Klaus Mohnke

On Vassiliev's knot invariants

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1994. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 37. pp. [169]--183.

Persistent URL: http://dml.cz/dmlcz/701554

ON VASSILIEV'S KNOT INVARIANTS

Klaus Mohnke

Abstract

The origin of this article is a lecture given by P.Cartier[1] in Zdikov in February 1993. It presents a construction of a probably 'complete' set of knot invariants based on ideas of Vassiliev, Sossinsky, Kontsevich, Bar–Nathan, and others. To have a convenient framework, we consider a vector space *V* for which the set of all knots is considered to be a basis. Then we construct a finite dimensional filtration in a more or less canonical way together with a natural basis which respects the filtration.

1 Introduction

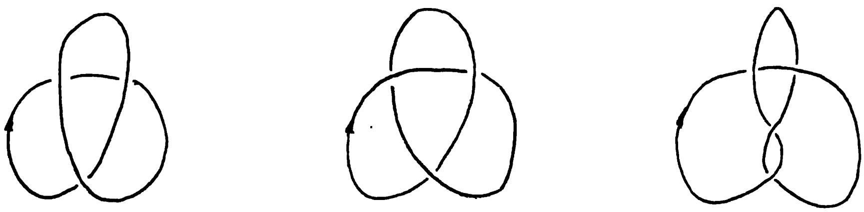
For more then a century the phenomenon of knots in three dimensional topology has been inspiring mathematicians and physicists. One of the first to study knots was the physicist P.G.Tait who proved properties of these objects, gave first tables, and made a series of conjectures at the end of the 19th century.

First of all, let us discuss what we (and mathematicians in general) will understand by the notion *knot*. Naively, you would probably think of it as a string *knotted* in order to keep your shoes at your feet or to keep a parcel together. But there is a problem with this picture: sometimes you want to put your shoes off or you want the person who gets the parcel to look in it and see what presents you have sent — you are very lucky to unknot your *knot*. In the case of the parcel a pair of scissors may help. So, in our case a knot will be a continuous embedding of the (oriented) circle $\mathbf{S}^1$ into the 3-sphere $\mathbf{S}^3$ (usually, one adds the point at infinity to the Euclidean 3-space because one is just interested in isotopy classes of such embeddings). But still we are not satisfied with this notion, uncontrollable degeneracies may occur (think about Peano curves or infinitely iterated knots). So, in our context knots will be *tame knots*, i.e. embeddings which are isotopic to finite polygonal embeddings. Now take your string, knot it, and glue the ends of the string together (using glue rather than a knot!). Now you can deform your knot (e.g. you can 'tighten' it or 'loosen' it) or you can move it in Euclidean space but you will still have the same knot (as even a non–mathematician would be convinced of). So, what we are actually looking at are equivalence classes of tame embeddings where the equivalence is given by isotopies. The resulting classes are nothing else as the path connected components of the space of tame embeddings with the obvious topology as a subset of the space of all polygons in Euclidean space. These classes of knots will simply be called knots throughout the lecture.

The next question is how to visualize a knot. One possibility is to take the famous belt which basically plays the role of the string. This is, unfortunately, not very convenient for books and articles and not for computations either. Just as bad is the representation via a polygon or a smooth embedding... So, we will use *regular planar diagrams* to represent

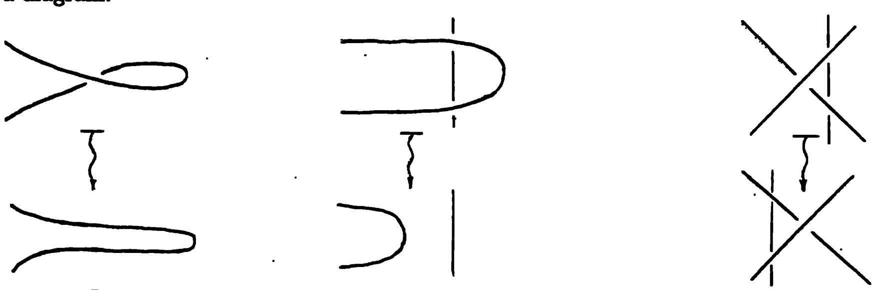
[1]Départment de Mathématique et Informatique, École Normale Supérieure, France

a knot as they were introduced by K.Reidemeister, i.e. special projections of the knot onto the Euclidean plane. Let us have a look at some examples:



unknot trefoil figure eight knot

We just require that the diagram is as tame as the knots we are considering, i.e. they are isotopic as immersions in the Euclidean plane to a polygonal immersion whose singularities are at most double points. In addition to that, we mark the crossings depending on whether we have an over– or an under–crossing of the knot in three space. Again, as with the isotopy classes, there are multiple choices of diagrams for each equivalence class of knots, depending on the choice of the embedding of the given knot as well as the projection. Considering two regular planar diagrams as equivalent if they are isotopic within the space of regular planar diagrams it is clear that to each knot there exist arbitrarily many diagrams representing it – just look at the following picture showing small parts of a diagram:



type I type II type III

But fortunately, this is all that can happen, according to the following

Theorem 1.1 (Reidemeister) *Two diagrams correspond to the same knot iff there is a finite sequence of Reidemeister moves I-III as depicted above transferring one of the diagrams into the other.*

Remark 1.2 *If we replace knots by ribbons which can be considered as knots together with an assigned 'twisting number', undergoing a Reidemeister move type I the ribbon gets a twist, i.e. a change of ± 1 in the 'twisting number' as you easily convince yourself by experiment.*

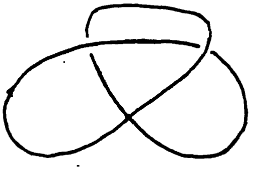
2 The Tautological Vector Space

Now let us start with the basic construction used for defining Vassiliev's invariants. Consider the Q-vector space $\mathcal{V}$ for which the set of all knots is a basis. Denote by $\mathcal{P}$ the vector space generated by all regular planar diagrams is a basis and by $\mathcal{R}$ the subspace of $\mathcal{P}$ which is generated by all differences $[D]-[D']$ where D and D' are diagrams transferrable into each other via a sequence of Reidemeister moves. Denote for a moment by Π the quotient $\mathcal{P}/\mathcal{R}$. Then Reidemeister's Theorem just states that

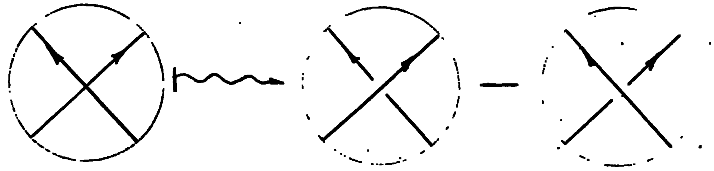
$$\Pi \cong \mathcal{V}.$$

After all, this remark seems as tautological as the vector space itself but, nevertheless, it is rather important for our purposes: remember that we will do all computations on the level of diagrams. We immediately have (tautological) invariants: the elements of the dual vector space. This is basically the kind of invariant we are going to talk about in these lectures and is of course as useless as a tautology could be!

Fortunately, there is much more structure hidden in this vector space $\mathcal{V}$. Let us consider not only ordinary knots but singular knots, i.e. immersions of S^1 into the Euclidean three space for which the only singularities are double points. The corresponding regular planar diagrams of knots with p double points are just diagrams where p crossings have no assigned under– or over–crossing.



Consider a singular knot with one double point. We can resolve the singularity in two ways –to the double point in the picture of a diagram of the singular knot we can assign the difference of the corresponding over– and under–crossing



In the diagram this can be done in a canonical way as the picture above shows, and it is easy to see that this does not depend on the particular choice of the diagram. So, we have a well–defined difference $\Delta K = K_+ - K_- \in \mathcal{V}$ for a singular knot K with just one singularity. Having a singular knot with p double points we can generalize this construction either inductively or by defining

$$\Delta_p K = \sum_{2^p \, possibilities} \pm[possibility] \qquad \in \mathcal{V},$$

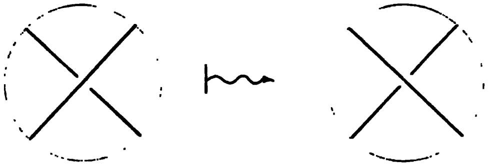
where 'possibility' means to dissolve the p singularities in a certain way analogously to the example $p=1$ and the sign of the summand depends on the number of the chosen under–crossings for dissolving the singularities. So, we can make the following

Definition 2.1 *We denote by $\mathcal{V}_p \subset \mathcal{V}$ the subspace generated by all possible $\Delta_p K$.*

One of the basic properties of $\mathcal{V}$ is the following

Proposition 2.2 *$\{\mathcal{V}_p\}_{p=0}^{\infty}$ is a finite dimensional filtration of the vector space $\mathcal{V}$, i.e. $\mathcal{V}_0 = \mathcal{V}$, $\mathcal{V}_{p+1} \subset \mathcal{V}_p$ and $\mathcal{A}_p = \mathcal{V}_p/\mathcal{V}_{p+1}$ is finite dimensional.*

Proof. The inclusion is clear: an element of $\mathcal{V}_{p+1}$ is a difference of two elements of $\mathcal{V}_p$. For the finite dimensionality observe that $dim(\mathcal{A}_0) = 1$ is just the statement that each knots is equivalent to the unknot if you allow not only Reidemeister moves but also the change of an under– into an over–crossing and vice versa (see the picture below).

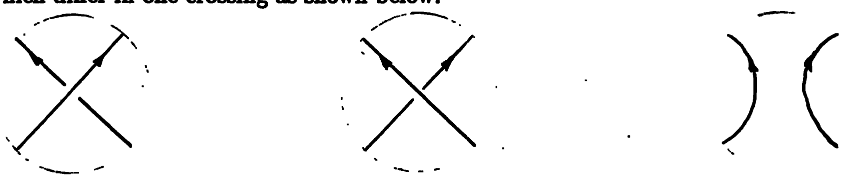


The statement for general p is slightly more subtle. Define $\mathcal{V}^p$ to be the vector space freely generated by all singular knots with exactly p double points. As for regular knots resolve one double point of $K \in \mathcal{V}^{p+1}$. Let $\mathcal{V}_1^p \subset \mathcal{V}^p$ be the vector space generated by all such differences. Applying the operation Δ_p on the remaining p double points we get the vector space $\mathcal{V}_p^1 = \mathcal{V}_{p+1}$. On the other hand $\mathcal{V}^p/\mathcal{V}_1^p$ is finite dimensional: The dimension is the number of different arrangements of pairs of points on a circle – see the Feynman Diagrams below. This is just the statement that with the additional equivalence relation of interchanging over– and under–crossing there is no 'knotting' of singular knots. But now the operation Δ_p generates $\mathcal{V}_p$ and with the equality above we have $dim(\mathcal{V}_p/\mathcal{V}_{p+1}) \leq dim(\mathcal{V}^p/\mathcal{V}_1^p) < \infty$. □

The finite dimensionality of these quotients makes it useful to restrict ourselves to elements of the dual $\mathcal{V}^*$ of finite type:

Definition 2.3 (Vassiliev-Invariants) *We call an element $\phi \in \mathcal{V}^*$ of* finite type *(or a* Vassiliev-Invariant*) iff there exists a $p \in \mathrm{N}$ such that $\phi_{|\mathcal{V}_{p+1}} = 0$.*

Actually a lot of such finite type invariants are well–known to topologists already. Let us consider e.g. the HOMFLY-polynomial. Take the diagrams of three links L_+, L_-, and L_0 which differ in one crossing as shown below.



L_+ L_- L_0

The polynomial is then characterized by the following relations:

$$xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0,$$
$$P_{unknot} = 1.$$

Setting $x = q^N$ $y = -q^{-N}$ and $z = (-1)^N(q - q^{-1})$ we obtain the Turaev polynomials $T(L,N)(q)$. Setting $q = 1 + c_1h + c_2h^2 + ...$ these polynomials translate into power series $T(L,N)(h)$ in h. Denote by $T_i(L,N)$ its coefficients. Then Joan Birman and Xiao-Son Lin proved in [BL] the following

Proposition 2.4 (Birman,Lin) *$T_p(L,N)$ is a Vassiliev invariant of order at most p.*

Proof. Extend $T_p(L,N)$ linearily to the whole vector space $\mathcal{V}$. This obviously defines an element in $\mathcal{V}^*$. Now we see from the first relation for the HOMFLY polynomial via comparison of coefficients that

$$T_p(L_+,N) - T_p(L_-,N) = \\ \Phi_p(T_{p-1}(L_+,N), ..., T_0(L_-,N), T_{p-1}(L_0,N), ...T_0(L_0,N)).$$

Φ_p is a linear functional on $\mathbf{R}$ depending on p, N, c_i. $\Phi_0 \equiv 0$ because $T_0(L,N) = 1$ for all knots L, integers N, and power series q. By iterating this procedure we see that

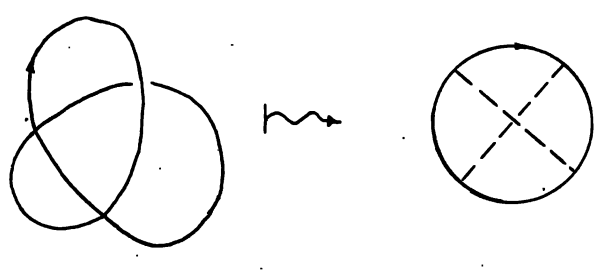
$$T_p(\Delta^{p+1}L, N) = 0$$

for all N, singular L with p double points and q as above. □

Now we want to assign to each knot its image in $\mathcal{V}/\mathcal{V}_p \cong A_0 \oplus A_1 \oplus \ldots \oplus A_{p-1}$. This isomorphism is, unfortunately, not canonical. Kontsevich constructs in his approach a linear map

$$\mathcal{V} \longrightarrow \prod_{p=0}^{\infty} \mathcal{A}_p \cong \varinjlim \mathcal{V}/\mathcal{V}_p = \hat{\mathcal{V}}.$$

The basic ingredients are Feynman diagrams. Assign to each singular knot a Feynman diagram in the following way:



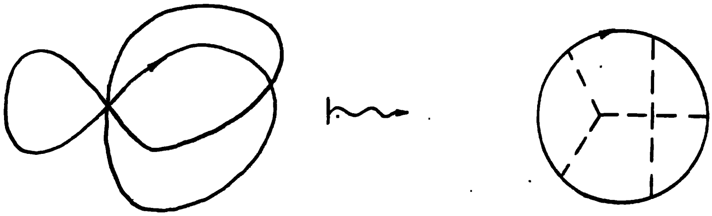
Now the image under the mapping $\mathcal{V}^p \longrightarrow \mathcal{A}_p = \mathcal{V}_p/\mathcal{V}_{p+1}$ just depends on the Feynman diagram as stated in the proof of the theorem.

From our construction follows that the Vassiliev invariant gives a 'complete' system of knot invariants if the following conjecture is true:

$$\bigcap_{p=0}^{\infty} \mathcal{V}_p = 0.$$

3 Feynman Diagrams

We consider the space $\mathcal{I}$ of smooth immersions $x : \mathbf{S}^1 \longrightarrow \mathbf{R}^3$. The set of singularities $F \subset \mathbf{S}^1$ is supposed to be finite and to contain only transverse selfintersections. Then we can assign to x a Feynman diagram via the equivalence relation $\theta_1 \sim \theta_2 \Leftrightarrow x(\theta_1) = x(\theta_2)$ (see the example below).



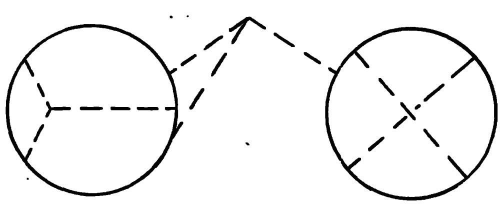
So, the set of Feynman diagrams with one closed Wilson line $\mathcal{D}$ defines a stratification of the topological space

$$\mathcal{I} = \coprod_{D \in \mathcal{D}} \mathcal{I}_D,$$

where $\mathcal{I}_D$ is the set of all immersions with the associated Feynman diagram D. Denote by $\emptyset$ the diagram just consisting of a closed Wilson line. Then $\mathcal{I}_\emptyset = \mathcal{I}_0$ is just the set of knots (without considering the isotopy relation). This set is disconnected and we have

$$\mathcal{V} \cong \mathrm{H}_0(\mathcal{I}_\emptyset; \mathbf{Q}),$$

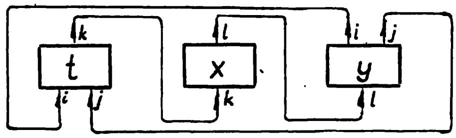
i.e. the path–connected components of $\mathcal{I}_0$ are the isotopy classes of knots.Denote by $\mathcal{I}_1$ the set of immersions with only double points, then $\mathcal{I}_0 \subset \mathcal{I}_1$.We can assign Feynman diagrams not just to singular knots but to singular links, too. All other constructions are working as well: e.g. the class of a link in the corresponding $\mathcal{V}_0/\mathcal{V}_1$ is determined by the number of its components.



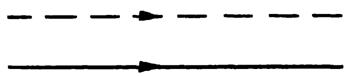
But recall, that we are actually interested in distinguishing components in $\mathcal{I}_0$. In doing so we aim to count the minimal numbers of $\mathcal{I}_1$–walls in order to move from one knot to the other. So, we should study the structure of $\mathcal{I}_1$ carefully. It turns out to be convenient to consider Feynman diagrams with dotted and solid lines and at most trivalent vertices vertices as in the example above. We grade such diagram by half the number of its vertices. Denote by $\mathcal{FD}^p$ the $\mathbf{Q}$–vector space freely generated by all trivalent diagrams of degree p.

Now consider a simple Lie algebra $\mathbf{g}$, a Casimir operator $c \in \mathbf{g} \otimes \mathbf{g}$ and its inverse $c^{-1} \in \mathbf{g}^* \otimes \mathbf{g}^*$ for which the contraction of $c \otimes c$ in the middle components via the natural pairing gives the identity endomorphism of $\mathbf{g}$. Moreover, assume we have a finite dimensional representation $R : \mathbf{g} \longrightarrow \mathrm{End}(V)$. There is a a general principle to write down tensors and there contraction in several variables in a graphical way (see [PR]):

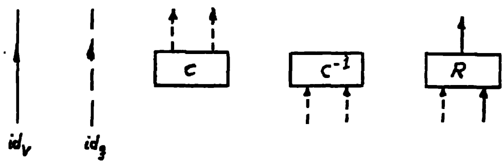
To describe the scalar $t^k_{ij} x^l_k y^{ij}_l$ we draw the following picture:



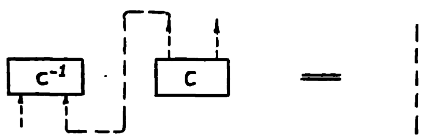
In our case we have two different types of components for a tensor:



For example we have defined already:



The statement that c^{-1} is the inverse of c is just expressed by the picture:

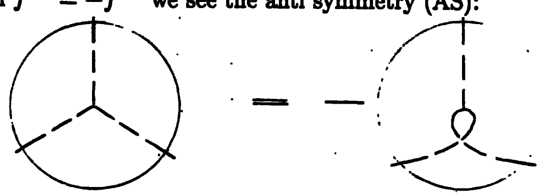


Now c and c^{-1} lower and raise indices, respectively. So we can drop the arrows for the dotted lines. In other words, given the ingredients of a Lie algebra, a Casimir, and a finite dimensional representation we can assign to each Feynman diagram a number. From now on we will specify the Casimir as $c = trace$. Given an orthornormal basis $\{e^a\}$ of g, we have a tensor

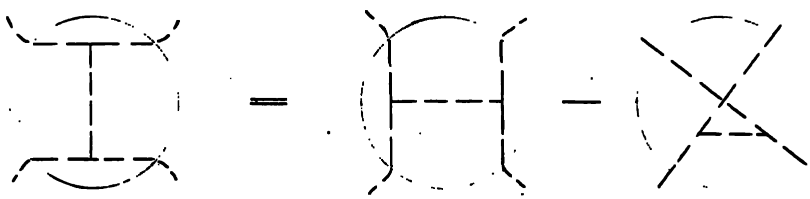
$$f^{abc} = Tr([e^a, e^b]e^c).$$

To a node where three dotted lines meet in our diagram we will assign the 'box' with the inscription 'f', and to a node where a dotted line meets a Wilson line we assign the inscription 'R'. Denoting the contraction operation according to our present rules we have some reductions:

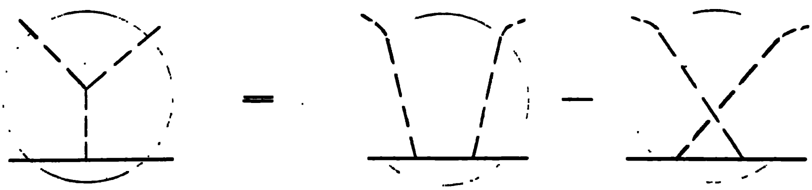
(i) from $f^{abc} = -f^{bac}$ we see the anti symmetry (AS):



(ii) from the Jacobi identity we conclude the IHX-relation:

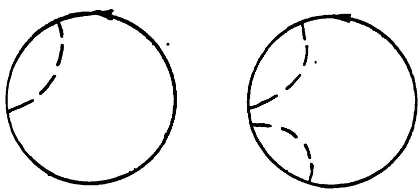


(iii) and, moreover, from the assumption that R is a homomorphism of Lie algebras $R : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ we obtain the STU-relation:

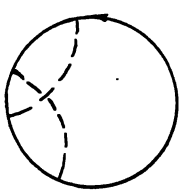


From this relation we easily deduce that all nodes with three dotted lines can be eliminated in a closed Feynman diagram of our type (i.e. each dotted line meets a Wilson line).

Moreover, we have simple representations for $Tr_V c$, $Tr_V c^2$, etc.

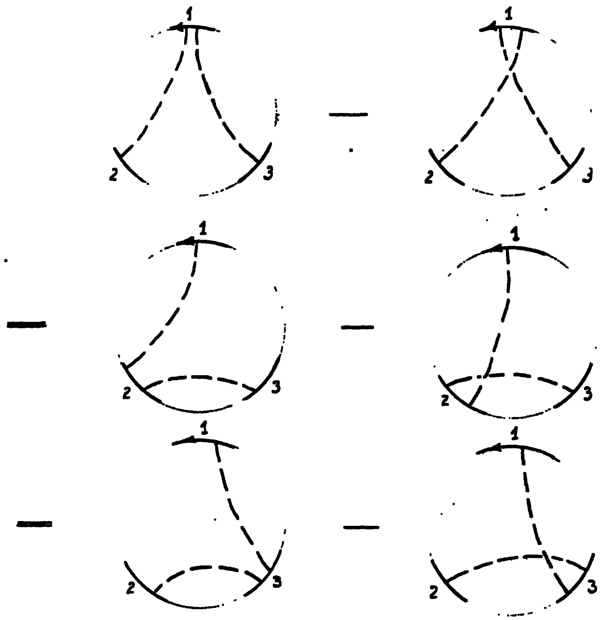


Note the difference to $Tr_V(\sum e^a e^b e^a e^b)$!

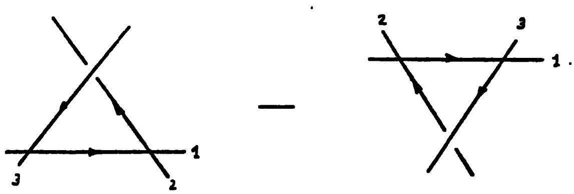


What we obtain are linear functionals on $\mathcal{FD}^p$ which obey the three relations (it actually turns out that STU determines AS and IHX). We denote by $\mathcal{T}^p$ the quotient of $\mathcal{FD}^p$ after these relations. Our aim is to construct linear functionals on the space $\mathcal{V}_p$. Consider the $\mathbf{Q}$-vector space generated by all Feynman diagrams without triple points, with one Wilson line and p dotted lines. This space is isomorphic to our $\mathcal{V}^p/\mathcal{V}_1^p$. Remember that after choosing planar diagrams for each different singular knot we had a well-defined mapping onto $\mathcal{V}_p/\mathcal{V}_{p+1}$. Now introduce on this vector space the following equivalence relations:

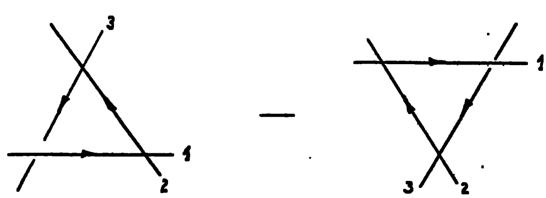
(i) the Yang–Baxter relation:



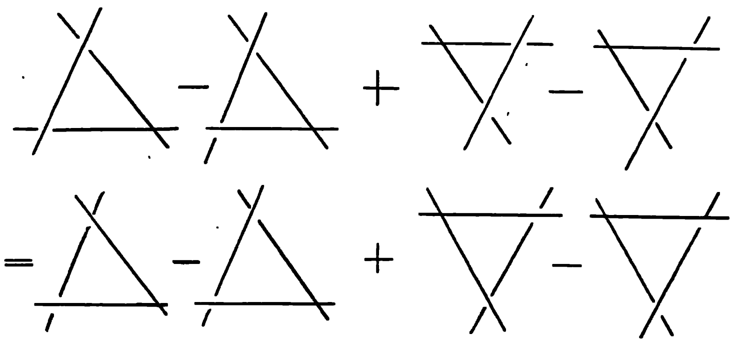
This is rather easy to explain. The first difference of diagrams in the above relation can be represented by a difference of singular knots which locally look like:



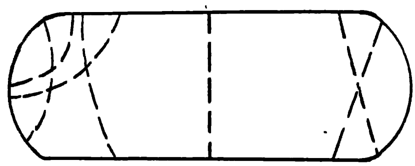
The second difference gives a similar picture, just with another point of the triangle being an overcrossing instead of an intersection.



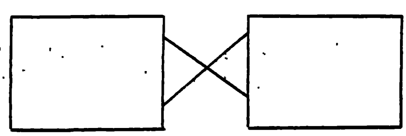
But now, resolving the different points in the two pictures above (possibly changing the overcrossings to undercrossings first, which is allowed because we are not interested in the particular character of nonsingular crossings in the diagram but only in the Feynman diagram itself) gives two differences of singular knots which are obviously the same.



(ii) the neglection of *separated diagrams*:



We can represent such a diagram by a singular knot which is trivially 'twisted':



It is clear that this maps to zero under the above map. Denote by $\mathcal{B}^p$ the quotient of $\mathcal{V}^p/\mathcal{V}^p_1$ by relation (i) and by $\mathcal{A}^p$ the quotient of $\mathcal{V}^p/\mathcal{V}^p_1$ by the two relations above. $\mathcal{B}^p$ and $\mathcal{T}^p$ are isomorphic (see [BN]). It is clear from the remarks that there is a well–defined surjective map

$$\iota_p : \mathcal{A}^p \longrightarrow \mathcal{A}_p.$$

Remark 3.1 *We want to construct from these linear functionals on the vector space of all closed Feynman diagrams linear functionals on $\mathcal{A}_p$. In order to get this, we will use the idea of Bar-Nathan's proof (see [BN]) that ι_p is in fact an isomorphism. It remains to make sure that our linear functionals on $\mathcal{V}^p/\mathcal{V}^p_1$ obey the relations (i) and (ii). An easy computation shows that the classical functionals coming from a finite–dimensional representation of a simple Lie algebra and the trace cannot give the desired invariants for knots. We have to renormalize them, i.e. to find a natural projection from the well defined functionals on $\mathcal{B}^p$ to functionals on $\mathcal{A}_p$ (see [BN] for details). This is a well-known phenomenon in Quantum Field Theory. It corresponds to removing the 'vacuum loop'. Given a simple Lie algebra g and a finite dimensional representation R we denote by $\phi_{g,R}$ the renormalized linear functional corresponding to these data. This is a well-defined functional on $\mathcal{A}^p$. It is not known whether these functionals generate all Vassiliev invariants or not. Computations up to degree 9 show coincidence.*

The rest of this lecture will be devoted to constructing a well–defined inverse of ι_p. This was basically done by Kontsevich. The aim is to define a collection of linear maps κ_p : $\mathcal{V} \longrightarrow \mathcal{A}^p$ such that

$$\begin{array}{lll} \text{(i)} & \kappa_{p|\mathcal{V}_{p+1}} & = 0, \\ \text{(ii)} & \text{For } \bar{\kappa}_p : \mathcal{V}_p/\mathcal{V}_{p+1} & = \mathcal{A}_p \longrightarrow \mathcal{A}^p \\ & \text{we have:} \quad \iota_p \kappa_p & = Id. \end{array}$$

We will use representation of knots via braid group elements to define κ_p.

4 Braid Groups,Knots and the Knishnik- Zamolodchikov Connection

Define

$$\tilde{X}^n = \mathbf{C}^n \setminus \bigcup_{i \neq j} \{t_i = t_j\}$$

and

$$X^n = \tilde{X}^n / \mathbf{S}_n.$$

Definition 4.1 *The* full and the pure braid group *are defined to be*

$$\mathbf{B}_n = \pi_1(X^n)$$

and

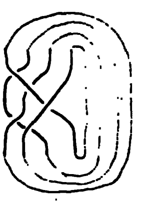
$$\mathbf{P}_n = \pi_1(\tilde{X}^n)$$

respectively.

The representation of $\mathbf{B}_n$ can be given canonically without specifying a point: an element $g \in \mathbf{B}_n$ is represented by a path starting and ending in $X^n \cap \mathbf{R}^n/\mathbf{S}_n$. This is possible because this real part is connected. The full braid group is generated by elements $\sigma_1, ..., \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \forall |i-j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \forall i \leq n-2.$$

We can form a link out of a braid in an obvious way :



This operation is called the *closure* of a braid. There are two operations (the so called Markov's moves) on a braid $b \in \mathbf{B}_n$ which do not change its closure:
(i) conjugation with an arbitrary element $g \in \mathbf{B}_n$,
(ii) *stabilization* of b into the larger braid group $\mathbf{B}_{n+1}$ via $b \mapsto b\sigma_n^{\pm 1}$

Proposition 4.2 (Markov) *The closures of two braids b and b' give the same links iff there is a finite sequence of Markov's moves or their inverses which change b into b'.*

Having a sequence of *traces*, i.e. maps $\{tr_n : \mathbf{B}_n \longrightarrow \mathbf{R}\}$ satisfying the relations

$$tr_n(b) = tr_{n+1}(b\sigma_n^{\pm 1}),$$
$$tr_n(bb') = tr_n(b'b).$$

for $b, b' \in \mathbf{B}_n$, we know, after all, that we obtain a knot invariant assigning to each knot the value of the trace for a braid representative of this knot. Let e.g. $R \in \mathbf{End}(V^{\otimes 2})$ be a solution of the Yang-Baxter equation. Then

$$\sigma_i \longmapsto s_{i,(i+1)} R_{i,(i+1)}$$

defines a representation of $\mathbf{B}_n$ in $V^{\otimes n}$. With a little luck the first condition is satisfied, too, for the usual matrix trace and we get a knot invariant. So, it semms natural to look for such representations and their traces. In the following we summarize some basic steps in the definition of the Knishnik-Zamolodchikov connection and a construction of Drinfeld. Consider the central descending sequence of the pure braid group:

$$\mathbf{P}_n = \mathbf{P}_n^1 \supset \mathbf{P}_n^2 \supset \ldots \quad .$$

Then each factor $\Gamma_n^k = \mathbf{P}_n / \mathbf{P}_n^k$ can be identified with a cocompact subgroup of a simply connected real nilpotent Lie group $\Gamma_n^k \subset G_n^k$. We have a sequence

$$G_n^1 \longleftarrow G_n^2 \longleftarrow \ldots \longleftarrow G_n^\infty,$$

where

$$G_n^\infty = \varprojlim G_n^k,$$

and, similarly, for the Lie algebras

$$\mathbf{g}_n^1 \longleftarrow \mathbf{g}_n^2 \longleftarrow \ldots \longleftarrow \mathbf{g}_n^\infty,$$

with a natural inclusion

$$\mathbf{P}_n \hookrightarrow G_n^\infty.$$

The exponential map

$$exp : \mathbf{g}_n^\infty \longrightarrow G_n^\infty$$

is a diffeomorphism. Thus, for each $b \in \mathbf{P}_n$ there is a uniquely defined element $\ln b \in \mathbf{g}_n^\infty$. There is an analogous construction for $\mathbf{B}_n \hookrightarrow \tilde{G}_n^\infty$ and again $\ln b \in \tilde{\mathbf{g}}_n^\infty$ for $b \in \mathbf{B}_n$.

Let us describe the Lie algebra $\mathbf{g}_n^\infty$ and the logarithm $\ln b$ in greater detail. With $\mathbf{g}_n^\infty(k) = Ker(\mathbf{g}_n^\infty \longrightarrow \mathbf{g}_n^k)$ we have:

$$\mathbf{g}_n^\infty \supset \mathbf{g}_n^\infty(1) \supset \mathbf{g}_n^\infty(2) \supset \ldots,$$

and we define

$$\mathbf{t}_n = \mathbf{t}_n(1) \oplus \mathbf{t}_n(2) \oplus \ldots,$$
$$\mathbf{t}_n(k) = \mathbf{g}_n^\infty(k-1) / \mathbf{g}_n^\infty(k).$$

These algebras have an explicit description. Its generators are well-defined elements $\{t_{ij}\}_{1\leq i<j\leq n} \subset H_1(\tilde{X}^n; \mathbf{R})$ satisfying the following relations:
(1) t_{ij} commutes with t_{kl} if i, j, k, l are pairwise distinct;
(2) t_{ij} commutes with $t_{ik} + t_{kj}$.
So $\mathbf{t}_{n+1}$ is a semi-direct product of $\mathbf{t}_n$ with the Lie algebra freely generated by $t_{1,n+1}, ..., t_{n,n+1}$. Now, the main problem is to find an explicit isomorphism

$$\mathbf{g}_n^\infty \xrightarrow{\sim} \prod_{k=1}^{\infty} \mathbf{t}_n(k) = \mathbf{t}_n.$$

This is basically done via Drinfeld's construction we are going to describe briefly. First recall the construction of the Knishnik-Zamolodchikov connection. Define the holomorphic closed 1-forms ω_{jk} on $\tilde{X}^n$ as

$$\omega_{jk} = \frac{dz_j - dz_k}{z_j - z_k}.$$

These form a basis of $H^1(\tilde{X}^n; \mathbf{R})$. According to Brieskorn and Arnold the only relation is

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0.$$

Having $T_{ij} \in \mathbf{End}(W)$ we define the connection on the trivial vector bundle with fibre W over $\tilde{X}^n$ by the 1-form

$$A = \sum_{j<k} T_{ij}\omega_{ij}.$$

If A is flat, which is equivalent to $A \wedge A = 0$, we call the connection Knishnik-Zamololdchikov connection. This is exactly the case if the T_{ij} satisfy the same relation as the t_{ij} do. So the T_{ij} are actually given by a representation in W of the algebra $\mathbf{t}_n$. For that reason Manin calls $\mathbf{t}_n$ the dual of $H^1(\tilde{X}^n; \mathbf{C})$. We have the following example: For a given simple Lie algebra $\mathbf{h}$ with a Casimir c and a representation $R : \mathbf{h} \longrightarrow \mathbf{End}(V)$ take $W = V^{\otimes n}$ and $T_{jk} = c_{ab}R^a \otimes R^b$ acting on the jth and kth position.

Choosing the Lie algebra and the representation properly we get the desired explicit isomorphism. Having a representation W of $\mathbf{t}_n$ we integrate (or exponentiate) this to a representation of $\mathbf{P}_n$ in W, which is given in terms of power series in the T_{ij}. So, formally we get an isomorphism $\tilde{\Phi}_n : \mathbf{P}_n \longrightarrow \hat{\mathcal{U}}\mathbf{t}_n$, where $\hat{\mathcal{U}}\mathbf{t}_n$ is the completion of the enveloping algebra, i.e. its elements are formal power series in $\mathbf{t}_n$. For this, one has to solve the differential equation

$$d\Phi = A\Phi.$$

This has locally a solution if A is flat and integrating it globally we get a representation of the pure braid group from the monodromy of the solution. Drinfeld explicitly solved this equation and computed the monodromy. Finally, he extended it to an isomorphism $\Phi_n : \mathbf{CB}_n \longrightarrow \mathbf{CS}_n \times \hat{\mathcal{U}}\mathbf{t}_n$. Hereby we used the canonical action of $\mathbf{S}_n$ on the second factor in the half-direct product.

Let us have a look at the simplest nontrivial case where $n = 3$. Then

$$\mathbf{t}_3 = \mathbf{C} << A, B >>,$$

and with

$$z = \frac{z_1 - z_2}{z_1 - z_3}$$

we have to solve

$$\frac{d\Phi}{dz} = \lambda(\frac{A}{z} + \frac{B}{1-z})\Phi.$$

The Kummer transformation for hypergeometric functions gives explicit solutions: With $G_0(z) \sim \exp(\lambda A \ln z)$ around 0 and $G_1(z) \sim \exp(\lambda B \ln(1-z))$ around $z = 1$ we have

$$G_1(z) = G_0(z)\Phi.$$

Therefore the holonomy can be calculated explicitely. Introducing an additional element C we have

$$\mathbf{C} << A, B >>= \mathbf{C} << A, B, C >> /\{A + B + C = 0\},$$

with $\mathbf{S}_3$ acting in the obvious way. Then we can lift the monodromy $\tilde{\Phi}_3$: $\mathbf{P}_3 \longrightarrow \mathbf{C} << A, B, C >> /\{A + B + C = 0\}$ to a homomorphism

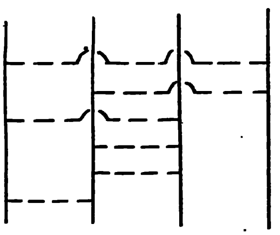
$$\begin{array}{rcl} \Phi_3 : \mathbf{CB}_3 & \longrightarrow & \mathbf{CS}_n \times \mathbf{C} << A; B, C >> /\{A + B + C = 0\} \\ & & \text{given by} \\ \sigma_1 & \mapsto & s_{12} \otimes \exp(\pi i A) \\ \sigma_2 & \mapsto & s_{23} \otimes \Phi \exp(2\pi i B)\Phi^{-1}. \end{array}$$

5 The Construction of the Map κ_p

The first construction was given by Kontsevich, writing down a complicated integral formula and using the Knishnik-Zamalodchikov connection with values in $\mathcal{A}^p$ to show its independence of various choices to be made (see e.g. [BN]). For a more combinatorial definition the reader is referred to [BN, C] The lecturer developed an alternative construction using a representing braid $b \in \mathbf{B}_n$ of the knot. We just present a vague sketch of the idea here.

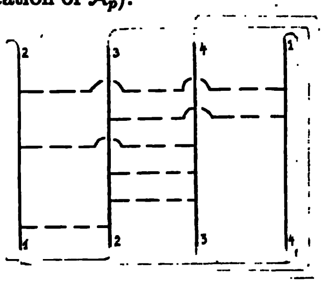
Given the extension of Drinfeld's trivialization of the Knishnik-Zamalodchikov connection Φ_n we get an element $\Phi_n(b) \in \mathbf{CS}_n \times \hat{\mathcal{U}}\mathbf{t}_n$. But this in turn determines again Feynman diagrams:

The enveloping algebra is generated by the monomials in $\{t_{ij}\}_{i,j=1}^n$. On the other hand each monomial in $\mathbf{CS}_n \times \hat{\mathcal{U}}\mathbf{t}_n$ gives a Feynman diagram with open Wilson lines:

$c_s \otimes$ 

$c_s \otimes t_{12} t_{23}^2 t_{13} t_{24} t_{14}$

So, the p-th homogenous part again defines an element in $\mathcal{A}_p$ via closing the 'open' diagram via the c_s if $c_s \in CS_n$ (according to the fact that the image of Φ is a formal power series with complex coefficients we only know that this element lives in the complexification of $\mathcal{A}_p$).



$c_s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ in the example above

We have to check that this gives a well-defined linear functional as claimed, i.e. we have to check invariance under Markov's moves in the braid group. This is basically done using the equivalence relations among the t_{ij}.

References

[BN] Dror Bar-Nathan: On the Vassiliev knot invariants; pre-preprint 1992

[BN1] Dror Bar-Nathan: Vassiliev homotopy string link invariants; pre-preprint 1992

[BL] Joan Birman, Xiao-Son Lin: Knot polynomials and Vassiliev's invariants; Colombia University preprint, 1991

[C] Pierre Cartier: Construction combinatoire des invariants de Vassiliev–Kontsevich des nœds; C.R.Acad.Sci.Paris, t.316, Série I, p. 1205–1210, 1993

[PR] R.Penrose, W.Rindler: Spinors and space time. vol.1 & 2; Cambridge Monographs in Mathematical Physics 1986

[Pi] Sergey Piunikhin: Combinatorial expression for universal Vassiliev link invariant, Harvard University preprint 1993

[S] A.B.Sossinsky: Feynman diagrams, link polynomials and Vassiliev knot invariants; Moscow State University preprint, 1992

Klaus Mohnke
Humboldt-Universität zu Berlin
Fachbereich Mathematik
Institut für Reine Mathematik
Ziegelstraße 13a
10099 Berlin
e-mail: kmohnke@mathematik.hu-berlin.de