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# SPACE-TIME DECOMPOSITIONS VIA DIFFERENTIAL FORMS

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ABSTRACT. Space-time decompositions of the physical quantities represented by differential forms on a Lorentzian manifold $(M, g; + - - -)$ with respect to arbitrary observer field and the decomposition of the standard operations acting on them are studied, making use of the ideas of the theory of connections on principal bundles. Simple explicit general formulas are given as well as their application to the Maxwell equations.

## 1. Introduction

There is rather extensive literature devoted to space-time split of the laws of physics in curved spacetime (cf. [1] for a review and also the references therein). According to Sec. 2. of [1] there are two rather different methods available to cope with the problem, viz. the congruence method and the hyperspace one.

Here we present a systematic method of 3+1 split within the *congruence* method using the language of *differential forms* on both (4 and 3+1) levels.

The use of forms within the 3+1 decomposition program can be traced back to the classical paper on geometrodynamics [2] (p.581), where it was applied, however, in the framework of the *hyperspace* method (cf. also [3], pp. 93-94); the time serves there as a parameter labeling the spacelike hypersurfaces and the time derivative of a form is interpreted as a differentiation with respect to a parameter.

The observer field approach similar to ours can be found in [4] (pp. 193-197). What we add here is the introduction of the (simply realized) operator hor (cf. Sec.3) and spatial exterior derivative within the general congruence approach (Sec. 4b). These objects turn out to be very convenient to manipulate with and enable one to derive very simple and at the same time general general decomposition formulas and rules.

The following data are assumed in the article : a 4-dimensional Lorentzian ($g$ of the signature + - - -) manifold $M$ with orientation ( $\equiv$ spacetime), and an *observer* (velocity) *field*, i.e. a future oriented vector field on $M$ obeying

$$g(V, V) \equiv \| V \|^2 = 1 \tag{1.1}$$

(the integral curves of $V$ provide then the congruence of proper-time parametrized worldlines of observers).

All the constructions are in fact local, i.e. it is enough that the objects mentioned

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above are available only in a domain $\mathcal{U} \subset M$ rather than globally and consequently no global properties of $M$ are assumed.

Proofs of some technical details are omitted here. The interested reader can find them in [11].

## 2. The decomposition of forms

For any vector field $W$ let us define the following standard operations on forms :

$$i_W : \Omega^p(M) \to \Omega^{p-1}(M)$$
$$j_W : \Omega^p(M) \to \Omega^{p+1}(M)$$

$$i_W\alpha(U, ...) := \alpha(W, U, ...) \tag{2.1}$$

$$j_W\alpha := \tilde{W} \wedge \alpha \qquad \tilde{W} \equiv g(W, .) \equiv \flat_g W \tag{2.2}$$

Then the following identity holds :

$$j_W i_U + i_U j_W = g(U, W)\ \hat{1}, \tag{2.3}$$

in particular

$$j_V i_V + i_V j_V = \ \hat{1} \equiv \text{identity on } \Omega(M). \tag{2.4}$$

Further, introducing

$$\mathcal{P} := i_V j_V \qquad \mathcal{Q} := j_V i_V \tag{2.5}$$

one checks easily that they represent the set of *projection operators* on $\Omega^p(M)$, i.e.

$$\mathcal{P}^2 = \mathcal{P} \quad \mathcal{Q}^2 = \mathcal{Q} \quad \mathcal{P}\mathcal{Q} = 0 = \mathcal{Q}\mathcal{P} \quad \mathcal{P} + \mathcal{Q} = \hat{1} \tag{2.6}$$

Then for any $\alpha \in \Omega^p(M)$ one has

$$\alpha = (\mathcal{Q} + \mathcal{P})\alpha = \tilde{V} \wedge i_V\alpha + i_V j_V\alpha \tag{2.7}$$

i.e. one obtains the *decomposition*

$$\alpha = \tilde{V} \wedge \hat{s} + \hat{r} \tag{2.8}$$

where

$$\hat{s} \equiv i_V\alpha \qquad \hat{r} \equiv i_V j_V\alpha \tag{2.9}$$

## 3. Operator hor and spatial forms

At any point $m \in M$ we define *vertical* (instantaneous time) direction - parallel to $V$ and *horizontal* (instantaneous 3-space) directions - perpendicular to $V$. Then for any vector there is the unique decomposition

$$U = U_{||} + U_{\perp} \equiv \text{ver } U + \text{hor } U$$

and one can define (in the spirit of the theory of connections on principal bundles, cf. [5] )

$$(\text{hor } \alpha)(U, W, ...) := \alpha(\text{hor } U, \text{hor } W, ...) \tag{3.1}$$

It turns out that this operation is realized explicitly as

$$\text{hor } \alpha = i_V j_V \alpha \equiv \mathcal{P}\alpha \equiv \hat{r} \tag{3.2}$$

so that the decomposition in (2.8) can be rewritten also as

$$\alpha = \tilde{V} \wedge i_V \alpha + \text{hor } \alpha \tag{3.3}$$

One can also introduce purely *spatial* (horizontal) *forms* as those satisfying

$$\alpha = \text{hor } \alpha \tag{3.4}$$

We shall denote the space of spatial p-forms by $\hat{\Omega}^p(M)$. ¿From (3.3) we obtain useful criterion

$$\alpha = \text{spatial form} \qquad \Leftrightarrow \qquad i_V \alpha = 0 \tag{3.5}$$

Then we see that $\hat{r}, \hat{s}$ in the decomposition (2.8) are spatial ((2.9) plus $i_V i_V = 0$).
Note : If a local orthonormal frame field $e_a \equiv (e_0 \equiv V, e_i)$ and its dual $e^a \equiv (e^0 \equiv \tilde{V}, e^i)$, are used and if

$$\alpha = \frac{1}{p!}\alpha_{a...b} e^a \wedge ... \wedge e^b, \tag{3.6}$$

then the decomposition (2.8) is just the split into two parts which do and do not contain respectively the basis 1-form $e^0 \equiv \tilde{V}$, i.e.

$$\alpha = e^0 \wedge \hat{s} + \hat{r}$$

where $\hat{s}, \hat{r}$, being spatial, do not already contain the local "time" basis 1-form $e^0$, but rather only the "spatial" basis 1-forms $e^i$; explicitly

$$\hat{s} = \frac{1}{(p-1)!}\alpha_{0i...j} \underbrace{e^i \wedge ... \wedge e^j}_{p-1} \qquad \hat{r} = \frac{1}{p!}\alpha_{k...j} \underbrace{e^k \wedge ... \wedge e^j}_{p} \tag{3.7}$$

## 4. The decomposition of the operations on forms

According to (2.8) any form on $(M, g, V)$ can be 3+1 decomposed as

$$\alpha = \tilde{V} \wedge \hat{s} + \hat{r}$$

so that the full information about $\alpha \in \Omega^p(M)$ is encoded (in an observer dependent way) into a *pair* of *spatial* forms $\hat{s} \in \hat{\Omega}^{(p-1)}(M)$ and $\hat{r} \in \hat{\Omega}^p(M)$. In this section we perform the decomposition of the standard *operations* on forms, viz. the *Hodge star* $*$ and the *exterior derivative* $d$ (some other important operators are then easily obtained by their combinations). By this we mean to introduce some "spatial" operations (acting directly on $\hat{s}, \hat{r}$ and dependent on the observer) producing the same effect as does the given operator acting on $\alpha$.

### 4a. The Hodge star

The horizontal subspace of a tangent space at each point inherits natural metric tensor $\hat{h}$ (with signature $+ + +$ by definition) and orientation (a frame $(e_1, e_2, e_3)$ is declared to be right-handed if $(V \equiv e_0, e_1, e_2, e_3)$ is right-handed). These data are just enough for the unique *spatial Hodge* operator

$$\hat{*} := *_{\hat{h}} : \hat{\Omega}^p(M) \to \hat{\Omega}^{3-p}(M) \tag{4a.1}$$

(it is to be applied only on spatial forms). Using the operator

$$\hat{\eta}\alpha := (-1)^{\deg \alpha} \alpha$$

one readily computes that the decomposition of the "full" Hodge star reads

$$*(\tilde{V} \wedge \hat{s} + \hat{r}) = \tilde{V} \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \tag{4a.2}$$

As an example, applying this to $1 \in \Omega^0(M)$ $(\hat{s} = 0, \hat{r} = 1)$ results to the decomposition of the 4-volume form

$$*1 \equiv \omega = \tilde{V} \wedge \hat{*}1 =: \tilde{V} \wedge \hat{\omega} \tag{4a.3}$$

where

$$\hat{\omega} := \hat{*}1 \tag{4a.4}$$

is the *spatial volume form*. In local orthonormal coframe field $e^a$ it is just

$$\omega \equiv e^0 \wedge e^1 \wedge e^2 \wedge e^3 = e^0 \wedge (e^1 \wedge e^2 \wedge e^3) \equiv \tilde{V} \wedge \hat{\omega}$$

## 4b. The exterior derivative

Let $\mathcal{D}$ be a spatial ($\equiv$ horizontal) domain (of any possible dimension), $\hat{b}$ a spatial form. Then

$$\int_{\mathcal{D}} d\hat{b} \overset{1.}{=} \int_{\partial\mathcal{D}} \hat{b} \qquad \text{due to Stokes' theorem}$$

$$\overset{2.}{=} \int_{\mathcal{D}} \text{hor } d\hat{b} \equiv \int_{\mathcal{D}} \hat{d}\hat{b} \qquad \text{since } \mathcal{D} \text{ is horizontal}$$

$\Rightarrow$

(4b.1) $$\int_{\mathcal{D}} \hat{d}\hat{b} = \int_{\partial\mathcal{D}} \hat{b}$$

where we introduced the *spatial exterior derivative*

(4b.2) $$\hat{d} := \text{hor } d \equiv i_V j_V \, d$$

(exactly like the *covariant* exterior derivative of forms on principal bundle with connection). Thus for spatial forms and domains the "full" operator $d$ in the Stokes formula can be replaced by $\hat{d}$. This means that $\hat{d}$ and $\hat{*}$ provide the basic building blocks for the "3-dimensional vector analysis" operations, being the natural generalizations of div, curl etc. used in Minkowski space (div $\sim \hat{*}\hat{d}\hat{*}$, curl $\sim \hat{*}\hat{d}$, ...). We emphasize that the validity of the *spatial Stokes formula* (4b.1) for $\hat{d}$ is essential for the usefulness and naturality of the latter, e.g. as a means to relate the usual differential 3+1 laws to the corresponding integral ones (like $\text{div}\mathbf{B} = 0 \leftrightarrow \oint \mathbf{B}.d\mathbf{S} = 0$).

So our task now is to express the action of the "full" $d$ operator in terms of $\hat{d}$ (and possibly some other ones) directly on $\hat{s}, \hat{r}$ present in the decomposition (2.8) of $\alpha$ . We have

$$d\alpha = d\tilde{V} \wedge \hat{s} - \tilde{V} \wedge d\hat{s} + d\hat{r}$$

so that we are to focus our attention to two particular issues, viz. $d$ of $\tilde{V}$ and $d$ of a spatial form.

The decomposition of the 2-form $d\tilde{V}$ according to (2.8) results in

(4b.3) $$d\tilde{V} = \tilde{V} \wedge \hat{a} + \hat{y}$$

with $\hat{a} \in \hat{\Omega}^1(M), \hat{y} \in \hat{\Omega}^2(M)$. The forms $\hat{a}, \hat{y}$ are the *kinematical characteristics* of the observer field $V$, which can be easily extracted from any given $V$ using (2.9). It turns out (see also [6], [7], [8]) that $\hat{a}$ equals to the *acceleration 1-form*

(4b.4) $$\hat{a} = g(\nabla_V V, .) \equiv g(a, .) \equiv \tilde{a}$$

($a \equiv \nabla_V V$ is the *acceleration field* corresponding to $V$) and the 2-form $\hat{y}$, the *vorticity form* (tensor) is the measure of the (non)integrability of the spatial (horizontal) distribution, i.e. it encodes whether or not the instantaneous 3-spaces mesh together to form a (local) spatial 3-domain $\mathcal{D}$ (or, equivalently, whether or not the *time synchronization* is possible). These properties of $\hat{a}$ and $\hat{y}$ are reflected in the terminology: $V$ is said to be *geodesic* if $\hat{a} = 0$, *irrotational* or *time-synchronizable* if $\hat{y} = 0$ and *proper-time synchronizable* if both $\hat{a}$ and $\hat{y}$ vanish (then $V = \partial_t, \tilde{V} = dt$ in adapted coordinates).

The computation of the action of $d$ on a spatial form, as well as on a general form $\alpha$ then, results in

$$d\hat{b} = \tilde{V} \wedge \mathcal{L}_V \hat{b} + \hat{d}\hat{b} \qquad \text{i.e.} \qquad d = j_V \mathcal{L}_V + \hat{d} \quad \text{on } \hat{\Omega}(M) \tag{4b.5}$$

($\hat{b} \in \hat{\Omega}^p(M)$) and

$$d(\tilde{V} \wedge \hat{s} + \hat{r}) = \tilde{V} \wedge (-\hat{d}\hat{s} + \mathcal{L}_V \hat{r} + \hat{a} \wedge \hat{s}) + (\hat{d}\hat{r} + \hat{y} \wedge \hat{s}) \tag{4b.6}$$

The formula (4b.6) gives the desired 3+1 decomposition of the "full" $d$ operator. Notice the explicit occurrence of both kinematical characteristics $\hat{a}$ and $\hat{y}$.

The spatial exterior derivative $\hat{d}$, being the modification of the "full" $d$, may not (contrary to $d$) be nilpotent (possibly $\hat{d}\hat{d} \neq 0$) in general. The computation shows, that this is really the case, viz.

$$\hat{d}\hat{d} = -\hat{y} \wedge \mathcal{L}_V \qquad \text{on } \hat{\Omega}(M) \tag{4b.7}$$

This may seem to contradict (4b.1), since the exterior derivative is uniquely defined by the Stokes formula [9] (and it *is* then nilpotent due to the nilpotence of the boundary operator). Note, however, that (4b.1) is valid (for *all* degrees of spatial forms) only if the spatial regions (of *all* possible dimensions) do exist, which needs $\hat{y} = 0$ for p=3 due to the Frobenius theorem; but then we have $\hat{d}\hat{d} = 0$.

## 5. Matrix notation

For the computation of more complex expressions (e.g. the *codifferential* in (5.3)) it is quite useful to introduce matrix realization of the operators. If the decomposition (2.8) of $\alpha$ is represented by a column

$$\alpha \equiv \tilde{V} \wedge \hat{s} + \hat{r} \ \leftrightarrow \ \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

then e.g.

$$*(\tilde{V} \wedge \hat{s} + \hat{r}) = \tilde{V} \wedge \hat{*}\hat{r} + \hat{*}\hat{\eta}\hat{s} \leftrightarrow \begin{pmatrix} \hat{*}\hat{r} \\ \hat{*}\hat{\eta}\hat{s} \end{pmatrix} \equiv \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

so that

$$* \ \leftrightarrow \ \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix}.$$

For the exterior derivative we obtain similarly

$$d(\tilde{V} \wedge \hat{s} + \hat{r}) \leftrightarrow \begin{pmatrix} -\hat{d}\hat{s} + \mathcal{L}_V \hat{r} + \hat{a} \wedge \hat{s} \\ \hat{d}\hat{r} + \hat{y} \wedge \hat{s} \end{pmatrix} \equiv \begin{pmatrix} -\hat{d} + \hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} \hat{s} \\ \hat{r} \end{pmatrix}$$

or

$$d \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix}.$$

In the same sense we can then express also other useful operations in terms of such matrices; for the sake of convenience we collect them here together :

(5.1) $$* \leftrightarrow \begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix} \qquad *^{-1} \leftrightarrow \begin{pmatrix} 0 & -\hat{*}\hat{\eta} \\ \hat{*} & 0 \end{pmatrix}$$

(5.2) $$\hat{\eta} \leftrightarrow \begin{pmatrix} -\hat{\eta} & 0 \\ 0 & \hat{\eta} \end{pmatrix} \qquad d \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix}$$

$$\delta := *^{-1}d * \hat{\eta} \leftrightarrow \begin{pmatrix} 0 & -\hat{*}\hat{\eta} \\ \hat{*} & 0 \end{pmatrix}\begin{pmatrix} -\hat{d}+\hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix}\begin{pmatrix} 0 & \hat{*} \\ \hat{*}\hat{\eta} & 0 \end{pmatrix}\begin{pmatrix} -\hat{\eta} & 0 \\ 0 & \hat{\eta} \end{pmatrix} =$$

(5.3) $$\begin{pmatrix} \hat{\delta} & \hat{*}(\hat{y} \wedge \hat{*}) \\ -\hat{*}\mathcal{L}_V\hat{*} & -\hat{\delta} + \hat{*}(\hat{a} \wedge \hat{*}\hat{\eta}) \end{pmatrix}$$

where

(5.4) $$\hat{\delta} := \hat{*}^{-1}\hat{d}\hat{*}\hat{\eta}$$

is the *spatial codifferential*,

(5.5) $$i_V \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad j_V \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(5.6) $$\text{hor} = i_V j_V \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(5.7) $$\mathcal{L}_V \equiv i_V d \;+\; d\, i_V \leftrightarrow \begin{pmatrix} \mathcal{L}_V & 0 \\ \hat{a} & \mathcal{L}_V \end{pmatrix}$$

## 6. The Maxwell equations

According to the standard conventions on the relationship between the components of the electromagnetic field 2-form $F \equiv \frac{1}{2}F_{ab}e^a \wedge e^b$ ($e_a$ is $g$-orthonormal frame) and the 3-space vectors of the electric and magnetic fields respectively

(6.1) $$F_{0\alpha} = E_\alpha = E^\alpha \qquad F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma}B^\gamma \equiv -\epsilon_{\alpha\beta\gamma}B_\gamma$$

($e_\alpha$ is $\hat{h}$-orthonormal frame; $\alpha, \beta \ldots$ run from 1 to 3, being raised and lowered by the *spatial* metric tensor $\hat{h}_{\alpha\beta} \equiv +\delta_{\alpha\beta} \equiv -\eta_{\alpha\beta}$), one can associate with the electric and magnetic fields the *spatial forms*

(6.2) $$\hat{E} = E_\alpha e^\alpha =: \mathbf{E}.d\mathbf{r} \qquad \hat{B} = B^\alpha dS_\alpha =: \mathbf{B}.d\mathbf{S} \qquad dS_\alpha := \frac{1}{2}\epsilon_{\alpha\beta\gamma}e^\beta \wedge e^\gamma$$

Then

$$F = \tilde{V} \wedge \hat{E} - \hat{B} \leftrightarrow \begin{pmatrix} \hat{E} \\ -\hat{B} \end{pmatrix} \tag{6.3}$$

(so that $\hat{s} = \hat{E}$, $\hat{r} = -\hat{B}$ here). Similarly the electric *4-current 1-form* decomposes to

$$j = j_a e^a = j_0 e^0 + j_i e^i \equiv \rho \tilde{V} - \hat{j} \leftrightarrow \begin{pmatrix} \rho \\ -\hat{j} \end{pmatrix} \qquad \hat{j} := j_\alpha e^\alpha = j^\alpha e^\alpha \tag{6.4}$$

Then

$$*F = \tilde{V} \wedge (-\hat{*}\hat{B}) - \hat{*}\hat{E} \leftrightarrow \begin{pmatrix} -\hat{*}\hat{B} \\ -\hat{*}\hat{E} \end{pmatrix} \tag{6.5}$$

$$*j = \tilde{V} \wedge (-\hat{*}\hat{j}) + \rho\hat{\omega} \leftrightarrow \begin{pmatrix} -\hat{*}\hat{j} \\ \rho\hat{\omega} \end{pmatrix} \tag{6.6}$$

and so the 3+1 decomposition of the Maxwell equations

$$d * F = -4\pi * j \tag{6.7}$$

$$dF = 0 \tag{6.8}$$

and the continuity equation

$$d * j = 0, \tag{6.9}$$

respectively result in

$$\hat{d}\hat{*}\hat{E} + \hat{y} \wedge \hat{*}\hat{B} = 4\pi\rho\hat{\omega} \tag{6.7a}$$

$$\hat{d}\hat{*}\hat{B} - \mathcal{L}_V \hat{*}\hat{E} - \hat{a} \wedge \hat{*}\hat{B} = 4\pi\hat{*}\hat{j} \tag{6.7b}$$

$$\hat{d}\hat{E} + \mathcal{L}_V \hat{B} - \hat{a} \wedge \hat{E} = 0 \tag{6.8a}$$

$$\hat{d}\hat{B} - \hat{y} \wedge \hat{E} = 0 \tag{6.8b}$$

and

$$\mathcal{L}_V(\rho\hat{\omega}) + \hat{d}\hat{*}\hat{j} - \hat{a} \wedge \hat{*}\hat{j} = 0 \tag{6.10}$$

In particular in the simplest situation, viz. for the irrotational ($\hat{y} = 0$), geodesic ($\hat{a} = 0$) observer field $V$ (then $V = \partial_t$, $\tilde{V} = dt$) we get

(6.7a') $$\hat{d}\hat{*}\hat{E} = 4\pi\rho\hat{\omega}$$

(6.7b') $$\hat{d}\hat{*}\hat{B} - \mathcal{L}_{\partial_t}\hat{*}\hat{E} = 4\pi\hat{*}\hat{j}$$

(6.8a') $$\hat{d}\hat{E} + \mathcal{L}_{\partial_t}\hat{B} = 0$$

(6.8b') $$\hat{d}\hat{B} = 0$$

and

(6.10') $$\mathcal{L}_{\partial_t}(\rho\hat{\omega}) + \hat{d}\hat{*}\hat{j} = 0$$

Since the equations (6.7a) - (6.10) are written in terms of differential forms and standard well-behaved operations with respect to integrals, one can readily write down their corresponding *integral versions* : let spatial domains of necessary dimensions exist (2-dimensional surface $\mathcal{S}$, 3-dimensional volume $\mathcal{D}$ - the latter case needs $\hat{y} = 0$, therefore we put $\hat{y} = 0$ in the equations where the integration over 3-dimensional domain is performed); then

(6.11a) $$\oint_{\partial\mathcal{D}} \hat{*}\hat{E} = 4\pi\int_{\mathcal{D}} \rho\hat{\omega} \equiv 4\pi Q$$

(6.11b) $$\oint_{\partial\mathcal{S}} \hat{*}\hat{B} - \frac{d}{d\tau}\bigg|_0 \int_{\Phi_\tau(\mathcal{S})} \hat{*}\hat{E} - \int_{\mathcal{S}} \hat{a} \wedge \hat{*}\hat{B} = 4\pi\int_{\mathcal{S}} \hat{*}\hat{j}$$

(6.12a) $$\oint_{\partial\mathcal{S}} \hat{E} + \frac{d}{d\tau}\bigg|_0 \int_{\Phi_\tau(\mathcal{S})} \hat{B} - \int_{\mathcal{S}} \hat{a} \wedge \hat{E} = 0$$

(6.12b) $$\oint_{\partial\mathcal{D}} \hat{B} = 0$$

and

(6.13) $$\frac{d}{d\tau}\bigg|_0 \int_{\Phi_\tau(\mathcal{D})} \rho\hat{\omega} + \oint_{\partial\mathcal{D}} \hat{d}\hat{*}\hat{j} - \int_{\mathcal{D}} \hat{a} \wedge \hat{*}\hat{j} = 0$$

where $\Phi_\tau$ is the (local) flow generated by $V$.

The equations (6.7a) - (6.10) can be also expressed in more familiar form, making

use of 3-dimensional vector analysis operators div, curl etc. [11].
Equivalently, if instead of (6.7)

$$\delta F = 4\pi j \tag{6.14}$$

is used, (6.7a), (6.7b) are to be replaced by

$$\hat{\delta}\hat{E} - \hat{*}(\hat{y} \wedge \hat{*}\hat{B}) = 4\pi\rho \tag{6.14a}$$

$$\hat{\delta}\hat{B} - \hat{*}\mathcal{L}_V\hat{*}\hat{E} - \hat{*}(\hat{a} \wedge \hat{*}\hat{B}) = 4\pi\hat{j} \tag{6.14b}$$

(they can be obtained directly by applying $\hat{*}$ on (6.7a), (6.7b), too.)
The decomposition of the 4-*potential* 1-form

$$A \leftrightarrow \begin{pmatrix} \phi \\ -\hat{A} \end{pmatrix} \tag{6.15}$$

gives

$$\begin{pmatrix} \hat{E} \\ -\hat{B} \end{pmatrix} \leftrightarrow F \equiv dA \leftrightarrow \begin{pmatrix} -\hat{d}+\hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} \phi \\ -\hat{A} \end{pmatrix} = \begin{pmatrix} -\hat{d}\phi + \phi\hat{a} - \mathcal{L}_V\hat{A} \\ \phi\hat{y} - \hat{d}\hat{A} \end{pmatrix} \tag{6.16}$$

so that

$$\hat{E} = -\hat{d}\phi + \phi\hat{a} - \mathcal{L}_V\hat{A} \tag{6.17}$$

$$\hat{B} = \hat{d}\hat{A} - \phi\hat{y} \tag{6.18}$$

Finally, the gauge transformation:

$$A \mapsto A' \equiv A + d\chi \leftrightarrow \begin{pmatrix} \phi \\ -\hat{A} \end{pmatrix} + \begin{pmatrix} -\hat{d}+\hat{a} & \mathcal{L}_V \\ \hat{y} & \hat{d} \end{pmatrix} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{6.19}$$

is

$$\phi \mapsto \phi' \equiv \phi + V\chi \tag{6.20}$$

$$\hat{A} \mapsto \hat{A}' \equiv \hat{A} - \hat{d}\chi \tag{6.21}$$

## 7. Conclusions and summary

In this article we presented a simple method of 3+1 decomposition of the physical equations written in terms of differential forms on spacetime $(M, g)$ with respect to a general observer field $V$.

The method consists of the decomposition of both forms and operations on them. The decomposition of forms is based technically on a simple identity (2.4), which can be interpreted in terms of projection operators on $\Omega^p(M)$. The decomposition of the operations on forms consists first in the decomposition (4a.2) of the Hodge star operator and then the decomposition of the exterior derivative $d$. Here the formalism mimics the approach used standardly in the theory of connections on principle bundle, viz. we first introduce the operator hor (projecting on the " spatial part" of the form; its simple realization is given by (3.2)) and then define the *spatial* exterior derivative as $\hat{d} :=$ hor $d$ (the counterpart of the *covariant* exterior derivative on principle bundle with connection). The decomposition of $d$ is then given by (4b.6). The essential property of $\hat{d}$, which makes it a useful object, is the validity of the *spatial Stokes formula* (4b.1). It provides the usual link between the differential and integral formulations of the physical laws respectively. The language of differential forms on both 4 and 3+1 levels turns out to be the most convenient tool for realization of this link, since forms are the objects directly present under the integral signs.

Let us mention, that also the quantities of physical interest which "are not" forms (energy-momentum tensor, Ricci and Einstein tensors,...) admit description in terms of forms [4]; it is then possible to apply the decomposition presented here also to them.

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