Robin C. Graham

Volume and area renormalizations for conformally compact Einstein metrics

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 19th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2000. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 63. pp. 31--42.

Persistent URL: http://dml.cz/dmlcz/701645

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 63 (2000) pp. 31-42

VOLUME AND AREA RENORMALIZATIONS FOR CONFORMALLY COMPACT EINSTEIN METRICS

C. ROBIN GRAHAM

1. INTRODUCTION

It has long been known that there are very close connections between the geometry of hyperbolic space $\mathbb{H}^{n+1}$ of $n+1$ dimensions and the conformal geometry of the n-sphere $\mathbb{S}^n$, viewed as the sphere at infinity of $\mathbb{H}^{n+1}$. In recent years it has been realized that it is fruitful to consider generalizations of some of these connections when $\mathbb{H}^{n+1}$ is replaced by a "conformally compact" Einstein manifold X of negative scalar curvature, and $\mathbb{S}^n$ is replaced by a compact conformal manifold M, the "conformal infinity" of X. Quite recently there has been a great deal of interest in the physics community in a correspondence (the so-called Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence) proposed by Maldacena [16] between string theory and supergravity on such X and supersymmetric conformal field theories on M. In this article we describe some new purely geometric invariants of conformally compact Einstein manifolds and of their minimal submanifolds which have been discovered via this correspondence.

The relevant notion of conformal infinity is that introduced by Penrose. A Riemannian metric g_+ on the interior X^{n+1} of a compact manifold with boundary $\overline{X}$ is said to be conformally compact if $\overline{g} \equiv r^2 g_+$ extends continuously (or with some degree of smoothness) as a metric to $\overline{X}$, where r is a defining function for $M = \partial X$, i.e. $r > 0$ on X and $r = 0$, $dr \neq 0$ on M. The restriction of $\overline{g}$ to TM rescales upon changing r, so defines invariantly a conformal class of metrics on M, the conformal infinity of g_+. We are concerned with conformally compact metrics g_+ which satisfy the Einstein condition $\mathrm{Ric}(g_+) = -ng_+$. At least near the hyperbolic metric, these can be parametrized by their conformal infinities: in [8] it is shown that each conformal structure on $\mathbb{S}^n$ sufficiently near the standard one is the conformal infinity of a unique (up to diffeomorphism) conformally compact Einstein metric on the ball near the hyperbolic metric.

The volume $\mathrm{Vol}(X)$ of any conformally compact manifold X is infinite. An appropriate renormalization of $\mathrm{Vol}(X)$ for X Einstein gives rise to the new volume invariants. In the physics setting, $\mathrm{Vol}(X)$ arises from a concrete procedure outlined by Witten

The paper is in final form and no version of it will be submitted elsewhere.

[22] and independently by Gubser, Klebanov, and Polyakov [10], following the suggestion of Maldacena, for calculating observables in a conformal field theory on M via supergravity and string theory on X. Under various limits and approximations, the partition function of a conformal field theory on M is given in terms of the gravitational action on X, which for an Einstein metric g_+ is proportional to the volume $\mathrm{Vol}(X)$.

The volume renormalization was carried out by Henningson and Skenderis in [12]. As shown in [7] and [8], each representative metric on M for the conformal infinity determines a special defining function r in a neighborhood of M. As $\epsilon \to 0$, the function $\mathrm{Vol}(\{r > \epsilon\})$ has an asymptotic expansion in negative powers of ϵ, and a $\log \epsilon$ term if n is even. The coefficients of the negative powers of ϵ depend on the representative conformal metric used to determine r. However, it turns out that if n is odd, then the constant term in the expansion is independent of this choice, so is a global invariant of the metric g_+. If n is even, the constant term is not invariant, giving rise to a so-called conformal anomaly. However, in this case the coefficient of the $\log \epsilon$ term is invariant, and in fact is given by the integral of a local curvature expression over M. The $\log \epsilon$ coefficient is therefore actually a conformal invariant of M, independent of which (X, g_+) might have been chosen with conformal infinity M.

Various of the conformal field theories to which the AdS/CFT correspondence applies contain observables associated to submanifolds N of M. According to the correspondence, in a suitable approximation the expectation value of such an observable can be calculated in terms of the area $A(Y)$ in the g_+ metric of minimal submanifolds Y of X with $\partial Y = N$. Existence theory for such minimal submanifolds is discussed for hyperbolic X in [1], [2]. As in the volume case, necessarily $A(Y) = \infty$, so one is led to consideration of renormalizing the area of a minimal submanifold. This renormalization was discussed in hyperbolic space for $\dim N = 1, 2$ in [3] and in general in [9]. If r is the special defining function associated to a conformal representative on M as above, then $\mathrm{Area}(Y \cap \{r > \epsilon\})$ has an expansion in negative powers of ϵ, and again a $\log \epsilon$ term if $k = \dim(N)$ is even. The invariance properties of the coefficients are similar to those above. If k is odd, then the constant term in the expansion is independent of the choice of conformal representative on M, so is a global invariant of the minimal submanifold Y. If k is even, there is a conformal anomaly for the constant term, but the $\log \epsilon$ coefficient is a conformal invariant of the submanifold N of M. One can calculate explicitly the $\log \epsilon$ coefficient for $k = 2$; it turns out to be a version on a general conformal manifold of the Willmore functional of a surface in conformally flat space. Even in the conformally flat case, this relationship between the Willmore functional of a surface and the renormalization of the area of a minimal extension seems to be of some interest. The Willmore functional is called the "rigid string action" in the physics literature ([3], [18]).

In §2. we review some of the basic properties of conformally compact Einstein metrics. In §3. we discuss the results of [12]: the derivation of the volume renormalization and resulting invariants and anomalies and the explicit identification of the $\log \epsilon$ coefficient and anomaly for $n = 4, 6$. We also calculate the renormalized volume for $\mathbb{H}^{n+1}$ when n is odd; it turns out that its sign depends on the parity of $(n+1)/2$. In §4. we review the area renormalization for minimal submanifolds, following [9].

We remark that in order to justify the derivation of the asymptotic expansions in ϵ of the volume and area, we have to assume that the Einstein metric g_+ and the minimal submanifold Y are sufficiently regular at infinity. Here sufficiently regular means that they have asymptotic expansions to high enough order, in general involving log terms, which formally solve the Einstein or minimal area equations. One expects that if the conformal structure on M and the submanifold N are smooth, then any conformally compact Einstein metric g_+ and minimal submanifold Y will have such regularity, assuming they take on the boundary data in a suitable sense. Some regularity results for minimal submanifolds of hyperbolic space are given in [11], [14], [15], [20]. (An error in [14] is corrected in [20].) A regularity theorem for Einstein metrics has been obtained by Skinner [19].

2. Conformally Compact Einstein Metrics

Let X be the interior of a compact manifold with boundary $\overline{X}$ of dimension $n+1$ as in the introduction and let g_+ be a conformally compact metric on X. Let r be a sufficiently smooth defining function for $M = \partial X$ defined near M and set $\overline{g} = r^2 g_+$. As discussed in the introduction, the conformal class $[\overline{g}|_{TM}]$ is an invariant of g_+, independent of any choices. The function $|dr|^2_{\overline{g}} = \overline{g}^{ij} r_i r_j$ extends to $\overline{X}$ and its restriction to M is independent of the choice of r, so defines a second invariant of g_+. The metric g_+ on X is complete and its sectional curvature is asymptotically constant at each boundary point–conformally transforming the curvature tensor shows that

$$R_{ijkl} = -(|dr|^2_{\overline{g}})(g_{ik}g_{jl} - g_{il}g_{jk}) + O(r^{-3})\,, \tag{2.1}$$

where here the curvature tensor R and metric g both refer to g_+, and our conventions are such that the above formula without the error term defines a curvature tensor of constant curvature $-(|dr|^2_{\overline{g}})$. It follows that the value of the invariant $|dr|^2_{\overline{g}}$ at a boundary point is the negative of the asymptotic sectional curvature of g_+ there.

We will assume that g_+ satisfies the normalized Einstein condition $\mathrm{Ric}(g_+) = -ng_+$. Contracting in (2.1) shows that in this case we have $|dr|^2_{\overline{g}} = 1$ on M.

In general, a choice of defining function r determines a representative metric $\overline{g}|_{TM} = (r^2 g_+)|_{TM}$ for the conformal structure on M. However, in the other direction, the conformal representative and this relation only determine $r \mod O(r^2)$. In the case when $|dr|^2_{\overline{g}} = 1$ on M, in particular when g_+ is Einstein, one can impose a second condition to determine r uniquely in a neighborhood of M.

Lemma 2.1. *A metric on M in the conformal infinity of g_+ determines a unique defining function r in a neighborhood of M such that $\overline{g}|_{TM}$ is the prescribed boundary metric and such that $|dr|^2_{\overline{g}} = 1$.*

Proof. Given any choice of defining function r_0, let $\overline{g}_0 = r_0^2 g_+$ and set $r = r_0 e^{\omega}$, so $\overline{g} = e^{2\omega}\overline{g}_0$ and $dr = e^{\omega}(dr_0 + r_0 d\omega)$. Thus

$$|dr|^2_{\overline{g}} = |dr_0 + r_0 d\omega|^2_{\overline{g}_0} = |dr_0|^2_{\overline{g}_0} + 2r_0(\nabla_{\overline{g}_0} r_0)(\omega) + r_0^2 |d\omega|^2_{\overline{g}_0}\,,$$

so the condition $|dr|^2_{\overline{g}} = 1$ is equivalent to

$$2(\nabla_{\overline{g}_0} r_0)(\omega) + r_0 |d\omega|^2_{\overline{g}_0} = \frac{1 - |dr_0|^2_{\overline{g}_0}}{r_0}\,. \tag{2.2}$$

This is a non-characteristic first order PDE for ω, so there is a solution near M with $\omega|_M$ arbitrarily prescribed. □

A defining function determines for some $\epsilon > 0$ an identification of $M \times [0, \epsilon)$ with a neighborhood of M in $\overline{X}$: $(p, \lambda) \in M \times [0, \epsilon)$ corresponds to the point obtained by following the integral curve of $\nabla_{\overline{g}} r$ emanating from p for λ units of time. For a defining function of the type given in the lemma, with $|dr|^2_{\overline{g}} = 1$, the λ-coordinate is just r, and $\nabla_{\overline{g}} r$ is orthogonal to the slices $M \times \{\lambda\}$. Hence, identifying λ with r, on $M \times [0, \epsilon)$ the metric $\overline{g}$ takes the form $\overline{g} = g_r + dr^2$ for a 1-parameter family g_r of metrics on M, and

$$g_+ = r^{-2}(g_r + dr^2)\,. \tag{2.3}$$

We explicitly identify a special defining function r and normal form (2.3) for the hyperbolic metric $g_+ = 4(1 - |x|^2)^{-2}\Sigma(dx^i)^2$ on the unit ball in $\mathbb{R}^{n+1}$. Notice that in general the condition $|dr|^2_{\overline{g}} = 1$ can be rewritten as $|d(\log \frac{1}{r})|^2_{g_+} = 1$, which is the eikonal equation for $\log \frac{1}{r}$ in the metric g_+. The distance function $d(x) =$ (hyperbolic distance from x to 0) satisfies the eikonal equation and also $d(x) \to \infty$ as $|x| \to 1$, so we take $\log \frac{1}{r} = d(x)$, i.e. $r = e^{-d(x)}$. Now it is a basic fact of hyperbolic geometry that $d(x) = \log \frac{1+|x|}{1-|x|}$, so $r = \frac{1-|x|}{1+|x|}$ is a special defining function for $\mathbb{H}^{n+1}$ as in Lemma 2.1. Then $\overline{g} = r^2 g_+ = 4(1+|x|)^{-4}\Sigma(dx^i)^2$, so the associated representative for the conformal structure is $g_0 = \frac{1}{4}(\text{usual metric on } \mathbb{S}^n)$. Writing $\Sigma(dx^i)^2$ in polar coordinates and expressing everything in terms of r gives $g_+ = r^{-2}\left((1 - r^2)^2 g_0 + (dr)^2\right)$, and therefore

$$g_r = (1 - r^2)^2 g_0\,. \tag{2.4}$$

We now impose the Einstein condition on a metric of the form (2.3). One can decompose the tensor $\mathrm{Ric}(g_+) + n g_+$ into components with respect to the product structure $M \times (0, \epsilon)$. A straightforward calculation shows that the vanishing of the component with both indices in M is given by

$$r g''_{ij} + (1 - n) g'_{ij} - g^{kl} g'_{kl} g_{ij} - r g^{kl} g'_{ik} g'_{jl} + \frac{r}{2} g^{kl} g'_{kl} g'_{ij} - 2r \mathrm{Ric}_{ij}(g_r) = 0\,, \tag{2.5}$$

where g_{ij} denotes the tensor g_r on M, $'$ denotes ∂_r, and $\mathrm{Ric}_{ij}(g_r)$ denotes the Ricci tensor of g_r with r fixed. As indicated in the introduction, we assume that g_r is sufficiently regular that its asymptotics may be calculated from (2.5) (and the equations for the other components of $\mathrm{Ric}(g_+) + n g_+$). Differentiating (2.5) $\nu - 1$ times with respect to r and setting $r = 0$ gives

$$(\nu - n)\partial_r^\nu g_{ij} - g^{kl}(\partial_r^\nu g_{kl}) g_{ij} = (\text{terms involving } \partial_r^\mu g_{ij} \text{ with } \mu < \nu)\,. \tag{2.6}$$

Beginning with the initial condition that g_r is a given representative metric at $r = 0$, we may use (2.6) inductively to solve for the expansion of g_r. So long as $\nu < n$, $\partial_r^\nu g|_{r=0}$ is uniquely determined at each step, and since the left-hand side of (2.5) respects parity in r, we have $\partial_r^\nu g|_{r=0} = 0$ for ν odd. However this breaks down for $\nu = n$. In that case, if n is odd, it follows from parity considerations that the right-hand side of (2.6) vanishes at $r = 0$, so $g^{kl}\partial_r^n g_{kl} = 0$ but the trace-free part of $\partial_r^n g_{kl}$ may be chosen arbitrarily. If n is even, then the right-hand side of (2.6) might have non-vanishing trace-free part, forcing the inclusion of a $r^n \log r$ term in the expansion for g_r with a trace-free coefficient. The trace of the r^n coefficient is determined but not its trace-free part. It can be shown that the remaining components of $\mathrm{Ric}(g_+)+ng_+$ give no further information to this order.

Summarizing, we see that for n odd, the expansion of g_r is of the form

$$g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + g^{(n-1)}r^{n-1} + g^{(n)}r^n + \dots , \tag{2.7}$$

where the $g^{(j)}$ are tensors on M, and $g^{(n)}$ is trace-free with respect to a metric in the conformal class on M. For j even and $0 \le j \le n-1$, the tensor $g^{(j)}$ is locally formally determined by the conformal representative, but $g^{(n)}$ is formally undetermined, subject to the trace-free condition. For n even the analogous expansion is

$$g_r = g^{(0)} + g^{(2)}r^2 + (\text{even powers}) + hr^n \log r + g^{(n)}r^n + \dots , \tag{2.8}$$

where now the $g^{(j)}$ are locally determined for j even and $0 \le j \le n-2$, h is locally determined and trace-free, the trace of $g^{(n)}$ is locally determined, but the trace-free part of $g^{(n)}$ is formally undetermined.

Of course, the determined coefficients in these expansions may be calculated by carrying out the indicated differentiations above and keeping track of the lower order terms at each stage. For example, for $n = 2$ one finds that $h = 0$ and

$$g^{ij}g^{(2)}_{ij} = -\frac{1}{2}R , \tag{2.9}$$

while for $n \ge 3$ one has $g^{(2)}_{ij} = -P_{ij}$, where

$$(n-2)P_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij} , \tag{2.10}$$

and R_{ij} and R denote the Ricci tensor and scalar curvature of the chosen representative g_{ij} of the conformal infinity.

In order to establish conformal invariance of the renormalized volume invariants, we will later need to use the following Lemma.

Lemma 2.2. *Let r and $\hat{r}$ be special defining functions as in Lemma 2.1 associated to two different conformal representatives. Then*

$$\hat{r} = re^{\omega} \tag{2.11}$$

for a function ω on $M \times [0,\epsilon)$ whose expansion at $r = 0$ consists only of even powers of r up through and including the r^{n+1} term.

Proof. We have $\hat{r} = e^{\omega}r$ where ω is determined by (2.2), which in this case becomes

$$2\omega_r + r(\omega_r^2 + |d_M\omega|^2_{g_r}) = 0 . \tag{2.12}$$

The Taylor expansion of ω is determined inductively by differentiating this equation at $r = 0$. Clearly $\omega_r = 0$ at $r = 0$. Consider the determination of $\partial_r^{k+1}\omega$ resulting from differentiating (2.12) an even number k times and setting $r = 0$. The term ω_r^2 gets differentiated $k-1$ times, so one of the two factors ends up differentiated an odd number of times, so by induction vanishes at $r = 0$. Now $|d_M\omega|^2_{g_r} = g_r^{ij}\omega_i\omega_j$, so the $k-1$ differentiations must be split between the three factors, so one of the factors must receive an odd number of differentiations. When an odd number of derivatives hits a ω_i, the result again vanishes by induction. But by (2.7) and (2.8), so long as $k-1 < n$, the odd derivatives of g_r vanish at $r = 0$. □

3. Volume Renormalization

Let g_+ be a conformally compact Einstein metric on X. As discussed above, a representative metric g on M for the conformal infinity of g_+ determines a special defining function r for M and an identification of a neighborhood of M in $\overline{X}$ with $M \times [0, \epsilon)$. In this identification, g_+ takes the form (2.3), where $g_0 = g$ is the chosen representative metric. Therefore the volume element dv_{g_+} is given by

$$dv_{g_+} = r^{-n-1} \left(\frac{\det g_r}{\det g} \right)^{1/2} dv_g dr . \tag{3.1}$$

From (2.7) and (2.8) and the properties stated there for the coefficients in those expansions, it follows that

$$\left(\frac{\det g_r}{\det g} \right)^{1/2} = 1 + v^{(2)} r^2 + (\text{even powers}) + v^{(n)} r^n + \dots , \tag{3.2}$$

where the $\dots$ indicates terms vanishing to higher order. All indicated $v^{(j)}$ are locally determined functions on M and $v^{(n)} = 0$ if n is odd.

Consider now the asymptotics of $\mathrm{Vol}_{g_+}(\{r > \epsilon\})$ as $\epsilon \to 0$. Pick a small number r_0 and express $\mathrm{Vol}(\{r > \epsilon\}) = C + \int_{\{\epsilon < r < r_0\}} dv_{g_+}$. Integrating (3.1) using (3.2) we obtain for n odd

$$\begin{aligned} \mathrm{Vol}(\{r > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + (\text{odd powers}) + c_{n-1} \epsilon^{-1} \\ + V + o(1) \end{aligned} \tag{3.3}$$

and for n even

$$\begin{aligned} \mathrm{Vol}(\{r > \epsilon\}) = c_0 \epsilon^{-n} + c_2 \epsilon^{-n+2} + (\text{even powers}) + c_{n-2} \epsilon^{-2} \\ + L \log \tfrac{1}{\epsilon} + V + o(1) . \end{aligned} \tag{3.4}$$

The coefficients c_i and L are integrals over M of local curvature expressions of the metric g. For example, $c_0 = \frac{1}{n} \mathrm{Vol}_g(M)$. Also,

$$L = \int_M v^{(n)} \, dv_g . \tag{3.5}$$

The renormalized volume is the constant term V in the expansion for $\mathrm{Vol}(\{r > \epsilon\})$, which a-priori depends on the choice g of representative conformal metric on M.

Theorem 3.1. *If n is odd, then V is independent of the choice of g.*
If n is even, then L is independent of the choice of g.

Proof. The special defining functions r and $\hat{r}$ associated to representative metrics g and $\hat{g}$ are related as in Lemma 2.2. We can solve (2.11) for r to give $r = \hat{r} b(x, \hat{r})$, where the expansion of b also has only even powers of $\hat{r}$ up through the $\hat{r}^{n+1}$ term. It is important to note that in this relation, the x still refers to the identification associated with r.

Set $\hat{\epsilon}(x, \epsilon) = \epsilon b(x, \epsilon)$. Then $\hat{r} > \epsilon$ is equivalent to $r > \hat{\epsilon}(x, \epsilon)$, so

$$\mathrm{Vol}(\{r > \epsilon\}) - \mathrm{Vol}(\{\hat{r} > \epsilon\}) = \int_M \int_\epsilon^{\hat{\epsilon}} dv_{g_+}$$

$$(3.6) \qquad = \int_M \int_\epsilon^{\hat{\epsilon}} \sum_{\substack{0 \le j \le n \\ j \text{ even}}} v^{(j)}(x) r^{-n-1+j} dr dv_g + o(1),$$

where we have used (3.1), (3.2). For n odd this is

$$\sum_{\substack{0 \le j \le n-1 \\ j \text{ even}}} \epsilon^{-n+j} \int_M \frac{v^{(j)}(x)}{-n+j} \left(b(x,\epsilon)^{-n+j} - 1 \right) dv_g + o(1).$$

Since $b(x,\epsilon)$ is even through terms of order $n+1$ in ϵ, it follows that this expression has no constant term as $\epsilon \to 0$. Similarly, when n is even, the r^{-1} term in (3.6) contributes $\log b(x,\epsilon)$, so there is no $\log \frac{1}{\epsilon}$ term as $\epsilon \to 0$. □

According to Theorem 3.1, for n odd the renormalized volume V is an absolute invariant of the conformally compact Einstein metric g_+. But this is not so if n is even. Suppose g and $\hat{g} = e^{2\Upsilon} g$ are two metrics in the conformal infinity of g_+, where $\Upsilon \in C^\infty(M)$. The difference $V_g - V_{\hat{g}}$ is the constant term in the expansion of (3.6). By the local determination of the $v^{(j)}$ and of the expansion of $b(x,\epsilon)$, we see that this anomaly takes the form

$$V_{\hat{g}} - V_g = \int_M \mathcal{P}_g(\Upsilon) dv_g,$$

where $\mathcal{P}_g$ is a polynomial nonlinear differential operator whose coefficients are polynomial expressions in g, its inverse, and its derivatives. Moreover, it is easy to see that the linear part in Υ of $\mathcal{P}_g(\Upsilon)$ is just $v^{(n)}\Upsilon$. Since this linear part measures the infinitesimal change under conformal rescalings, $V_{\hat{g}} - V_g$ is determined by knowledge of $v^{(n)}$ for general g. In summary, for n even, the fundamental object is the function $v^{(n)}$—its integral over M is by (3.5) the conformal invariant L, and multiplication by it gives the infinitesimal anomaly, which determines the full anomaly.

It is straightforward to carry out the calculations indicated above to identify $v^{(n)}$ and $\mathcal{P}_g$ in low dimensions. For $n = 2$ one obtains

$$v^{(2)} = -\frac{1}{4} R, \qquad \mathcal{P}_g(\Upsilon) = -\frac{1}{4}(R\Upsilon + \Upsilon_i \Upsilon^i),$$

so $L = -\pi\chi(M)$, where $\chi(M)$ denotes the Euler characteristic of M.

For $n = 4$ one obtains

$$v^{(4)} = \frac{1}{8}[(P_i{}^i)^2 - P_{ij}P^{ij}],$$

$$\mathcal{P}_g(\Upsilon) = v^{(4)}\Upsilon + \Upsilon_{ij}\Upsilon^i\Upsilon^j - P_{ij}\Upsilon^i\Upsilon^j - \frac{1}{4}(\Upsilon_i\Upsilon^i)^2 + P_j{}^j\Upsilon_i\Upsilon^i.$$

The Gauss-Bonnet Theorem for $n = 4$ reads

$$32\pi^2\chi(M) = \int_M [|W|^2 - 8P_{ij}P^{ij} + 8(P_i{}^i)^2] dv_g,$$

where

$$W_{ijkl} = R_{ijkl} - (P_{ik}g_{jl} + P_{jl}g_{ik} - P_{il}g_{jk} - P_{jk}g_{il})$$

is the Weyl conformal curvature tensor. Therefore

$$L = \frac{\pi^2}{2}\chi(M) - \frac{1}{64}\int_M |W|^2 dv_g.$$

For $n = 6$ one obtains

$$v^{(6)} = \frac{1}{48}[-P^{ij}B_{ij} + 3P_i{}^i P_{kl}P^{kl} - 2P_{ij}P_k{}^i P^{jk} - (P_i{}^i)^3],$$

where

$$B_{ij} = P_{ij,k}{}^k - P_{ik,j}{}^k - P^{kl}W_{kijl}.$$

Again there is an explicit realization of $L = \int_M v^{(6)} dv_g$ as a linear combination of the Euler characteristic and the integral of a local conformal invariant. Define

$$C_{ijk} = P_{ij,k} - P_{ik,j}$$

and set

$$V_{ijklm} = W_{ijkl,m} + g_{im}C_{jkl} - g_{jm}C_{ikl} + g_{km}C_{lij} - g_{lm}C_{kij}$$

and

$$U_{ijkl} = C_{jkl,i} - P_i{}^m W_{mjkl}.$$

Then

$$I = |V|^2 - 16W_{ijkl}U^{ijkl} + 16|C|^2$$

is a conformal invariant in general dimension $n \geq 3$; it is the norm-squared of the first covariant derivative of the curvature tensor of the ambient metric of [7]. One can calculate that for $n = 6$,

$$L = -\frac{\pi^3}{6}\chi(M) + \frac{1}{2304}\int_M J dv_g,$$

where

$$J = -3I + 7W_{ijkl}W^{ij}{}_{pq}W^{klpq} + 4W_{ijkl}W^{ipkq}W^j{}_p{}^l{}_q.$$

For $\mathbb{H}^{n+1}$, using (2.4) it is possible to calculate the invariants V for n odd and L for n even. From (2.4) one obtains

$$\left(\frac{\det g_r}{\det g_0}\right)^{1/2} = (1 - r^2)^n,$$

so recalling that $4g_0$ is the usual metric on $\mathbb{S}^n$, it follows from (3.1) that

$$\text{Vol}(\{r > \epsilon\}) = 2^{-n}\text{Area}(\mathbb{S}^n)\int_\epsilon^1 r^{-n-1}(1 - r^2)^n dr. \tag{3.7}$$

For n odd, write

$$\int_\epsilon^1 r^{-n-1}(1 - r^2)^n dr = -\frac{1}{n}\int_\epsilon^1 (1 - r^2)^n d(r^{-n})$$

$$= \frac{1}{n}\epsilon^{-n}(1 - \epsilon^2)^n - 2\int_\epsilon^1 r^{-n+1}(1 - r^2)^{n-1} dr.$$

The boundary term has no constant term in ϵ, so upon applying the same procedure inductively it follows that $\int_\epsilon^1 r^{-n-1}(1 - r^2)^n dr$ has constant term

$$\frac{(-2)^{\frac{n+1}{2}} n(n-1)\dots(\frac{n+1}{2})}{n(n-2)\dots 1}\int_0^1 (1 - r^2)^{\frac{n-1}{2}} dr.$$

Collecting the constants, one finds

$$V = (-1)^{\frac{n+1}{2}} \frac{\pi^{\frac{n+2}{2}}}{\Gamma(\frac{n+2}{2})}.$$

For $n = 2m$ even, expand $(1 - r^2)^n$ using the binomial theorem; it follows that the $\log \frac{1}{\epsilon}$ coefficient in the expansion of $\int_\epsilon^1 r^{-n-1}(1-r^2)^n dr$ is $(-1)^m \binom{n}{m}$. Substituting into (3.7) and simplifying gives

$$L = (-1)^m \frac{2\pi^m}{m!}.$$

A more familiar setting for conformal anomalies is in the study of functional determinants of conformally invariant differential operators. The invariance properties of V are reminiscent of those for the functional determinant of the conformal Laplacian, which is conformally invariant in odd dimensions but which has an anomaly in even dimensions ([17]). We remark that the AdS/CFT correspondence predicts that the volume anomaly for $n = 4$ is a particular linear combination of functional determinant anomalies on scalars, spinors, and 1-forms; this prediction was confirmed in [12]. The properties of the invariant L are, on the other hand, similar to those for the constant term in the expansion of the integrated heat kernel for the conformal Laplacian, which vanishes in odd dimensions but in even dimensions is a conformal invariant obtained by integrating a local expression in curvature ([4], [17]).

4. Area Renormalization

Let (X^{n+1}, g_+) be a conformally compact Einstein manifold with conformal infinity $(M, [g])$ as above. In this section we describe the renormalization of the area of minimal submanifolds $Y \subset X$ of dimension $k+1$, $0 \le k \le n-1$, which extend regularly to $\overline{X}$. Set $N = \overline{Y} \cap M$. We assume that N is a smooth submanifold of M. We will outline the arguments and refer to [9] for details.

First one must study the asymptotics of Y near M. Locally near a point of N, coordinates $(x^\alpha, u^{\alpha'})$ for M may be chosen, where $1 \le \alpha \le k$ and $1 \le \alpha' \le n-k$, so that $N = \{u = 0\}$ and so that $\partial_{x^\alpha} \perp \partial_{u^{\alpha'}}$ on N with respect to a metric in the conformal infinity of g_+. Choose a representative metric g for the conformal infinity and recall that this choice determines by Lemma 2.1 a defining function r for M and an identification of a neighborhood of M in $\overline{X}$ with $M \times [0, \epsilon)$. This identification determines an extension of the x^α and $u^{\alpha'}$ into X, and together with r these form a local coordinate system on $\overline{X}$. We consider submanifolds Y which in such coordinates may be written as a graph $\{u = u(x,r)\}$. One can calculate the minimal surface equation for Y explicitly as a system of differential equations for the unknowns $u^{\alpha'}(x,r)$. These equations together with the boundary condition $u(x, 0) = 0$ are used to study the asymptotics of $u(x,r)$ at $r = 0$. One finds (see [9]) that for k odd

$$u = u^{(2)}r^2 + (\text{even powers}) + u^{(k+1)}r^{k+1} + u^{(k+2)}r^{k+2} + \dots, \tag{4.1}$$

and for k even

$$u = u^{(2)}r^2 + (\text{even powers}) + u^{(k)}r^k + wr^{k+2}\log r + u^{(k+2)}r^{k+2} + \dots, \tag{4.2}$$

where the $u^{(j)}$ and w are functions of x, all of which are locally determined except for $u^{(k+2)}$, and the ... indicates terms vanishing to higher order. Observe in particular that the minimal submanifold Y is determined to order $k+2$ by $N = \partial Y$, that the expansion of u is even in r to order $k+2$, and that the irregularity in the expansion occurs at order $k+2$. The consequence $\partial_r u = 0$ at $r = 0$ has the geometric interpretation that Y intersects M orthogonally, a fact very familiar from the geometry of geodesics in hyperbolic space. For the case $k = 0$ of geodesics it turns out that necessarily $w = 0$, and the local indeterminancy in this case of $u^{(2)}$ is a reflection of the familiar fact that at the boundary a geodesic may have any asymptotic curvature measured with respect to the smooth metric $\overline{g}$.

Next one calculates the metric induced on Y by the conformally compact Einstein metric g_+. The area form da_Y of Y takes the form

$$da_Y = r^{-k-1}\left[1 + a^{(2)}r^2 + (\text{even powers}) + a^{(k)}r^k + \dots\right] da_N dr\,, \tag{4.3}$$

where the ... indicates terms vanishing to higher order and da_N denotes the area form on N with respect to the chosen conformal representative g on the boundary. All indicated $a^{(j)}$ are locally determined functions on N and $a^{(k)} = 0$ if k is odd. A key observation in establishing (4.3) is that since the induced metric depends only on u and its first coordinate derivatives, the local indeterminacy and irregularities at order $k+2$ in u and those at order n in the metric g_r given by (2.7), (2.8) do not enter into the asymptotics of the area form to the indicated order. The evenness of $r^{k+1}da_Y$ then follows from that of g_r and of u.

Now we can consider the asymptotics of $\text{Area}_{g_+}(Y \cap \{r > \epsilon\})$ as $\epsilon \to 0$. Pick a small number r_0 and express $\text{Area}(Y \cap \{r > \epsilon\}) = C + \int_{Y\cap\{\epsilon<r<r_0\}} da_Y$. By (4.3) we obtain for k odd

$$\begin{aligned}\text{Area}(Y \cap \{r > \epsilon\}) = b_0\epsilon^{-k} + b_2\epsilon^{-k+2} + (\text{even powers}) + b_{k-1}\epsilon^{-1}\\ + A + o(1)\end{aligned}$$

and for k even

$$\begin{aligned}\text{Area}(Y \cap \{r > \epsilon\}) = b_0\epsilon^{-k} + b_2\epsilon^{-k+2} + (\text{even powers}) + b_{k-2}\epsilon^{-2}\\ + K\log\tfrac{1}{\epsilon} + A + o(1)\,.\end{aligned} \tag{4.4}$$

Observe that

$$K = \int_N a^{(k)}\, da_N\,. \tag{4.5}$$

The analogue of Theorem 3.1 is the following, which is proved by a similar argument.

Theorem 4.1. *If k is odd, then A is independent of the choice of g. If k is even, then K is independent of the choice of g.*

Therefore, for k odd, a minimal submanifold of X has a well-defined invariant renormalized area A. For k even, the $\log\frac{1}{\epsilon}$ coefficient K is a conformal invariant of the submanifold N of M given according to (4.5) by the integral of an expression determined locally by the geometry of $N \subset M$ with respect to the metric g.

Analogously to the volume case, there is a conformal anomaly for A when k is even. If $\hat{g} = e^{2\Upsilon}g$ is a conformally related metric, then the local determination of the coefficients $a^{(j)}$ in (4.3) and of the defining function $\hat{r}$ as in Lemma 2.1 implies that

$$A_{\hat{g}} - A_g = \int_N \mathcal{Q}_N(\Upsilon) da_N$$

for a differential expression $\mathcal{Q}_N$ determined locally by the geometry of $N \subset M$. One interesting difference from the volume anomaly is that the linearization of $\mathcal{Q}_N(\Upsilon)$ need not be just $a^{(k)}\Upsilon$–it can in general involve derivatives of Υ as well. However it is clear from rescaling in (4.4) that $\mathcal{Q}_N(\Upsilon) = a^{(k)}\Upsilon$ for Υ constant.

The invariant K and the anomaly for the lowest dimensional cases $k = 0, 2$ are calculated in [9]. For $k = 0$, Y is a union of geodesics in X and N consists of finitely many points. Of course a point has no geometry and the conclusions are rather trivial; one finds that K is the number of boundary points, $\mathcal{Q}$ evaluates Υ at a boundary point, and the anomaly is given by $A_{\hat{g}} - A_g = \sum_{p\in N} \Upsilon(p)$. To describe the $k = 2$ results recall that the second fundamental form of $N \subset M$ with respect to the metric g is the symmetric form $B^{\gamma'}_{\alpha\beta}$ on TN with values in $TN^{\perp}$ defined by $B(X,Y) = (\nabla_X Y)^{\perp}$ for vectors $X, Y \in TN$; here ∇ denotes the Levi-Civita covariant derivative of g_{ij} and $\perp$ the component in $TN^{\perp}$. On N, the metric g_{ij} decomposes into two pieces $g_{\alpha\beta}$ and $g_{\alpha'\beta'}$. The mean curvature vector of N is $H^{\gamma'} = g^{\alpha\beta}B^{\gamma'}_{\alpha\beta}$. The tensor P given by (2.10) also decomposes into pieces with respect to the decomposition $TM = TN \oplus (TN)^{\perp}$; we denote by $P_{\alpha\beta}$ its component with both indices in TN (*not* the corresponding tensor for the induced metric $g_{\alpha\beta}$). Then for $k = 2$ one finds

$$K = -\frac{1}{8}\int_N (|H|^2 + 4g^{\alpha\beta}P_{\alpha\beta}) da_N \tag{4.6}$$

and

$$\mathcal{Q}_N(\Upsilon) = -\frac{1}{8}(|H|^2 + 4g^{\alpha\beta}P_{\alpha\beta})\Upsilon + \frac{1}{4}(H^{\gamma'}\Upsilon_{\gamma'} - \Upsilon_i\Upsilon^i).$$

The quantity defined by (4.6) is therefore a conformal invariant of a surface N in a conformal manifold M. For conformally flat space this reduces to a multiple of the Willmore functional (for which, see, e.g., [5]). Other generalizations of the Willmore functional to curved conformal spaces are given in [6] and [21].

A different conformal anomaly associated to a surface in a conformal 6-manifold is discussed in [13].

References

[1] M. Anderson, *Complete minimal varieties in hyperbolic space*, Invent. Math. **69** (1982), 477–494.

[2] M. Anderson, *Complete minimal hypersurfaces in hyperbolic n-manifolds*, Comment. Math. Helv. **58** (1983), 264–290.

[3] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, *Operator product expansion for Wilson loops and surfaces in the large N limit*, Phys. Rev. D **59** (1999), 105023, hep-th/9809188.

[4] T. Branson and B. Ørsted, *Conformal indices of Riemannian manifolds*, Comp. Math. **60** (1986), 261–293.

[5] R. Bryant, *Surfaces in conformal geometry*, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math. **48**, Amer. Math. Soc. (1988), 227–240.

[6] B.-Y. Chen, *Some conformal invariants of submanifolds and their applications*, Bol. U.M.I. **10** (1974), 380–385.

[7] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Elie Cartan et les Mathematiques d'Aujourd'hui*, Asterisque (1985), 95–116.

[8] C.R. Graham and J. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.

[9] C.R. Graham and E. Witten, *Conformal anomaly of submanifold observables in AdS/CFT correspondence*, Nucl. Phys. B **546** (1999), 52–64, hep-th/9901021.

[10] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B **428** (1998), 105–114, hep-th/9802109.

[11] R. Hardt and F.-H. Lin, *Regularity at infinity for area-minimizing hypersurfaces in hyperbolic space*, Invent. Math. **88** (1987), 217–224.

[12] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, J. High Ener. Phys. **07** (1998), 023, hep-th/9806087; *Holography and the Weyl anomaly*, hep-th/9812032.

[13] M. Henningson and K. Skenderis, *Weyl anomaly for Wilson surfaces*, hep-th/ 9905163.

[14] F.-H. Lin, *On the Dirichlet problem for minimal graphs in hyperbolic space*, Invent. Math. **96** (1989), 593–612.

[15] F.-H. Lin, *Asymptotic behavior of area-minimizing currents in hyperbolic space*, Comm. Pure Appl. Math. **42** (1989), 229–242.

[16] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231–252, hep-th/9711200.

[17] T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, J. Diff. Geom. **25** (1987), 199–222.

[18] A.M. Polyakov, *Fine structure of strings*, Nucl. Phys. B **268** (1986), 406–412.

[19] D. Skinner, PhD. thesis, in preparation.

[20] Y. Tonegawa, *Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space*, Math. Z. **221** (1996), 591–615.

[21] J. Weiner, *On a problem of Chen, Willmore, et.al.*, Ind. Univ. Math. J. **27** (1978), 19–35.

[22] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998), 253–290, hep-th/9802150.

Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195-4350, U.S.A.
E-mail: robin@math.washington.edu