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Abstract

M. Anderson and Ch. G. Torre [1] had proved, that only generalised symmetries admitted by the vacuum Einstein equations consist of the diffeomorfism symmetry that is inherent in the Einstein equations and a trivial scaling symmetry. There are no conservation laws associated with these symmetries.

The method of the approximate symmetry groups developed by Ibragimov [2] gives us the chance to construct from symmetries of vacuum equations the symmetries of Einstein equations with non zero tensor of momentum-energy. This method is based on the infinitesimal perturbations of vacuum that will be stable with chosen symmetries.

1 Introduction

There are lot of works about the applications of point and generalised symmetries of partial differential equations. Main regions of application are for example:

- finding new solutions from known one
- finding corresponding conservation's law
- the question of integrability

The basic problem for the physics approach is the fundamental sensitivity of the developed symmetry groups to any small perturbations of initial set of equations. We can find an exact group of point or general symmetries of given set of equations, but we do not know what happens if this set of equations will be perturbed under some even small correction of the equation. These perturbations destroy a group of symmetries in general. We do not have any tools in the frame of standard theory to study this problem.

The point symmetry of the equations

$$\Delta_\beta(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1 \ldots\, i_k}) = 0 \tag{1}$$

is an ordinary vector field on jet-space of independent and dependent variables

$$\mathbf{Z} = A^i(x^j, u^\beta)\frac{\partial}{\partial x^i} + B^\alpha(x^j, u^\beta)\frac{\partial}{\partial u^\alpha}, \tag{2}$$

[0]The paper is in final form and no version of it will be submitted elsewhere

which satisfies a condition

$$prZ(\Delta_\beta) = 0 \tag{3}$$

on equation manifold (1). The prolongation of the vector field $\mathbf{Z}$ is defined as [5]

$$pr\mathbf{Z} = A^i D_i + \sum_{k=0}^{\infty} D_{i_1 i_2 \dots i_k}(B^\alpha - u_i^\alpha A^i)\frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_k}}. \tag{4}$$

Point symmetries are in one-to-one correspondence with first-order evolutionary symmetries

$$\mathbf{Y} = B^\beta(x^i, u^\alpha, u_i^\alpha)\frac{\partial}{\partial u^\beta}. \tag{5}$$

The generalised symmetries are connected with generalised vector field $\mathbf{Z}$, where the coefficients A_i and B_j are functions of x^j, u^β and the derivatives $u^\beta_{i_1 i_2 \dots i_k}$ to some arbitrary but finite order. This vector field must satisfy a condition

$$pr\mathbf{Z}(\Delta_\beta) = 0 \tag{6}$$

not only on the equation manifold, but on prolonged equation manifold defined by the equations

$$\Delta_\beta = 0 \tag{7}$$

$$D_{i_1 i_2 \dots i_k}\Delta_\beta = 0. \tag{8}$$

2 The approximate symmetry method

This method was developed by Ibragimov [2]. Method is based on the approximate groups of transformations and the main task is to find this group for the given set of approximate equations.

We can start with a perturbed equation

$$\Delta^0_\beta + \varepsilon\Delta^1_\beta \approx 0. \tag{9}$$

If this approximate equation is approximate invariant under infinitesimal transformation

$$\mathbf{X} = \mathbf{X}^0 + \varepsilon\mathbf{X}^1, \tag{10}$$

then the operator $\mathbf{X}^0$ must be admitted with the equation

$$\Delta^0_\beta = 0. \tag{11}$$

Inverse theorem does not take place.

If operator $\mathbf{X}^0$ of an unperturbed equation is prolonged to operator $\mathbf{X}^0 + \varepsilon\mathbf{X}^1$ of a perturbed equation, then we call this operator *stable symmetry*.

Now we can write an approximate analogy of standard Lie theorem [2]:

Theorem 1 *Let transformations $z' \approx f(z,\varepsilon)$ form an approximate group with the tangential vector field*

$$\xi(z,\varepsilon) \approx \frac{\partial f(z,\varepsilon,a)}{\partial a}\,|_{a=0}\,. \tag{12}$$

Then the function $f(z,\varepsilon,a)$ satisfies an approximate equation

$$\frac{\partial f(z,\varepsilon,a)}{\partial a} \approx \xi(f(z,\varepsilon,a),\varepsilon). \tag{13}$$

On the other hand, for any function $\xi(z,\varepsilon) \neq 0$ solution $z' \approx f(z,\varepsilon)$ of approximate Lie problem

$$\frac{dz'}{da} \approx \xi(z',a) \tag{14}$$

$$z'\,|_{a=0} \approx z \tag{15}$$

forms an approximate one-parameter group.

The equation above one call the approximate Lie equation.

3 The definition of invariance

Definition 1 *The approximate equation*

$$\Delta(z,\varepsilon) \approx 0 \tag{16}$$

we call invariant under approximate group of transformation $z' \approx f(z,\varepsilon,a)$ if

$$\Delta(f(z,\varepsilon,a),\varepsilon)) \approx 0 \tag{17}$$

for all z that satisfy the approximate equation (16).

Computing of approximate symmetries leads now to next theorem:

Theorem 2 *An approximate equation (16) is invariant under approximate group of transformation $z' \approx f(z,\varepsilon,a)$ with an infinitesimal operator*

$$\mathbf{Z} = \xi^i(z,\varepsilon)\frac{d}{dz^i} \tag{18}$$

$$\xi^i = \frac{\partial f}{\partial a}\,|_{a=0} + O(\varepsilon^p), \tag{19}$$

if and only if

$$\mathbf{Z}\Delta(z,\varepsilon)\mid_{\Delta(z,\varepsilon)\approx 0}\approx 0. \tag{20}$$

4 Symmetries of vacuum Einstein equation

Anderson, Torre [1] , Ibragimov [3] have proved, that the only generalised symmetries admitted by the vacuum Einstein equations consist of the diffeomorphism symmetry, that is inherent in the Einstein equations and a trivial scaling symmetry. More precisely, they proved the following theorem:

Theorem 3 *Let*

$$h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1\ldots i_k}) \tag{21}$$

be the components of a k^{th}-order generalised symmetry of vacuum Einstein equations $R_{ij} = 0$ in four space-time dimensions. Then there is a constant c and a generalised vector field

$$X^i = X^i(x^i, g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1\ldots i_k}) \tag{22}$$

such that, modulo the Einstein equations

$$h_{ab} = cg_{ab} + \nabla_a X_b + \nabla_b X_a. \tag{23}$$

5 Approximate symmetries of Einstein equations

We'll study the Einstein equations in the next section. Let

$$G_{ij} - \varepsilon T_{ij} \approx 0 \tag{24}$$

be an approximate form of Einstein equations with some given tensor of energy-momentum. As we show above, these equations admit a generalised symmetries of the evolutionary vector field on the bundle of Lorentz metrics

$$\mathbf{Y} = \mathbf{Y}^0 + \varepsilon \mathbf{Y}^1, \tag{25}$$

where

$$\mathbf{Y}^0 = h^0_{ab}(x^i, g_{ij}, g_{ij,k_1}, \ldots, g_{ij,k_1\ldots k_n}, \ldots)\frac{\partial}{\partial g_{ab}} \tag{26}$$

is an evolutionary field of vacuum Einstein equations (exactly) and $\mathbf{Y}^1$ takes the form

$$\mathbf{Y}^1 = h^1_{ab}(x^i, g_{ij}, g_{ij,k_1}, \ldots, g_{ij,k_1\ldots k_n}, \ldots)\frac{\partial}{\partial g_{ab}}. \tag{27}$$

Now we can get an algorithm how to find an approximate symmetries:

1. Computing of formal linearization of the vacuum Einstein equations

$$pr\mathbf{Y}(G_{ij}) = 0. \tag{28}$$

2. We choose an appropriate subgroup of generalised symmetries of vacuum Einstein equations and solve equation

$$pr\mathbf{Y}^0(G_{ij}) = \lambda(x^i, g_{ij}, \ldots)G_{ij}. \tag{29}$$

3. The function λ from the last equation we use for building of equation for $\mathbf{Y}^1$

$$pr\mathbf{Y}^1(G_{ij}) + pr\mathbf{Y}^0(T_{ij}) = \lambda(x^i, g_{ij}, \ldots)T_{ij}. \tag{30}$$

Whenever G_{ij} and its covariant derivatives vanish and $T^k_{i;k} = 0$ holds.

What can we get by the above described procedure:

- We can choose a known tensor of energy-momentum and observe a changing of the selected subgroup of generalised symmetries, that is, if its operator $\mathbf{Y}^0$ is prolonged to the operator $\mathbf{Y} = \mathbf{Y}^0 + \varepsilon\mathbf{Y}^1$ (there may be prolongation with $\mathbf{Y}^1 = 0$). This procedure is important for studying of the fluctuations of the vacuum solution. This fluctuation can be changed to exact solution of Einstein equations (that is, ε is not small yet).
- We can choose a subgroup of generalised symmetries and take this symmetries stable under the fluctuation. As a result of the procedure we get a condition for the form of the tensor of energy-momentum.

I have chosen subgroup $SO(3)$ of the spatial rotations

$$X^{\alpha\beta} = x^\beta\frac{\partial}{\partial x^\alpha} - x^\alpha\frac{\partial}{\partial x^\beta}, \quad \alpha, \beta = 1, 2, 3 \tag{31}$$

computed a formal linearization of Einstein equations

$$pr\mathbf{Y}^0(G_{ij}) = \frac{1}{2}\{[g^{ac}(\delta^d_i\delta^b_j + \delta^b_i\delta^d_j - g_{ij}g^{bd}) - g^{cd}(\delta^a_i\delta^b_j - g_{ij}g^{ab}) - g^{ab}\delta^c_i\delta^d_j]\nabla_c\nabla_d h_{ab} + (g_{ij}g^{ka}g^{mb}R_{km} - R\delta^a_i\delta^b_j)h_{ab}\} \tag{32}$$

and wrote the Einstein's tensor in terms of metric and its derivatives

$$
\begin{gathered}
G_{ik} = \tfrac{1}{2} g^{jl}(g_{kl,ij} + g_{il,jk} - g_{ik,jl} - g_{jl,ik}) + \\
\tfrac{1}{4} g^{jm} g^{ln}(3g_{ij,k}g_{ln,m} + 3g_{jk,i}g_{ln,m} - 3g_{ik,j}g_{ln,m} + 6g_{il,j}g_{kn,m} - \\
6g_{il,j}g_{mk,n} + 2g_{ik,l}g_{mn,j} - 2g_{li,k}g_{mn,j} - 2g_{lk,i}g_{mn,j} - g_{jl,i}g_{mn,k}) - \\
\tfrac{1}{2} g_{ik}[g^{jm}g^{ln}(g_{mn,jl} - g_{ln,mj}) - \\
g^{jl}g^{mn}g^{ab}(\tfrac{3}{2}g_{ma,j}g_{lb,n} - \tfrac{5}{4}g_{ma,j}g_{nb,l} + g_{lm,j}g_{nb,a} + \tfrac{3}{4}g_{mn,j}g_{ab,l} - 2g_{lm,j}g_{ab,n})]
\end{gathered}
\tag{33}
$$

For manipulation with these terrible expressions is very useful the package Ricci for algebraic system MATHEMATICA [6] written by John M. Lee.

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