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CYCLIC OPERADS AND HOMOLOGY OF GRAPH COMPLEXES

MARTIN MARKL*

Abstract

We will consider $\mathcal{P}$-graph complexes, where $\mathcal{P}$ is a cyclic operad. $\mathcal{P}$-graph complexes are natural generalizations of Kontsevich's graph complexes – for $\mathcal{P} = \mathcal{A}ss$ it is the complex of ribbon graphs, for $\mathcal{P} = \mathcal{C}omm$ the complex of all graphs. We construct a 'universal class' in the cohomology of the graph complex with coefficients in a theory. The Kontsevich-type invariant is then an evaluation, on a concrete cyclic algebra, of this class.

We also explain some results of M. Penkava and A. Schwarz on the construction of an invariant from a cyclic deformation of a cyclic algebra. Our constructions are illustrated by a 'toy model' of tree complexes.

1. Introduction.

In [4], M. Kontsevich constructed, for any cyclic A_∞-algebra, an element in the cohomology $H^*(\mathcal{M}_{g,n}; \mathbf{C})$ of the coarse moduli space of smooth algebraic curves of genus g with n unlabeled punctures. His construction is based on a certain combinatorial representation of $\mathcal{M}_{g,n}$ – the graph complex – and involves an A_∞-algebra. It resembles the state-sum model for the Jones polynomial of a link (in fact, it *is* a state sum). The aim of this note is to give a conceptual understanding of the existence of these classes.

Because of the resemblance mentioned above, it would be helpful to summarize the progress in the understanding the quantum-group-type invariants of links.
1st step. The simplest state-sum model based on the canonical R-matrix of the quantum sl_2.

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2nd step. A state-sum model related to the quantization of a general (semi-simple) Lie algebra.
3rd step. For a k-linear rigid braided monoidal category $\mathcal{C}$, each tangle T (= 'open' link) can be interpreted as a morphism in $\mathcal{C}$, i.e. as an element of $\mathcal{C}(V^{\odot m}, V^{\odot n})$, where m (resp. n) denotes the number of input (output) strings of T. A link is a closed tangle ($m = n = 0$) and we get an element of $\mathcal{C}(1, 1)$ (1 is the unit element of $\mathcal{C}$) which is a *number*, because $\mathcal{C}(1, 1) = \mathbf{k}$.

The construction of M. Kontsevich mentioned above would correspond to the 1st step of the above imaginary list. To accomplish the remaining two steps we need first a generalization of graph complexes to more general types of algebras. This generalization was described, for Lie and commutative associative algebras, by Kontsevich himself [5], but 'graph complexes' can be defined for algebras over an arbitrary *cyclic* (the cyclicity is absolutely essential) operad. Such a generalization was, in fact, given in [2] – every cyclic operad can be naturally considered as a modular operad, and the appropriate 'graph complex' is the Feynman transform of this operad introduced in the above mentioned paper.

We will then show that there exists a 'universal cohomology class' (Proposition 4.2) such that the invariant related to a concrete algebra with an invariant scalar product is a specialization (or evaluation) of this universal class (§4.3, §4.4). The universality means that the class 'contains' all special invariants. The construction of this class was made possible by a very explicit understanding of the structure of k-linear PROPs or 'theories' achieved in [7, 6]. This class is not only 'universal' but also the 'simplest possible' in the sense that it uses only 'generic' properties of objects.

As the first approximation of the understanding we offer the following comment. We are going to construct a complex and *simultaneously* a class in the homology of this complex. The following analogy is helpful. For any vector space V, the tensor product $V^* \otimes V$ of the dual $V^* = \mathrm{Hom}(V, \mathbf{k})$ and V contains the 'canonical element' $\eta \in V^* \otimes V$. If we pick a basis $(e_i)_{i\in I}$ of V, we may give a 'state-sum-type' definition of η as $\eta := \sum_{i\in I} e_i^* \otimes e_i$, where $(e_i^*)_{i\in I}$ is the dual basis. A 'categorical' definition says just that η corresponds, by duality, to the identity map $\mathbb{1} : V \to V$.

Observe that neither of the two 'parts' ('V^*'-part and 'V'-part) of η can exists independently. In our analogy, the 'V'-part is a coloring of a graph, while the 'V^*'-part is the coefficient of the cycle representing our canonical class, the coloring being given by an element of our cyclic operad, and the coefficient by an element of a *theory* which is, in an appropriate sense, dual to the operad. This explains why we will construct *simultaneously* both the coloring and the coefficient.

As the second step we offer our toy model – the tree complex (Section 2), which is very easy to understand. The general case is basically the same, only more technical, as we take into the account the symmetric group action and the cyclic structure.

As an application of our approach, we will try in Section 5 to elucidate the construction of a cycle out of an infinitesimal deformation of a cyclic algebra, as in [9, 8].

Let us finish the introduction with the following speculations. The 'graph complex' for a (normal, non-cyclic) operad is the tree complex (= disguised bar construction), the 'graph complex' for a cyclic operad is Kontsevich's graph complex (a special case

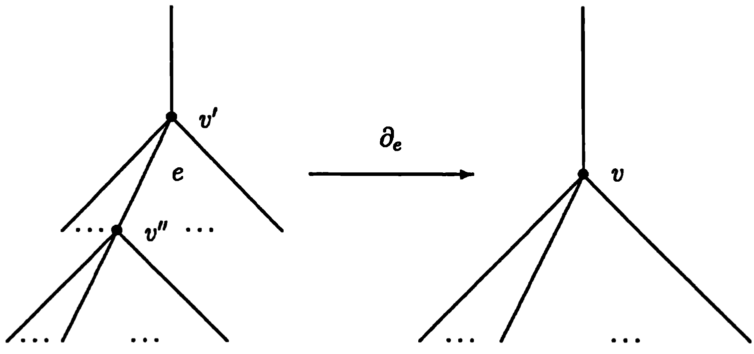

Figure 1: If the vertex v' is colored by $c' \in \mathcal{P}(k)$ and the vertex v'' is colored by $c'' \in \mathcal{P}(l)$, then the resulting vertex v is colored by $c := c'(1, \ldots, c'', \ldots, 1) \in \mathcal{P}(k + l - 1)$.

of the Feynman transform), while the 'graph complex' of a modular operad is the Feynman transform. We believe that the construction of the classes can be somehow carried over also to the last case. We will see that, for a Koszul (non-cyclic) operad, the class in the tree complex can be very explicitly described. What is the homological property (if there is such a property) of cyclic and/or modular operads which would made such a calculation possible also in the two remaining cases?

We assume that all algebraic objects are defined over a fixed field $\mathbf{k}$. For a vector space V, let V^* denote its dual, $V^* := \mathrm{Hom}(V, \mathbf{k})$. For a natural number k, $\mathbf{S}_k$ (resp. $\mathbf{C}_k$) denotes the symmetric group on k elements (resp. the cyclic group of order k). All calculations are made only up to signs and degrees.

2. Warming up.

In this section we describe the toy model of our construction – the tree complex of an operad, which is, in fact, a bar construction in a disguise. The construction is easy enough not to frustrate the reader by unnecessary details, but it illustrates well all the basic tricks – the formulation for a general operad (§2.1), the construction of an universal cycle (§2.2) and the evaluation at a concrete algebra (§2.3). Moreover, in some lucky cases the homology class of the universal cycle can be explicitly described (§2.4).

2.1. *Autumn (colored) trees.* Let $\mathcal{P}$ be an operad and T a rooted tree. We say that T is $\mathcal{P}$-colored, if each vertex v of T with k input edges is 'colored' by an element of $\mathcal{P}(k)$. Let $\mathcal{T}_i^{\mathcal{P}}(n)$ be the set of all $\mathcal{P}$-colored trees with n input edges and i inner edges.

Let $\mathsf{T} \in \mathcal{T}_i^{\mathcal{P}}(n)$ and let e be an inner edge of T, joining vertices v' and v''. Define $\partial_e(\mathsf{T}) \in \mathcal{T}_{i-1}^{\mathcal{P}}(n)$ to be the colored tree which is, as a tree, obtained by the collapsing of the edge e, while the coloring of the resulting new vertex is the obvious composition of the corresponding colorings at v' and v'', as indicated on Figure 1. The differential $\partial : \mathcal{T}_i^{\mathcal{P}}(n) \to \mathcal{T}_{i-1}^{\mathcal{P}}(n)$ is defined by $\partial(\mathsf{T}) := \sum \pm\partial_e(\mathsf{T})$, where the summation is taken over all inner edges of the tree T. The condition $\partial^2 = 0$ is an easy consequence of the axioms of an operad. The complex

$$(1) \qquad \mathcal{T}^{\mathcal{P}}_0(n) \xleftarrow{\partial} \mathcal{T}^{\mathcal{P}}_1(n) \xleftarrow{\partial} \cdots \xleftarrow{\partial} \mathcal{T}^{\mathcal{P}}_{n-3}(n) \xleftarrow{\partial} \mathcal{T}^{\mathcal{P}}_{n-2}(n)$$

is basically the bar construction $(B(\mathcal{P}), d_B)$ on the operad $\mathcal{P}$ (see [3]), with the opposite grading.

2.2. *The universal cycle.* Let $\Omega(\mathcal{P}^*) = (\mathcal{F}(\mathcal{P}^*), \partial_\Omega)$ be the cobar construction on the dual cooperad $\mathcal{P}^*$ and let $\mathcal{Q}$ be the operad for $\Omega(\mathcal{P}^*)$-algebras with trivial differential (Appendix 6.1). For any $n \geq 1$ and $0 \leq i \leq n-2$ there exists a universal cycle $\xi_i(n) \in C_i(\mathcal{T}^{\mathcal{P}}_*(n); \mathcal{Q}(n)) := \mathcal{T}^{\mathcal{P}}_i(n) \otimes \mathcal{Q}(n)$. It is constructed as follows.

To any vertex v of T with k input edges attach the 'canonical element' $\eta \in \mathcal{P}^*(k) \otimes \mathcal{P}(k)$ corresponding, by duality, to the identity map $\mathcal{P}(k) \to \mathcal{P}(k)$. Now decorate the vertices of T with these canonical elements and interpret the '$\mathcal{P}$'-part as a coloring of the vertex, and the '$\mathcal{P}^*$'-part as an element of $\mathcal{Q}$ under the canonical monomorphism $\mathcal{P}^* \hookrightarrow \mathcal{Q}$ (Appendix 6.1). Composing the '$\mathcal{P}^*$'-parts as indicated by the tree, we get an element of $\mathcal{Q}(n)$, where n is the number of input edges of T. The summation over all such trees gives $\xi_i(n)$. It follows easily that is satisfies $\partial(\xi_i(n)) = 0$, thus the construction gives rise to an element

$$[(\xi_i(n)] \in H_i(\mathcal{T}^{\mathcal{P}}_*(n); \mathcal{Q}(n)).$$

2.3. *The evaluation.* If $A : \mathcal{Q} \to \mathrm{End}(V)$ is a $\mathcal{Q}$-algebra, we may evaluate at A to obtain a cycle $A(\xi_i(n)) \in C_i(\mathcal{T}^{\mathcal{P}}_*(n); \mathrm{Hom}(V^{\otimes n}, V))$ which in turn gives an element $A[(\xi_i(n)] \in H_i(\mathcal{T}^{\mathcal{P}}_*(n); \mathrm{Hom}(V^{\otimes n}, V))$.

2.4. *The Koszul case.* The reader familiar with the theory of Koszul operads may find interesting the following explicit description of the element $[(\xi_i(n)]$. If the operad $\mathcal{P}$ is Koszul [3], then, by the very definition of the Koszulness and the universal coefficient formula,

$$H_i(\mathcal{T}^{\mathcal{P}}_*(n); \mathcal{Q}(n)) = \begin{cases} \mathcal{P}^{!*}(n) \otimes \mathcal{Q}(n), & \text{for } i = n-2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}^!$ is the Koszul dual of the operad $\mathcal{P}$. There exists a natural inclusion $\mathcal{P}^!(n) \hookrightarrow \mathcal{Q}(n)$ and $[(\xi_i(n)]$ is the image of the canonical element of $\mathcal{P}^{!*}(n) \otimes \mathcal{P}^!(n)$ under the induced inclusion $\mathcal{P}^{!*}(n) \otimes \mathcal{P}^!(n) \hookrightarrow \mathcal{P}^{!*}(n) \otimes \mathcal{Q}(n)$.

3. Graph complexes.

As we have already observed, graph complexes are special cases of the Feynman transform F introduced by E. Getzler and M. Kapranov in [2], so we could just say that

$$(2) \qquad \mathcal{G}^{\mathcal{P}}(n) := \bigoplus_{g \geq 0} \mathsf{F}\mathcal{P}(g, n),$$

where $\mathcal{G}^{\mathcal{P}}(n)$ is the graph complex we are going to use, the natural number n denotes the number of external edges and g refers to the 'genus'. We would like, however, to consider separately also the 'nonsymmetric' variant of the construction. Again, because a nonsymmetric operad can be considered (after tensoring with the regular representation of the symmetric group) as a symmetric operad, definition (2) would apply to this case as well, but this approach would obscure the *ribbon* structure of the underlying graph. We also need a notation, that is why we decided to include an explicit definition here.

3.1. *Symmetric vs. nonsymmetric.* We distinguish two cases – the nonsymmetric case and the symmetric case. In the nonsymmetric case we work with ribbon graphs and nonsymmetric cyclic operads (§3.3), while in the symmetric case we work with the ordinary cyclic operads (in the sense of [1]) and ordinary graphs.

The conceptual explanation of this dichotomy is the following. The vertices of our graphs are colored by elements of an operad. A 'color' of a vertex v must behave well under the group of local symmetries of the graph at v. How does this group look? For a general graph, it is the group $\mathbf{S}_{k+1}$ permuting the (half)edges at v, $k+1$ being the number of these edges. This means that the 'color' at v must admit a $\mathbf{S}_{k+1}$-symmetry, and we necessary arrive at the notion of a cyclic operad. In a ribbon graph, the set of (half)edges at v has a preferred cyclic order. The group $\mathbf{S}_{k+1}$ is the semidirect product of $\mathbf{S}_k$ and the cyclic group $\mathbf{C}_{k+1}$, and the cyclic order of the edges fixes the $\mathbf{S}_k$-part, thus t he 'color' at v must admit a $\mathbf{C}_{k+1}$-symmetry. The corresponding notion is that of a nonsymmetric cyclic operad, see §§3.3, 3.4 and 3.5 for details.

3.2. *Graphs.* As usual, a graph consists of edges and vertices. We suppose that all vertices are at least trivalent. Let vert(G) denote the set of vertices of the graph G and let edg(G) be the set of edges of G. For a vertex $v \in$ vert(G) let edg(v) denote the set of half edges (there may be loops in the graph!) at the vertex v. A ribbon graph is a graph such that a cyclic order on the set edg(v) is given, for any $v \in$ vert(G). We allow our graph G to have also some external edges; we denote the set of all these external edges by leg(G). Let $\mathcal{G}_i(n)$ denote the set of graphs with n external edges and i internal edges attached to two distinct vertices.

3.3. *Cyclic operads.* Following [1], a cyclic operad is an operad $\mathcal{P}$ such that the usual action of the symmetric group $\mathbf{S}_n$ on $\mathcal{P}(n)$ is extended to an action of the symmetric group $\mathbf{S}_{n+1}$. This extension has, of course, to satisfy appropriate axioms. There exists an (almost) obvious nonsymmetric version where each $\mathcal{P}(n)$ has an action of the cyclic group $\mathbf{C}_{n+1}$. We call these objects nonsymmetric cyclic operads.

3.4. *$\mathcal{P}$-colored graphs.* If I is a cyclically ordered set of $n+1$ elements, then the cyclic group $\mathbf{C}_{n+1}$ acts on the set of all cyclic-order preserving maps $f : \{0,1,\ldots,n\} \to I$. If V is a $\mathbf{C}_{n+1}$ space we put, as in [3],

$$V((I)) := \left(\bigoplus_{f:\{0,1,\ldots,n\}\to I} V \right)_{\mathbf{C}_{n+1}} \quad \text{(the set of coinvariants).}$$

By a $\mathcal{P}$-colored graph we mean a graph G such that each vertex v is 'colored' by an element of $\mathcal{P}((\text{edg}(v))$, where the cyclic order on edg(v) i s given by the ribbon

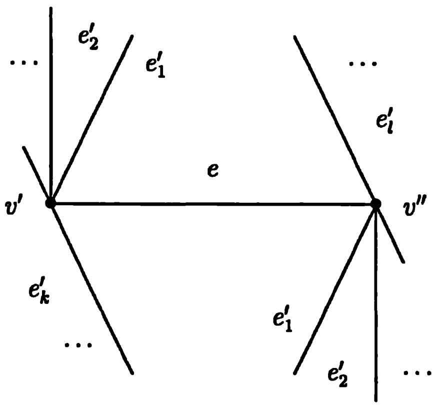

Figure 2: An edge e joining the vertices v' and v''.

structure of the graph. The symmetric variant of this definition is obvious. Denote by $\mathcal{G}_i^{\mathcal{P}}(n)$ the set of $\mathcal{P}$-colored graphs with n external and i internal edges attached to two distinct vertices.

3.5. *Contracting an edge.* Let us discuss the nonsymmetric case first. Let $\mathsf{G} \in \mathcal{G}_i^{\mathcal{P}}(n)$ be a $\mathcal{P}$-colored graph and let $e \in \mathrm{edg}(\mathsf{G})$ be an edge attached to two distinct vertices. Define the $\mathcal{P}$-colored graph $\partial_e(\mathsf{G}) \in \mathcal{G}_{i-1}^{\mathcal{P}}(n)$ as follows. As a graph it coincides with the graph G/e obtained by collapsing out the edge e from G, with the induced cyclic order on the resulting vertex v.

Before going further, we need some notation. Let e join (distinct) vertices v' and v'' and let $\mathrm{edg}(v') = \{e, e'_1, e'_2, \dots, e'_k\}$, $\mathrm{edg}(v'') = \{e, e''_1, e''_2, \dots, e''_l\}$ (in this cyclic order). This means that $\mathrm{edg}(v) = \{e''_l, e'_1, \dots, e'_k, e''_1, \dots, e''_{l-1}\}$ (in this cyclic order), see Figure 2. Let $c' \in \mathcal{P}((\mathrm{edg}(v')))$ and $c'' \in \mathcal{P}((\mathrm{edg}(v'')))$ be the colorings of the vertices v' and v'', respectively.

Define $f' : \{0, 1, \dots, k\} \to \mathrm{edg}(v')$ by $f'(0) = e$, and $f'(i) = e'_i$ for $1 \leq l \leq k$, and let $r' \in \mathcal{P}(k+1)_{f'}$ be a representative for c'. Similarly, let $f'' : \{0, 1, \dots, l\} \to \mathrm{edg}(v'')$ be given by $f''(0) = e''_l$, $f''(1) = e$ and $f''(j) = e''_{j-1}$ for $2 \leq j \leq l$, and let $r'' \in \mathcal{P}(l+1)_{f''}$ be a representative for c''. We then define the coloring c of v to be the equivalence class of $r''(r', 1, \dots, 1)$ in $\mathcal{P}(k+l-1)_f$, where $f(0) = e''_l$, $f(i) = e'_i$, for $1 \leq i \leq k$, and $f(k+j) = e''_j$, for $1 \leq j \leq l-1$. It follows from the cyclicity of the operad $\mathcal{P}$ that the coloring c does not depend on the particular choice of the representatives c' and c''. The symmetric case is even easier, because we do not need not to pay any attention to the cyclic order of the edges.

3.6. *Graph complex.* The differential $\partial : \mathcal{G}_i^{\mathcal{P}}(n) \to \mathcal{G}_{i-1}^{\mathcal{P}}(n)$ is defined by $\partial(\mathsf{G}) = \sum \pm\partial_e(\mathsf{G})$, the sum being taken over all edges attached to two distant vertices. The condition $\partial^2 = 0$ follows from the fact that $\mathcal{P}$ is an operad. We call $\mathcal{G}_*^{\mathcal{P}}(n) = (\mathcal{G}_*^{\mathcal{P}}(n), \partial)$ the $\mathcal{P}$-graph complex. For $n = 0$ we write simply $\mathcal{G}_*^{\mathcal{P}}$ instead of $\mathcal{G}_*^{\mathcal{P}}(0)$.

3.7. *Examples.* The nonsymmetric operad $\mathcal{A}ss$ for associative algebras is a nonsymmetric cyclic operad. Because $\mathcal{A}ss(n) = \mathbf{k}$ for each $n \geq 1$, the coloring contains no information and the complex $\mathcal{G}_*^{Ass} = (\mathcal{G}_*^{Ass}, \partial)$ is the complex of ribbon graphs introduced in [4].

Similarly, the symmetric cyclic operad *Comm* for commutative algebras has $Comm(n) = \mathbf{k}$ and the complex $\mathcal{G}_*^{Comm} = (\mathcal{G}_*^{Comm}, \partial)$ is the complex of (all) graphs considered in [4, 8].

For the symmetric cyclic operad *Lie*, the graph complex $\mathcal{G}_*^{Lie} = (\mathcal{G}_*^{Lie}, \partial)$ was constructed by M. Kontsevich in [5]. We may imagine a *Lie*-graph as a graph whose vertices are colored by $(k-1)!$ 'colors' representing a basis of $Lie(k)$, where $k+1$ is the number of edges at the vertex.

4. The cycle.

4.1. *The 'state sum'.* Let $\mathbf{T}$ be the theory describing cyclic $\Omega(\mathcal{P}^*)$-algebras with trivial differential (Appendix 6.1). The universal cycle $\xi_i(n) \in C_i(\mathcal{G}_*^{\mathcal{P}}(n); \mathbf{T}(n,0))$ is defined as follows. Decorate each vertex v of $\mathsf{G} \in \mathcal{G}_i(n)$ with the 'canonical element' of $\mathcal{P}^*(k) \otimes \mathcal{P}(k)$, $k = \mathrm{ord}\{\mathrm{edg}(v)\}$. As in the case of trees, we interpret the '$\mathcal{P}$'-part as the coloring of the vertex v and the '$\mathcal{P}^*$'-part as an element of $\mathbf{T}(k+1,0)$, under the canonical monomorphism $\mathcal{P}^*(*) \hookrightarrow \mathbf{T}(*+1,0)$ (Appendix 6.2). Now compose these '$\mathcal{P}^*$'-parts as elements of $\mathbf{T}$, using $\nu \in \mathbf{T}(0,2)$ as a 'propagator' along the edges of G. This gives an element of $\mathbf{T}(n,0)$. The requisite $\xi_i(n)$ is then the summation over all graphs $\mathsf{G} \in \mathcal{G}_i(n)$.

The central statement is the following proposition.

Proposition 4.2. *The chain* $\xi_i(n) \in C_i(\mathcal{G}_*^{\mathcal{P}}(n); \mathbf{T}(n,0))$ *is a cycle,* $\partial(\xi_i(n)) = 0$. *This means that it determines a homology class*

$$[\xi_i(n)] \in H_i(\mathcal{G}_*^{\mathcal{P}}(n); \mathbf{T}(n,0)).$$

4.3. *The cycle defined by a cyclic algebra.* Let $B = (V, A, h, \nu)$ be a cyclic $\Omega(\mathcal{P}^*)$-algebra as in §6.2, i.e. a map $B : \mathbf{T} \to \mathbf{E}nd(V)$ of theories. The evaluation at B gives a cycle $B(\xi_i(n)) \in C_i(\mathcal{G}_*^{\mathcal{P}}(n); \mathrm{Hom}(V^{\otimes n}, \mathbf{k}))$ which in turn defines the class

$$B([\xi_i(n)]) \in H_i(\mathcal{G}_*^{\mathcal{P}}(n); \mathrm{Hom}(V^{\otimes n}, \mathbf{k})).$$

Extremely important is the case $n = 0$ when $\mathrm{Hom}(V^{\otimes n}, \mathbf{k}) = \mathbf{k}$. We get elements

$$B(\xi_i) \in \mathcal{G}_i^{\mathcal{P}} \text{ and } B([\xi_i]) \in H_i(\mathcal{G}_*^{\mathcal{P}}).$$

4.4. *Examples.* If $\mathcal{P} = Ass$, the nonsymmetric operad for associative algebras, Proposition 4.2 gives the 'universal element'

$$[\xi] \in H(\mathcal{M}_{g,n}; \mathbf{T}(0,0)).$$

A cyclic $\Omega(Ass^*)$-algebra is an A_∞-algebra with nondegenerate invariant scalar product. The evaluation at such an algebra gives the classes in $H(\mathcal{M}_{g,n}; \mathbf{k})$ constructed by M. Kontsevich in [4].

If $\mathcal{P} = Comm$, the symmetric operad for associative commutative algebras, then a cyclic $\Omega(Comm^*)$-algebra is an L_∞ (or strong homotopy Lie) algebra with nondegenerate invariant scalar product. Evaluation at $[\xi_i]$ then gives the classes constructed in [8].

5. Lacunar graphs.

The construction of M. Kontsevich of an element in $H^*(\mathcal{M}_{g,n};\mathbf{C})$ requires a cyclic A_∞-algebra; we already know that this element is an evaluation at the universal element of Proposition 4.2. In [9], M. Penkava and A. Schwarz showed that, if we restrict to a suitable subcomplex of the graph complex, we may construct a similar invariant out of an infinitesimal deformation of a cyclic associative algebra. M. Penkava then generalized in [8] this construction to the L_∞ (strong homotopy Lie) algebra case. Here we explain an almost obvious generalization of these results to algebras over an arbitrary cyclic operad and give also a conceptual explanation of the construction.

Since we work with deformations, we need an independent variable t. We work over the extended coefficient ring $\mathbf{k}[t]$, the ring of polynomials in t. We will use the notation $\mathcal{P}[t]$ for $\mathcal{P}\otimes_{\mathbf{k}}\mathbf{k}[t]$, etc.

Fix $k\geq 3$ and consider (for a fixed i) the subspace $\mathcal{G}_i'^{\mathcal{P}}(n)$ of $\mathcal{G}_i^{\mathcal{P}[t]}(n)=\mathcal{G}_i^{\mathcal{P}}(n)[t]$ consisting of $\mathcal{P}[t]$-colored graphs such that

(i) all vertices are trivalent except exactly one which is $(k+1)$-valent,
(ii) trivalent vertices are colored by elements of $\mathcal{P}(2)=\mathcal{P}(2)\cdot t^0\subset\mathcal{P}(2)[t]$ and
(iii) the $(k+1)$-valent vertex is colored by an element of $\mathcal{P}(k)\cdot t^1\subset\mathcal{P}(k)[t]$.

Now construct the universal cycle $\xi_i(n)\in C_i(\mathcal{G}_*^{\mathcal{P}[t]}(n),\mathrm{T}(n,0)[t])$ and restrict it to a cycle $\xi_i'(n)\in C_i(\mathcal{G}_*'^{\mathcal{P}}(n),\mathrm{T}(n,0)[t])$. The crucial observation is that $\partial(\xi_i'(n))$ consists of at most linear terms in t. So, to evaluate at $\xi_i'(n)$, a map $A:\mathrm{T}[t]\to\mathrm{E}nd(V)[t]$ which is a map of theories <u>*modulo* t^2</u> is enough!

A moment's reflection shows that an infinitesimal deformation of a cyclic $\mathcal{P}^!$-algebra into a cyclic $\Omega(\mathcal{P}^*)$-algebra gives such a map. We may conclude this paragraph by observing that infinitesimal deformations of cyclic algebras over a Koszul cyclic operad are governed by the cyclic cohomology which was constructed, for a general cyclic operad, in [3].

6. Appendix.

6.1. *Algebras over* $\Omega(\mathcal{P}^*)$. Let $\mathcal{P}$ be an operad and let $\Omega(\mathcal{P}^*)$ be the cobar construction [3] on the dual cooperad $\mathcal{P}^*$, $\mathcal{P}^*=\{\mathcal{P}^*(n)\}_{n\geq1}$, with $\mathcal{P}^*(n)=\mathcal{P}(n)^*=\mathrm{Hom}(\mathcal{P}(n),\mathbf{k})$. This means that $\Omega(\mathcal{P}^*)=(\mathcal{F}(\mathcal{P}^*),\partial_\Omega)$, where $\mathcal{F}(\mathcal{P}^*)$ is the free operad on the collection $\mathcal{P}^*$ and the differential ∂_Ω is induced by the structure maps of the cooperad $\mathcal{P}^*$.

As usual, an algebra over $\Omega(\mathcal{P}^*)$ is a differential vector space (V,d_V) together with a map $A:\Omega(\mathcal{P}^*)\to\mathrm{End}(V,d_V)$ of differential operads; here $\mathrm{End}(V,d_V)$ is the endomorphism operad of (V,d_V). As we work with graphs having at least trivalent vertices, we consider only the case $d_V=0$. Such algebras can be described as algebras over the operad $\mathcal{Q}=\mathcal{F}(\mathcal{P}^*)/(\partial_\Omega(\mathcal{P}^*))$ (= the free operad on $\mathcal{P}^*$ modulo the ideal generated by $\partial_\Omega(\mathcal{P}^*)$). As an easy consequence of the quadraticity of the differential ∂_Ω we see that the canonical projection $\mathcal{P}^*\to\mathcal{F}(\mathcal{P}^*)$ induces a monomorphism $\mathcal{P}^*\hookrightarrow\mathcal{Q}$ of collections.

6.2. *Cyclic algebras.* Let A be an $\Omega(\mathcal{P}^*)$-algebra as above. We say that A is cyclic if there exists a symmetric bilinear product $h = \langle -|-\rangle$ on V such that

$$\langle A(\phi)(x_1,\ldots,x_n)|x_{n+1}\rangle = \langle x_1|A(\phi)(x_2,\ldots,x_{n+1})\rangle, \tag{3}$$

for any $\phi \in \mathcal{P}^*(n)$, and $x_1,\ldots,x_{n+1} \in V$.

Thus, a cyclic algebra is an object of the form $B = (V, A, h, \nu)$, where $A : \mathcal{Q} \to \mathrm{End}(V)$ is a $\mathcal{Q}$-algebra structure on V, $h : V \otimes V \to \mathbf{k}$ is a scalar product and $\nu : \mathbf{k} \to V \otimes V$ is the 'inverse matrix' of h in the sense that

$$(h \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \nu) = (\mathbb{1} \otimes h) \circ (\nu \otimes \mathbb{1}) = \mathbb{1}.$$

Such objects form an equationally given category which is not algebraic, i.e. it can not be described as a category of algebras over an operad, but rather as a category of algebras over a k-linear PROP. In [7, 6] we used the name 'theory' for a k-linear PROP. Although this terminology is obviously not the best one, we will use this name here. We refer to [7, 6] for a very thorough introduction to k-linear PROPs.

Just recall that a theory is a sequence $\mathbf{T} = \{\mathbf{T}(m,n);\ m,n \geq 0\}$ of k-linear spaces, each $\mathbf{T}(m,n)$ encoding operations with m inputs and n outputs, as the $\mathcal{P}(n)$-part of an operad $\mathcal{P}$ encodes operations with n inputs and just one output. For any vector space V there exists the 'endomorphism theory' $\mathbf{E}nd(V)$, with $\mathbf{E}nd(V)(m,n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$. An algebra over $\mathbf{T}$ is then a map $B : \mathbf{T} \to \mathbf{E}nd(V)$ of theories.

Let $\mathbf{T}$ be the theory describing cyclic $\mathcal{Q}$-algebras. The theory $\mathbf{T}$ is, in a well defined sense, generated by the operad $\mathcal{Q}$ and by the elements $h \in \mathbf{T}(2,0)$ and $\nu \in \mathbf{T}(0,2)$. The correspondence $q \mapsto h(q, \mathbb{1})$ defines a map $\mathcal{Q}(k) \to \mathbf{T}(k+1,0)$. The composition of this map with the inclusion $\mathcal{P}^*(k) \to \mathcal{Q}(k)$ gives the inclusion $\mathcal{P}^*(k) \hookrightarrow \mathbf{T}(k+1,0)$. The elements in the image of this map are, due to (3), invariant under the action of the cyclic group.

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