + a^0, \ \underline{y}^i = \underline{y}^i(x,y), \ \underline{z} = e^{2\pi i \vartheta(x,y)} z, \tag{1.3}$$

where $a, a^0 \in \mathbb{R}$, $a > 0$. We set $\tilde{a} = 1/a$. The function $\vartheta(x,y)$ represents a *change of gauge*. The transformation relations for coefficients of a Hermitian connections are

$$\underline{\text{ч}}_i = (\text{ч}_j + 2\pi \partial_j \vartheta) \frac{\partial y^j}{\partial \underline{y}^i}, \tag{1.4}$$

$$\underline{\text{ч}}_0 = (\text{ч}_0 + 2\pi \partial_0 \vartheta)\tilde{a} + (\text{ч}_j + 2\pi \partial_j \vartheta) \frac{\partial y^j}{\partial \underline{x}^0}. \tag{1.5}$$

Let us note that the quantum bundle Q can be viewed as an associated gauge-natural vector bundle, [11], defined on the category $\mathcal{PB}_{(1,3)}(U(1,\mathbb{C}))$ of principal $U(1,\mathbb{C})$-bundles over fibred manifolds with 1-dimensional bases and 3-dimensional fibres. The transformation relations (1.4) and (1.5) then define the action of the group $W^{1,1}_{(1,3)}U(1,\mathbb{C}) = G^1_{(1,3)} \underset{S}{\times} T^1_4 U(1,\mathbb{C})$ on the standard fibre of linear connections on Q. Here $\underset{S}{\times}$ denotes the semidirect product of groups and $G^k_{(1,3)}$ is the subgroup in the kth order differential group G^k_4 corresponding to fibred diffeomorphisms.

Let us consider the pullback bundle $\pi^\uparrow : Q^\uparrow := JE \times_E Q \to JE$ of the quantum bundle $\pi : Q \to E$, with respect to $t^1_0 : JE \to E$. Let us recall that a connection $\text{ч} : Q^\uparrow \to T^*JE \otimes_{JE} TQ^\uparrow$ is said to be the universal connection of the system of connections $\xi : JE \times_E Q \to T^*E \otimes_E TQ$ if, for every section $o : E \to JE$, the associated connection $\xi(o) : Q \to T^*E \otimes_E TQ$ of the system is obtained from ч by pullback according to the formula $\xi(o) = o^*\text{ч}$.

A connection $\text{ч}: Q^\uparrow \to T^*JE \otimes_{JE} TQ^\uparrow$ is said to be a *quantum connection* if, [4, 5],

Q1) ч is Hermitian,

Q2) ч is a universal connection,

Q3) the curvature of ч is given by

$$R_{\text{ч}} = i\,\Omega \otimes \mathbf{1}\,.$$

The condition Q3 implies, that the coefficients ч_λ of a quantum connection are given by coefficients of a Poicaré-Cartan 1-form

$$\Theta = -(\frac{1}{2} G^0_{ij} x^i_0 x^j_0 - A_0) d^0 + (G^0_{ij} x^j_0 + A_i) d^i$$

associated to the phase 2-form Ω, where $A = A_\lambda d^\lambda$ is a potential given by a chosen quantum connection.

The potential A is not a 1-form. If we express the transformation relations for A_λ, we get from (1.4) and (1.5) that A is a section of a 1-order gauge-natural bundle (called *potential bundle*) defined on the category $\mathcal{PB}_{(1,3)}(U(1,\mathbb{C}))$. The action of the group $W^{1,1}U(1,\mathbb{C})$ on the standard fibre of the potential bundle is then given by

$$\underline{A}_i = \left(A_j - G^0_{jp}\tilde{a}^p_r a^r_0 + 2\pi\vartheta_j\right)\tilde{a}^j_i\,, \tag{1.6}$$

$$\begin{aligned}\underline{A}_0 = {}& \left(A_0 + \frac{1}{2}G^0_{pq}\tilde{a}^p_r a^r_0 \tilde{a}^q_s a^s_0 + 2\pi\vartheta_0\right)\tilde{a} \\ & + \left(A_j + 2\pi\vartheta_j\right)\tilde{a}^j_0\,,\end{aligned} \tag{1.7}$$

where $(a, a^i_\lambda, \vartheta, \vartheta_j)$ are coordinates on the group $W^{1,1}_{(1,3)}U(1,\mathbb{C})$ and tilde denotes the inverse element.

A pair $(\boldsymbol{Q}, \text{ч})$ is said to be a *quantum structure* over spacetime.

In what follows we assume a quantum structure exists.

2. Quantum Lagrangians

Let us consider a section $\Psi \in \mathcal{S}(\boldsymbol{Q})$, its pullback on $\boldsymbol{JE}$ (denoted by the same symbol) and a quantum connection ч. The covariant differential of Ψ with respect to ч is a fibred morphism over $\boldsymbol{E}$

$$\nabla[\text{ч}]\Psi : \boldsymbol{JE} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}$$

and the *time-like* and the *space-like covariant differentials* of Ψ are

$$\overset{\circ}{\nabla}\Psi = \text{д} \lrcorner \nabla\Psi : \boldsymbol{JE} \to \mathbb{T}^* \otimes \boldsymbol{Q}, \quad \check{\nabla}\Psi : \mathbb{J}\boldsymbol{E} \to \mathbb{V}^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q}\,.$$

Then, for any section $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we obtain the following invariant fibred morphisms over $\boldsymbol{E}$

$$\overset{\circ}{\mathcal{L}}(\Psi) = \frac{1}{2}dt \wedge \left(h(\Psi, i\overset{\circ}{\nabla}\Psi) + h(i\overset{\circ}{\nabla}\Psi, \Psi)\right) : \boldsymbol{JE} \to \Lambda^4 T^*\boldsymbol{E},$$

$$\check{\mathcal{L}}(\Psi) = \frac{1}{2}dt \wedge (\bar{G} \otimes h)(\check{\nabla}\Psi, \check{\nabla}\Psi) : \boldsymbol{JE} \to \Lambda^4 T^*\boldsymbol{E},$$

and the *canonical quantum Lagrangian* is a unique (up to a multiplicative factor) linear combination of the above morphisms which projects on $\boldsymbol{E}$, namely

$$\mathcal{L}_{can}(\Psi) = \overset{\circ}{\mathcal{L}}(\Psi) - \check{\mathcal{L}}(\Psi)$$

with coordinate expression

$$\begin{aligned}\mathcal{L}_{can}(\Psi) = {}& \frac{1}{2}\left(i(\bar{\psi}\partial_0\psi - \psi\partial_0\bar{\psi}) + iG^{pq}_0 A_q(\psi\partial_p\bar{\psi} - \bar{\psi}\partial_p\psi)\right. \\ & \left. - G^{pq}_0\partial_p\bar{\psi}\partial_q\psi + \bar{\psi}\psi(2A_0 - G^{pq}_0 A_p A_q)\right) v^0\,,\end{aligned} \tag{2.1}$$

where $v^0 = v(u^0) = \sqrt{|g|}d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

From the point of view of natural geometry, [11, 12], the canonical quantum Lagrangian is a natural operator transforming vertical metrics, sections of the quantum bundle and potentials associated with quantum structures into volume forms on $\boldsymbol{E}$. This operator is of order one with respect to sections of the quantum bundle. In [9] we have classified all natural quantum Lagrangians of the above type by using Hermitian product with values in complex numbers and the Planck constant $\hbar$. According to [4], we now discuss the classification of natural Lagrangians under different conditions: we use the Hermitian product with values in vertical volumes forms, we assume also the dependence of natural quantum Lagrangians on spacetime connections (up to finite order k), we use physical dimensions of all objects and assume Lagrangians covariant with respect to changes of bases in unit spaces $\mathbb{L}$ and $\mathbb{T}$. Moreover, we use constants m, $\hbar$ (via the normalized metric).

According to the general theory of natural differential operators, [11, 12], any natural quantum Lagrangian is of the form $\mathcal{L} = f v = f_0 v^0$, where f_0 is a real $\mathbb{T}\otimes\mathbb{L}^{*3}$-valued function defined on standard fibres of bundles in question, i.e. the bundle of normalized vertical metrics on $\boldsymbol{E}$, the bundle of 1-jet prolongation of $\boldsymbol{Q}$, the bundle of potentials and k-jet prolongation of the bundle of linear connections on $\boldsymbol{E}$. Then f_0 is invariant with respect to the Lie group $W^{k+2,1}_{(1,3)}U(1,\mathbb{C}) = G^{k+2}_{(1,3)} \underset{S}{\times} T^1_4 U(1,\mathbb{C})$ acting on these standard fibres where the action on the standard fibre of $J_1\boldsymbol{Q}$ can be deduced from (1.3) and the action on the standard fibre of the potential bundle is given by (1.6) and (1.7). The others actions are standard, [11, 12]. I.e. f_0 is of the type

$$f_0(G^{ij}_0, z, \bar{z}, z_\lambda, \bar{z}_\lambda, A_\lambda, K_\mu{}^\lambda{}_\nu, K_\mu{}^\lambda{}_{\nu,\rho_1}, \dots, K_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_k}) . \tag{2.2}$$

We have

Lemma 2.1. *Any $W^{k+2,1}U(1,\mathbb{C})$-equivariant $\mathbb{T}\otimes\mathbb{L}^{*3}$-valued function (2.2) is a linear combination (with real coefficients) of two functions*

$$\begin{aligned}(f_1)_0 &= G^{ij}_0 z_i \bar{z}_j + i(z\bar{z}_0 - \bar{z}z_0)\\ &\quad - iG^{ij}_0 A_j(z\bar{z}_i - \bar{z}z_i) + (G^{ij}_0 A_i A_j - 2A_0)z\bar{z} ,\\ (f_2)_0 &= G^{ij}_0 R_i{}^p{}_{pj} z\bar{z} ,\end{aligned}$$

where $R_\mu{}^\lambda{}_{\nu\kappa}$ is the formal curvature of $K_\mu{}^\lambda{}_\nu$.

PROOF. Let us recall that the formal curvature of $K_\mu{}^\lambda{}_\nu$ is given by

$$R_\mu{}^\lambda{}_{\nu\kappa} = K_\mu{}^\lambda{}_{\nu,\kappa} - K_\mu{}^\lambda{}_{\kappa,\nu} + K_\rho{}^\lambda{}_\kappa K_\mu{}^\rho{}_\nu - K_\rho{}^\lambda{}_\nu K_\mu{}^\rho{}_\kappa \tag{2.3}$$

and its formal covariant derivative is given by

$$\begin{aligned}R_\mu{}^\lambda{}_{\nu\kappa;\rho} &= R_\mu{}^\lambda{}_{\nu\kappa,\rho} - K_\rho{}^\lambda{}_\sigma R_\mu{}^\sigma{}_{\nu\kappa} + K_\rho{}^\sigma{}_\mu R_\sigma{}^\lambda{}_{\nu\kappa}\\ &\quad + K_\rho{}^\sigma{}_\nu R_\mu{}^\lambda{}_{\sigma\kappa} + K_\rho{}^\sigma{}_\kappa R_\mu{}^\lambda{}_{\nu\sigma} .\end{aligned} \tag{2.4}$$

The formal higher order covariant derivatives can be defined in the same manner.

Further let us define the 1st order quantum formal covariant derivative of z by

$$z_{;\lambda} = z_\lambda - iA_\lambda z . \tag{2.5}$$

The formal curvature of K and its formal covariant derivatives are transformed under the action of $G^1_{(1,3)}$ as tensors and $z_{;\lambda}$ is transformed by the action of the group $W^{1,0}_{(1,3)}U(1,\mathbb{C})$ given by

$$\underline{z}_{;0} = e^{2\pi i\vartheta}(z_{;\mu}\tilde{a}^\mu_0 - \frac{i}{2} z G^0_{pq}\tilde{a}^p_r a^r_0 \tilde{a}^q_s a^s_0 \tilde{a}) , \tag{2.6}$$

$$\underline{z}_{;i} = e^{2\pi i\vartheta}(z_{;j} + izG^0_{pj}\tilde{a}^p_r a^r_0)\tilde{a}^j_i . \tag{2.7}$$

Now we can use the orbit reduction theorem, [11], and to express f_0 in the form $f_0 = g_0 \circ F$, where F is the mapping given by (2.3) – (2.5) and

$$g_0(G_0^{ij}, z, \bar{z}, z_{;\lambda}, \bar{z}_{;\lambda}, R_\mu{}^\lambda{}_{\nu\kappa}, R_\mu{}^\lambda{}_{\nu\kappa;\rho_1}, \dots, R_\mu{}^\lambda{}_{\nu\kappa;\rho_1;\dots;\rho_{k-1}}) \tag{2.8}$$

is a $W^{1,0}_{(1,3)}U(1,\mathbb{C})$-equivariant $\mathbb{T} \otimes \mathbb{L}^{*3}$-valued function.

From the homogeneous function theorem, [11, 12], applied on a change of base in $\mathbb{L}$ and a change of gauge we get that g_0 has to be in the form

$$g_0 = a_0 z\bar{z} + b_0^\lambda z\bar{z}_{;\lambda} + c_0^\lambda z_{;\lambda}\bar{z} + d_0^{\lambda\mu} z_{;\lambda}\bar{z}_{;\mu}\,, \tag{2.9}$$

where all coefficients are $G^1_{(1,3)}$-invariant and depend on G_0^{ij}, the formal curvature and its formal covariant derivatives. Then, if we consider global solutions only, we get from homogeneous function theorem

$$a_0 = a\, G_0^{ij} R_i{}^k{}_{kj}, \quad b_0^0 = b, \quad b_0^i = 0, \quad c_0^0 = c, \quad c_0^i = 0;$$
$$d_0^{i0} = d_0^{0i} = 0, \quad d_0^{00} = 0, \quad d_0^{ij} = d\, G_0^{ij}\,,$$

where $a, \dots, d$ are constants.

So we have

$$g_0 = a\, G_0^{ij} R_i{}^k{}_{kj} z\bar{z} + b\, z\bar{z}_{;0} + c\, z_{;0}\bar{z} + d\, G_0^{ij} z_{;i}\bar{z}_{;j}\,. \tag{2.10}$$

If we consider the equivariancy of (2.10) with respect to elements $(\delta^\lambda_\mu, a^p_0)$ in $G^1_{(1,3)}$ we get

$$b = id, \quad c = -id, \quad \text{and } a,\ d \text{ are arbitrary real numbers}. \tag{2.11}$$

Substituting (2.5) and (2.11) into (2.10) we get Lemma 2.1. □

Now we can prove

Theorem 2.1. *All 1st order (with respect to sections of the quantum bundle) natural quantum Lagrangians induced by the gravitational and quantum structure of spacetime are of the form*

$$\mathcal{L}(\Psi) = a\, \mathcal{L}_{can}(\Psi) - b\frac{\hbar}{2m} R dt \wedge h(\Psi, \Psi)\,,$$

where $\mathcal{L}_{can}(\Psi)$ is the canonical quantum Lagrangian, R is the scalar curvature of the vertical metric connection $\varkappa$ and a, b are real numbers.

PROOF. Theorem 2.1 is the direct consequence of the above Lemma 2.1. It is easy to see that the natural Lagrangian corresponding to the equivariant function $(f_1)_0$ is a constant multiple of the canonical quantum Lagrangian. Moreover, the vertical restriction of the Ricci tensor of spacetime connection K coincides with the Ricci tensor of the vertical restriction of K which, by the assumption $d\Omega = 0$, coincides with the vertical connection $\varkappa$ generated by g. Then $G_0^{ij} R_i{}^p{}_{pj} = \frac{\hbar_0}{m} R$, where R is the scalar curvature of $\varkappa$, and we get the Theorem 2.1. □

Remark 2.1. Polynomiality (in sections of the quantum bundle) of natural quantum Lagrangians is a consequence of physical dimensions of our objects. Without using these dimensions we would obtain much more complicated solutions, [9]. Moreover, physical dimensions of objects restrict also the order of derivatives of spacetime connection on the 1st order, i.e. $\mathcal{L}(\Psi)$ depends only on the curvature and does not depend on its covariant derivatives.

Remark 2.2. Since the vertical restriction of K is given by the metric g, all natural quantum Lagrangians described in Theorem 2.1 depend only on the vertical metric and its derivatives up to order two and they do not depend on the horizontal part of the spacetime connection K.

Remark 2.3. Comparing the results of classification in [9] and the classification of Theorem 2.1 we see that we have lost one natural Lagrangian in the list of base Lagrangians in [9]. Namely it is the vertical differential of the Hermitian product contracted by the metric, i.e. $\langle \bar{g}, (\check{d}h(\Psi,\Psi), \check{d}h(\Psi,\Psi))\rangle$. In our new situation, the Hermitian product $h(\Psi,\Psi)$ is a vertical volume form and its vertical differential vanishes.

Remark 2.4. Let us remark that the constant $-b/2$ in Theorem 2.1 we have chosen only to obtain later the expression of the Schrödinger equation corresponding to classical situation.

3. Euler-Lagrange operator and Schrödinger equation

Let us consider a quantum Lagrangian $\mathcal{L} = \ell_0\, d^0 \wedge d^1 \wedge d^2 \wedge d^3$ and recall that the *Euler-Lagrange morphism* $\mathcal{E}(\mathcal{L}) : J_2\boldsymbol{Q} \to V^*\boldsymbol{Q} \otimes \Lambda^4 T^*\boldsymbol{E}$ associated with $\mathcal{L}$ is given in coordinates by

$$\mathcal{E}(\mathcal{L}) = \left(\frac{\partial \ell_0}{\partial \bar{z}} - D_\lambda \frac{\partial \ell_0}{\partial \bar{z}_\lambda}\right) d^z \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3\,,$$

where D_λ is the formal derivative with respect to x^λ. By using the identification of $V^*\boldsymbol{Q}$ with $\boldsymbol{Q}\times\boldsymbol{Q}^*$ and the isomorphism $h^\sharp : \boldsymbol{Q}^* \to \mathbb{L}^3 \otimes \boldsymbol{Q}$ we can express the Euler-Lagrange morphism as

$$^\sharp\mathcal{E}(\mathcal{L}) : J_2\boldsymbol{Q} \to \mathbb{L}^3 \otimes \boldsymbol{Q} \otimes \Lambda^4 T^*\boldsymbol{E}.$$

Then, for any section $\Psi \in \mathcal{S}(\boldsymbol{Q})$, we have the *Euler-Lagrange operator* $^\sharp\mathcal{E}(\mathcal{L})(\Psi) = {}^\sharp\mathcal{E}(\mathcal{L}) \circ j^2\Psi$ associated with natural quantum Lagrangian of Theorem 2.1 in the form

$$\begin{aligned} {}^\sharp\mathcal{E}(\mathcal{L})(\Psi) = \Big[a\Big(&i(\partial_0\psi - iA_0\psi + \frac{1}{2}\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\psi) \\ &+ \frac{1}{2}G_0^{pq}(\partial_p\partial_q\psi - 2iA_p\partial_q\psi - i\psi\partial_q A_p - A_pA_q\psi) \\ &+ \frac{1}{2}G_0^{pq}\varkappa_p{}^h{}_q(\partial_h\psi - iA_h\psi)\Big) - b\frac{\hbar_0}{2m}R\psi\Big]\,\mathbf{b}\otimes v^0. \end{aligned} \tag{3.1}$$

We define the *Schrödinger operator* associated with a natural quantum Lagrangian $\mathcal{L}$ to be the sheaf morphism

$$\mathcal{O}_{Sch}(\mathcal{L}) := \langle \bar{v}; {}^\sharp\mathcal{E}(\mathcal{L})\rangle : \mathcal{S}(Q) \to \mathcal{S}(\mathbb{T}^* \otimes Q)\,, \tag{3.2}$$

Let us consider a section (an observer) $o : \boldsymbol{E} \to J\boldsymbol{E} \hookrightarrow \mathbb{T}^* \otimes T\boldsymbol{E}$ and let us define the divergence of o as a $\mathbb{T}^*$-valued function given by

$$L_o\eta = \operatorname{div}(o)\eta : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E} \tag{3.3}$$

which, in coordinates adapted to o, has a coordinate expression

$$\operatorname{div}(o) = \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,. \tag{3.4}$$

Further we have the (observed) Laplacian

$$\overset{o}{\Delta}(\Psi) = \langle \bar{g}; \nabla[o^{*}ч \otimes K]\nabla[o^{*}ч]\Psi\rangle : \boldsymbol{E} \to \boldsymbol{Q}$$

with coordinate expression, in coordinates adapted to o,

$$\begin{aligned}\overset{o}{\Delta}(\Psi) &= \big(g^{pq}(\partial_p - iA_p)(\partial_q - iA_q) + g^{pq}\varkappa^h_{pq}(\partial_h - iA_h)\big)(\Psi)\\ &= g^{pq}\big(\partial_p\partial_q\psi - 2iA_p\partial_q\psi - i\psi\partial_q A_p - A_pA_q\psi\big)\\ &\quad + g^{pq}\varkappa^h_{pq}\big(\partial_h\psi - iA_h\psi)\big)\mathbf{b}\end{aligned}$$

which implies that the Schrödinger operator (3.2) associated with natural quantum Lagrangian of Theorem 2.1 can be expressed as

$$\mathcal{O}_{Sch}(\mathcal{L}(\Psi)) = u^0 \otimes \left(a\big(i\,(\nabla_0 + \frac{1}{2}\mathrm{div}(o)) + \frac{\hbar_0}{2m}\overset{o}{\Delta}\big) - b\frac{\hbar_0}{2m}R\right)(\Psi) \tag{3.5}$$

and the *generalized Schrödinger equation* can be written in the form

$$i\,(\nabla_0 + \mathrm{div}(o))\Psi = -\frac{\hbar_0}{2m}\left(\overset{o}{\Delta} - kR\right)(\Psi)\,, \tag{3.6}$$

where a real constant $k = b/a$, $a \neq 0$, is arbitrary and there is no distinguished value for k which comes from naturality of the construction.

Let us note that even if the operators ∇_0, $\mathrm{div}(o)$ and $\overset{o}{\Delta}$ depend on an observer o, the Schrödinger operator (3.5) is observer independent.

Now we shall classify all natural operators $\mathcal{O}_{Sch} : \mathcal{S}(\boldsymbol{Q}) \to \mathcal{S}(\mathbb{T}^* \otimes \boldsymbol{Q})$ of the Schrödinger type, i.e. we shall classify all second order operators depending on the vertical metric field, the spacetime connection and its derivatives of finite order k and the field of potentials and its first order derivatives. Such operators are given by covariant mapping (over $\boldsymbol{E}$)

$$f : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q} \tag{3.7}$$

parametrized by the coordinates on the standard fibres of bundles of normalized metrics, 1st jet prolongation of the bundle of potentials and kth jet prolongation of the bundle of linear connections on $\boldsymbol{E}$. Any covariant mapping (3.7) is of the form $f = f_0 u^0 \otimes \mathbf{b}$, where

$$f_0(G_0^{ij}, z, \bar{z}, z_\lambda, \bar{z}_\lambda, z_{\lambda\mu}, \bar{z}_{\lambda\mu}, A_\lambda, A_{\lambda,\mu}, K_\mu{}^\lambda{}_\nu, K_\mu{}^\lambda{}_{\nu,\rho_1}, \dots, K_\mu{}^\lambda{}_{\nu,\rho_1\dots\rho_k}) \tag{3.8}$$

is an $W^{k+2,2}_{(1,3)}U(1,\mathbb{C})$-invariant $\mathbb{T} \otimes \mathbb{L}^{*3/2}$-valued function.

We have

Lemma 3.1. *Any* $W^{k+2,2}_{(1,3)}U(1,\mathbb{C})$*-invariant* $\mathbb{T} \otimes \mathbb{L}^{*3/2}$*-valued function (3.8) is of the form*

$$\begin{aligned}f_0 = a'\big(2i(z_0 - iA_0 z - \frac{1}{2}K_0{}^k{}_k z) + G_0^{ij}K_i{}^k{}_j(z_k - iA_k z)\\ + G_0^{ij}(z_{ij} - 2iA_i z_j - iA_{i,j}z - A_iA_j z)\big) + b' G_0^{ij} R_i{}^k{}_{kj} z\,,\end{aligned} \tag{3.9}$$

where a', b' *are complex constants.*

PROOF. Let us consider the formal curvature, its formal covariant derivative up the order $(k-1)$ and the 1st order quantum formal covariant derivative of z defined by (2.3) - (2.5).

Now let us define the 2nd order quantum formal covariant derivative of z by

$$z_{;\lambda;\mu} = (z_{;\lambda})_\mu - iA_\mu z_{;\lambda} + K_\lambda{}^\rho{}_\mu z_{;\rho}\,. \tag{3.10}$$

From (2.6) we can deduce that $z_{;\lambda;\mu}$ is transformed by the action of the Lie group $W^{2,0}_{(1,3)}U(1,\mathbb{C})$.

Now we can use the orbit reduction theorem and to express f_0 in the form $f_0 = g_0 \circ F$, where F is the mapping given by (2.3) - (2.5), (3.10) and

$$\begin{aligned} g_0(G_0^{ij}, z, \bar{z}, z_{;\lambda}, \bar{z}_{;\lambda}, z_{;\lambda;\mu}, \bar{z}_{;\lambda;\mu}, K_\mu{}^\lambda{}_\nu, \\ R_\mu{}^\lambda{}_{\nu\kappa}, R_\mu{}^\lambda{}_{\nu\kappa;\rho_1}, \dots, R_\mu{}^\lambda{}_{\nu\kappa;\rho_1;\dots;\rho_{k-1}}) \end{aligned} \tag{3.11}$$

is a $W^{2,0}_{(1,3)}U(1,\mathbb{C})$-equivariant $\mathbb{T}\otimes\mathbb{L}^{*3/2}$-valued function.

From the homogeneous function theorem applied on a change of base in $\mathbb{L}$ and a change of gauge we get that g_0 has to be in the form

$$g_0 = a_0 z + b_0^\lambda z_{;\lambda} + c_0^{\lambda\mu} z_{;\lambda;\mu}\,, \tag{3.12}$$

where all coefficients are $G^2_{(1,3)}$-invariant and depend on G_0^{ij}, $K_\mu{}^\lambda{}_\nu$, the formal curvature and its formal covariant derivatives. Then, if we consider global solutions only, we get from homogeneous function theorem

$$\begin{aligned} a_0 &= a_1\, G_0^{ij} R_i{}^k{}_{kj} + a_2\, G_0^{ij} K_i{}^p{}_j K_p{}^q{}_q \\ &\quad + a_3\, G_0^{ij} K_i{}^p{}_p K_j{}^q{}_q + a_4\, G_0^{ij} K_i{}^p{}_q K_j{}^q{}_p + a_5\, K_0{}^p{}_p; \\ b_0^0 &= b_1,\ \ b_0^i = b_2\, G_0^{pq} K_p{}^i{}_q + b_3\, G_0^{pi} K_p{}^q{}_q; \\ c_0^{00} &= 0,\ \ c_0^{0i} = c_0^{i0} = 0,\ \ c_0^{ij} = c\, G_0^{ij}\,, \end{aligned}$$

where $a_i, \dots, c$ are constants.

So we have

$$\begin{aligned} g_0 &= a_1\, G_0^{ij} R_i{}^k{}_{kj} z + a_2\, G_0^{ij} K_i{}^p{}_j K_p{}^q{}_q z \\ &\quad + a_3\, G_0^{ij} K_i{}^p{}_p K_j{}^q{}_q z + a_4\, G_0^{ij} K_i{}^p{}_q K_j{}^q{}_p z + a_5\, K_0{}^p{}_p z \\ &\quad + b_1\, z_{;0} + b_2\, G_0^{pq} K_p{}^i{}_q z_{;i} + b_3\, G_0^{pi} K_p{}^q{}_q z_{;i} + c\, G_0^{ij} z_{;i;j}\,. \end{aligned} \tag{3.13}$$

If we consider the equivariancy of (3.13) with respect to the kernel of the group homomorphism $p_1^2 : G^2_{(1,3)} \to G^1_{(1,3)}$, we get

$$\begin{aligned} &a_2 = a_3 = a_4 = b_2 = b_3 = 0; \\ &a_5 = -ic \quad \text{and} \quad a_1, b_1, c \quad \text{are arbitrary complex numbers.} \end{aligned}$$

Hence (3.13) reduces to

$$g_0 = a_1\, G_0^{ij} R_i{}^k{}_{kj} z + b_1 z_{;0} + c(G_0^{ij} z_{;i;j} - iK_0{}^p{}_p z)\,. \tag{3.14}$$

Finally we consider the equivariancy of (3.14) with respect to elements $(\delta^\lambda_\mu, a^p_0)$ in $G^1_{(1,3)}$ and we get

$$b_1 = 2ic\,, \qquad \text{and} \quad a_1,\, c \quad \text{are arbitrary}. \tag{3.15}$$

Substituting (2.5), (3.10) and (3.15) (where we have put $a_1 = b'$, $c = a'$) into (3.14) we get Lemma 3.1. □

Theorem 3.1. *All 2nd order natural operators of Schrödinger type are of the form*

$$\mathcal{O}_{Sch}(\Psi) = u^0 \otimes \left(a\big(i\,(\nabla_0 + \frac{1}{2}\mathrm{div}(o)) + \frac{\hbar_0}{2m}\overset{o}{\Delta}\big) - b\frac{\hbar_0}{2m}R \right)(\Psi), \tag{3.16}$$

where R is the scalar curvature of the vertical metric connection, o is an observer and a, b are complex numbers.

Proof. From the condition $d\Omega = 0$, [4], we have

$$K_0{}^k{}_k = -\frac{1}{2}g^{ij}\partial_0 g_{ij} = -\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}$$

which implies that the operator corresponding to (3.9) (for $a = 2a'$, $b = -b'/2$) is (3.16). □

Remark 3.1. From Theorem 3.1 it follows that any natural operator of Schrödinger type is linear (with respect to sections of the quantum bundle) and is associated with a natural quantum Lagrangian described by Theorem 2.1. Moreover, the corresponding generalized Schrödinger equation is of the type (3.6), where the constant k can be complex.

Remark 3.2. Let us note that by using different methods the canonical quantum Lagrangian and the corresponding covariant Schrödinger equation were studied for instance in [3, 13, 14]. Our result concerning the canonical quantum Lagrangian and the corresponding covariant Schrödinger equation corresponds to [3].

4. Higher order natural quantum Lagrangians

In Theorem 2.1 we have classified all 1st order natural quantum Lagrangians. Naturally, there is a question if higher order natural quantum Lagrangians exist. The answer is positive, at least in the second order. If we consider the Schrödinger operator $\mathcal{O}_{Sch}(\mathcal{L}_{can})$ associated with the canonical quantum Lagrangian $\mathcal{L}_{can}$, then it is easy to see that

$$\mathcal{L}_{Sch}(\Psi) = \frac{1}{2}\big(dt \wedge h(\Psi, \mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi))) + dt \wedge h(\mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi)), \Psi)\big) \tag{4.1}$$

is the 2nd order natural quantum Lagrangian. Moreover, by using the same method as in the proof of Lemma 3.1, we can classify all 2nd order Lagrangians and we get

Theorem 4.1. *All 2nd order (with respect to sections of the quantum bundle) natural quantum Lagrangians induced by the gravitational and quantum structure of spacetime are of the form*

$$\mathcal{L}(\Psi) = a\,\mathcal{L}_{can}(\Psi) - b\frac{\hbar}{2m}R\,dt \wedge h(\Psi, \Psi) + c\,\mathcal{L}_{Sch}(\Psi)$$

where a, b, c are real numbers. □

Now, if we associate the Schrödinger operator with the second order natural quantum Lagrangian $\mathcal{L}_{Sch}$ we get

$$\mathcal{O}_{Sch}(\mathcal{L}_{Sch}(\Psi)) = \mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi)), \tag{4.2}$$

which implies that the Schrödinger operator associated with 2nd order natural quantum Lagrangians from Theorem 4.1 is

$$\mathcal{O}_{Sch}(\mathcal{L}(\Psi)) = (a+c)\mathcal{O}_{Sch}(\mathcal{L}_{can}(\Psi)) - b\frac{\hbar}{2m}R(\Psi), \tag{4.3}$$

i.e. it is the operator from Theorem 3.1.

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